A NEW HIGH SCHOOL ALGEBRA

A. S. SINHA

WITH ANSWERS

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WITH ANSWERS

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PREFACE

This book has been prepared on modern lines so as to meet the requirements of those preparing for the High School and Matriculation Examinations of the Indian Universities.

The treatment is thorough, and special care has been taken to deal with the difficulties of students of average ability. The processes of Algebra have throughout been identified with those of Arithmetic. Easy problems explaining the use of the various notations have been set from the very beginning. A large number of oral examples have been given, specially at the introductory stage, to make the students familiar with the elementary principles of Algebra. The examples are mostly original, while some of them have been taken from English and Indian Examination papers. In framing new examples, the Indian conditions with which the students are supposed to be familiar have been kept in view. The examples are sufficiently numerous. A good number of representative examples have been fully worked out.

The first part of the book is a mere translation of my New Middle School Algebra originally written in Hindi and Urdu.

My best thanks are due to Miss Lila Wati Jhanwar, M.A. for her valuable suggestions and assistance in the preparation of the book and to Principal A. D. Banerji, M.A., who was kind enough to go through the manuscript. I also acknowledge my indebtedness to Mrs. F. I. Badcock for the careful reading of proofs. I am also under obligation to the Educational Boards of U. P., C. P. and Rajputana & C. I., whose examination papers of recent years I have given at the end of the book.

DEHRA DUN,
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ANAND SWARUP SINHA

PART II

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PART I

PART I

CHAPTER I

ALGEBRA. SIGNS OF OPERATION

Algebra. Similarity to and Difference from Arithmetic. Signs of Addition, etc. Coefficients. Expressions. Terms.

1. Algebra is that branch of Mathematics which treats of the relation of numbers. Both Arithmetic and Algebra treat of numbers. In Arithmetic all the numbers used are represented by the *digits* 1, 2, 3,...9, while in Algebra the numbers used are also represented by the *letters* of the alphabet a, b, c,...x, y, z. In Arithmetic the digits have fixed values which do not vary, while in Algebra each letter may represent any value which we may assign to it at any particular time or in any particular problem.

Thus, in Arithmetic, $1+2$ is always equal to 3, whereas, in Algebra $a+b$ shall be equal to 3 only when either a is equal to 1, and b is equal to 2, or when a is equal to 2, and b is equal to 1.

It is thus clear that Algebra is simply a generalized form of Arithmetic. They should not be regarded as separate subjects. They are both parts of the *Science of Numbers.*

2. Signs of Operation. The students are already familiar with the meanings and use of the signs of operation '+', '−', '×', '÷', '=', etc., which have the same meanings in Algebra as in Arithmetic.

(i) The sign '+' is read **plus** and is called **the sign of addition**. As in Arithmetic the sum of 4 and 3 is represented by $4+3$, similarly, in Algebra the sum of a and b is represented by $a+b$. When this sign is placed between two or more numbers, it signifies that the values of those numbers have to be *added*. Thus, $a+b$ (read *a plus b*) means that the value represented by a is to be added to that represented by b. If a represents 4 and b represents 3, then the value of $a+b$ will be equal to $4+3$ *i.e.*, 7. Similarly, if c represents 5, $a+b+c$ will be equivalent to $4+3+5$ *i.e.*, 12.

When no sign appears before a number, the sign '+' is understood to be preceding it. Thus, a may be taken as $+a$.

(ii) The sign '−' is read **minus**, and is called **the sign of subtraction.** As in Arithmetic the difference of 4 and 3 is indicated by $4-3$, similarly, in Algebra the difference of a and b is indicated by $a-b$. When it is placed before a number, it signifies that the number is to be *subtracted.* Thus, $a-b$ (read *a minus b*) means that the value represented by b is to be subtracted from that represented by a. If a represents 4 and b represents 3, then $a-b$ will be equal to $4-3$ *i.e.*, 1.

(iii) The sign '×' is read **into** and is called **the sign of multiplication**. When it is placed between two numbers, it indicates that the two numbers are to be multiplied together. Thus, $a \times b$ (read *a into b*) means that the value represented by a is to be multiplied by that represented by b. If a represents 2 and b represents 3, then $a \times b$ will be equal to 2×3 *i.e.*, 6. Similarly, if c represents 4, $a \times b \times c$ will be equivalent to $2 \times 3 \times 4$ *i.e.*, 24.

As in Arithmetic, the product of two numbers is also expressed by placing a **dot** or a **point** between them, thus, 2.3 signifies 2×3; similarly, in Algebra the product of two numbers can also be expressed by placing a point between them. Thus, $a.b$ means $a \times b$.

The sign of multiplication is sometimes omitted in Algebra when numbers are not represented by figures or digits. Thus, the product of a and b is written as ab, and the product of a, b and c is written as abc. But this is not the case in Arithmetic where digits, when written side by side, have got *place values*. Thus, 23 does not mean 2×3, but is equivalent to 2 tens and 3 units, while in Algebra, ab means $a \times b$ and not a tens and b units.

(iv) The sign '÷' is read **divided by** and is called **the sign of division.** When it is placed between two numbers, it indicates that the former is to be divided by the latter. Thus, $a \div b$ (read *a divided by b*) means that the value represented by a is to be divided by that represented by b. If a represents 20 and b represents 4, then $a \div b$ will be equal to $20 \div 4$ *i.e.*, 5.

$a \div b$ is also expressed as $\frac{a}{b}$ or a/b.

(v) The sign '=' is read **is equal to,** and is called **the sign of equality**. Thus, $2+3=5$ indicates that the values of the numbers on either side of the sign are equal. Thus, $a+b=c$, if the sum of the values of a and b is equal to the value of c, *i.e.*, if a represents 2, b represents 3 and c represents 5, then $a+b=c$.

(vi) The sign '$\therefore$' represents **therefore,** and '$\because$' represents **because.**

3. When two or more numbers are multiplied together, then each number is called a **factor** of the product, and each factor is called a **coefficient** of the product of the other factors. Thus, x, y and z are the factors of the product xyz, and in the product xyz, x is the coefficient of yz, y is the coefficient of xz, and z is coefficient of xy.

4. When one of the numbers represented by a letter x be multiplied by another number represented by a digit 3, then in the product $3x$, 3 is called the **numerical coefficient** of x. Similarly, in the product $5a$, 5 is the numerical coefficient of a, and in $23xy$, 23 is the numerical coefficient of xy.

The numerical coefficient is generally placed first. When there is no numerical factor in a product, the numerical coefficient may be supposed to be unity. Thus, in x and xy, the numerical coefficients of x and xy are in each case unity.

5. When numbers or letters and figures are connected by the signs '+' and '−', they are said to form an **expression,** and each part of an expression is called its **term.** Thus, the expression $3+a-b$ is formed of the terms 3, a and b. Similarly, the expression $2a+3bc$ is formed of the terms $2a$ and $3bc$.

An expression is **simple,** when it consists of only one term, such as $2a$. An expression is **compound,** when it consists of two or more terms, such as $2a+b$, $2a+3b-5d$.

6. Terms containing the same letters and differing only in their numerical coefficients are called **like terms**. Thus, $6a$, a and $2a$ are like terms, and $3xy$ and $4xy$ are also like terms.

EXAMPLES I

(*Examples* 23—32 *may be taken orally*)

Write down in signs

1. a into b. **2.** x into y. **3.** 5 into a.

4. l divided by b. **5.** The sum of x, y and z.

6. The product of a, b and c.

7. The sum of 8 and 7 is equal to 15.

8. The sum of a and a is equal to $2a$.

9. The difference of 9 and 7 is equal to 2.

10. The difference of $3x$ and $2x$ is equal to x.

11. The sum of 10 and 8 is equal to the sum of 13 and 5.

12. The sum of a and b is equal to the sum of x and y.

13. The difference of 10 and 6 is equal to the difference of 8 and 4.

14. The difference of x and y is equal to the difference of a and b.

Can the sign '×' be omitted in the following ?

15. $a \times b$. **16.** $3 \times a$. **17.** 3×2. **18.** $x \times y \times z$.

19. What is the difference between 32 and 3×2 ?

20. What is the difference between xy and $x \times y$?

21. What is the difference between 32 and 23 ?

22. What is the difference between xy and yx ?

23. If $x=10$, find the value of

(i) $x+5$. (ii) $x-5$. (iii) $x \times 5$. (iv) $x \div 5$.

(v) $2x+8$. (vi) $5x-10$. (vii) $8x \times 2$. (viii) $4x \div 8$.

24. If $x=9$ and $y=3$, find the value of

(i) $x+y$. (ii) $x-y$. (iii) $x \times y$. (iv) $x \div y$.

(v) $4x+3y$. (vi) $2x-4y$. (vii) $5xy$. (viii) $5x \div 15$.

25. Find the value of $a+b-c$ when

(i) $a=5$, $b=4$, $c=3$. (ii) $a=27$, $b=0$, $c=15$.

(iii) $a=\frac{3}{4}$, $b=\frac{1}{4}$, $c=\frac{1}{2}$.

26. Find the value of $3ab+2cd$ when

(i) $a=5$, $b=2$, $c=1$, $d=4$. (ii) $a=3$, $b=4$, $c=2$, $d=11$.

27. Find the value of $xy-3z$ when
(i) $x=4$, $y=2$, $z=1$. (ii) $x=8$, $y=7$, $z=12$.

28. Find the value of $4xy-3wz$ when
(i) $x=5$, $y=3$, $z=4$, $w=2$. (ii) $x=1$, $y=10$, $z=4$, $w=3$.

29. Find the value of $\frac{a}{b}+c$ when
(i) $a=6$, $b=3$, $c=1$. (ii) $a=14$, $b=7$, $c=10$.

30. Find the value of $\frac{a}{b}-c$ when
(i) $a=6$, $b=3$, $c=1$. (ii) $a=18$, $b=3$, $c=4$.

31. Find the value of $\frac{a}{b}+\frac{c}{d}$ when
(i) $a=5$, $b=10$, $c=4$, $d=8$. (ii) $a=3$, $b=6$, $c=16$, $d=4$.

32. Find the value of $\frac{a}{b}-\frac{c}{d}$ when
(i) $a=5$, $b=10$, $c=4$, $d=8$. (ii) $a=10$, $b=5$, $c=3$, $d=3$.

33. Express 24 algebraically when $a=2$ and $b=4$.

Solution.

Since 24 is equal to 2 tens and 4 units,

$$\begin{aligned}\therefore\ 24 &= 10\times 2+4\\ &= 10\times a+b\\ &= 10\,a+b.\end{aligned}$$

34. Express 35 algebraically when $a=3$ and $b=5$.

35. Express 27 algebraically when $x=2$ and $y=7$.

36. Express 84 and 48 algebraically when $x=4$ and $y=8$.

37. Express 135 algebraically when $x=1$, $y=3$ and $z=5$.

Solution.

Since 135 is equal to 1 hundred, 3 tens and 5 units,

$$\begin{aligned}\therefore\ 135 &= 100+30+5\\ &= 100\times 1+10\times 3+5\\ &= 100\times x+10\times y+z\\ &= 100x+10y+z.\end{aligned}$$

38. Express 234 algebraically when $a=2$, $b=3$ and $c=4$.

39. Express 514 and 415 algebraically when $x=4$, $y=1$ and $z=5$.

40. What is the difference between $5\frac{3}{4}$ and $a\frac{b}{c}$ when $a=5$, $b=3$ and $c=4$?

41. Write down the numerical coefficient of
(i) $3a$. (ii) a. (iii) $100a$. (iv) $\frac{1}{2}a$.

42. Write down the coefficient of x in
(i) $5x$. (ii) ax. (iii) $abcx$. (iv) $10px$.

43. Write down the numerical coefficient of
(i) $6ax$. (ii) xyz. (iii) $20lm$. (iv) $\frac{3}{5}xy$.

44. Which are like terms in the following expressions ?
(i) $6a+3x+5a+8x$. (ii) $3xy+4ab+\frac{1}{2}xy+ab+2xy$.

45. Express x rupees (i) in annas, (ii) in pies.

Solution.

(i) Since 5 rupees $=16\times5$ annas $=80$ annas.
Similarly x ,, $=16\times x$,, $=16x$,,

(ii) Since 3 rupees $=192\times3$ pies $=576$ pies.
Similarly x ,, $=192\times x$,, $=192x$ pies.

46. Express x hours (i) in minutes, (ii) in seconds.

47. Express a miles (i) in yards, (ii) in feet, (iii) in inches.

48. Express y metres (i) in decimetres, (ii) in centimetres, (iii) in millimetres.

49. Express x square yards (i) in square feet, (ii) in square inches.

50. If a boy walks at the rate of 3 miles an hour, how far will he walk in 4 hours ?

51. If a boy walks at the rate of 3 miles an hour, how far will he walk in x hours ?

52. If a boy walks at the rate of y miles an hour, how far will he walk in 4 hours ?

53. If a boy walks at the rate of y miles an hour, how far will he walk in x hours ?

54. If one dozen oranges cost Rs. 2, what will x oranges cost ?

55. The length and breadth of a rectangle are l yards and b yards respectively, find its area.

56. The area of a rectangle is a square yards and its length is l feet, find its breadth.

57. Rama has a pens, Krishna b pens and Gopal c pens. How many pens have they altogether ? What does the answer become when $a=5$, $b=3$ and $c=9$?

58. A farmer took x goats to the market for sale and sold y goats. How many goats did he bring back ? What does the answer become when $x=50$ and $y=35$?

59. A man took a horses to the market for sale and returned with b horses. How many horses did he sell ? What does the answer become when $a=75$ and $b=40$?

60. A man walks x miles on the first day, y miles on the second, z miles on the third and w miles on the fourth. How many miles did he walk in the four days ? What does the answer become when $x=5$, $y=6$, $z=4$ and $w=7$?

CHAPTER II

EASY ADDITION AND SUBTRACTION

Addition of Like Terms

7. The process of addition in Algebra is the same as that in Arithmetic with this difference that in Algebra we make use of letters besides figures.

For example, 4 men and 5 men are together equal to 9 men,

i.e., 4 men + 5 men = 9 men.

If for the word 'men', its first letter 'm' is written, the above statement can be written as

$$4m + 5m = 9m.$$

Thus, we see that it makes no difference whether we write 'men' or 'm'.

Similarly, we can have

$$4x + 5x = 9x,$$

$$7a + 8a = 15a, \text{ etc.}$$

We have learnt in Arithmetic that only things of the same kind are added together. For example, 5 books and 6 books are together equal to 11 books ; and 2 rupees, 5 rupees and 8 rupees are together equal to 15 rupees,

i.e., 5 books + 6 books = 11 books,

and 2 rupees + 5 rupees + 8 rupees = 15 rupees.

Writing 'b' for 'books' and 'r' for 'rupees', the above can be written algebraically thus

$$5b + 6b = 11b,$$

$$\text{and } 2r + 5r + 8r = 15r.$$

From the above we see that in Algebra also only like terms can be added together. In Arithmetic 5 books cannot be added to 3 pens nor a certain number of houses can be added to a certain number of men, similarly in Algebra $5b$ cannot be added to $3p$. Their sum is expressed as $5b + 3p$, for b cannot be added to p. Similarly, the sum of x and y which are unlike terms is expressed as $x + y$. But if $2x$ and $5x$ which are like terms are to be added, their sum is equal to $7x$, obtained by adding their numerical coefficients.

Hence, we see that *the sum of a number of like terms is a single similar like term whose numerical coefficient is equal to the sum of the numerical coefficients of all the like terms.*

EXAMPLE 1. *Find the sum of* 3x *and* 5x.

The sum of the numerical coefficients of the two terms is $3+5=8$.

$\therefore\ 3x+5x=8x.$

EXAMPLE 2. *Simplify* 3xy + xy + 2xy.

The sum of the numerical coefficients of the three terms

$=3+1+2=6.$

$\therefore\ 3xy+xy+2xy=6xy.$

EXAMPLE 3. *Find the sum of* 5a, b, 6a, 7b *and* 2a.

Since $5a$, $6a$ and $2a$ are like terms of one kind and b and $7b$ are like terms of the second kind,

$\therefore\ 5a+6a+2a=13a,$

and $b+7b=8b.$

Hence the sum $=5a+b+6a+7b+2a$

$=5a+6a+2a+b+7b$

$=13a+8b.$

NOTE. $13a$ and $8b$ cannot be added together as they are not like terms.

EXAMPLES II

(*Examples* 1—22 *may be taken orally*)

Add together

1. 2 yards and 7 yards.
2. $2y$ and $7y$.
3. $4x$ and $8x$.
4. a and $3a$.
5. a, a and a.
6. $3x$, $5x$ and $8x$.
7. x and $\frac{1}{2}x$.
8. $\frac{1}{2}x$ and $\frac{1}{3}x$.
9. $\frac{1}{2}b$, $\frac{1}{3}b$ and $\frac{1}{3}b$.
10. x, $\frac{1}{3}x$ and $\frac{2}{3}x$.
11. $4b$, $5b$, $8b$ and $10b$.
12. c, $10c$, $20c$ and $30c$.
13. x, $10x$, $100x$ and $1000x$.
14. p, $3p$, $5p$, $7p$, $9p$ and $11p$.

Simplify

15. $d+3d+8d+10d$.

16. $x+2x+3x+4x+5x+6x$.

17. $a+\frac{1}{3}a+\frac{1}{4}a+\frac{2}{3}a+\frac{3}{4}a$.

18. $ab+ab+ab+ab$.

19. $xy+7xy+8xy$.

20. $xyz+2xyz+4xyz$.

21. $2abc+\frac{1}{2}abc+\frac{1}{4}abc$.

22. $abcd+3abcd+9abcd$.

Simplify the following by combining like terms

23. $2a+3x+5a+9x$.

24. $a+3b+\frac{1}{2}a+4b+\frac{3}{2}a$.

25. $a+2b+7c+3a+4c+7b+c$.

26. $a+3b+2a+5c+4a+7b+3c$.

27. $2xy+yz+xy+5yz+4xy+3yz$.

28. $ab+2bc+3ca+4ab+5bc+6ca+7ab+8bc+9ca$.

Find the sum of

29. x and y.

30. a, b and c.

31. ab, bc and ca.

32. Rama has x oranges and Krishna y, how many oranges have they together ?

33. Kailash travelled x miles on cycle, y miles on foot and z miles on horse, how many miles did he travel altogether ?

34. Shyama travelled x miles in the morning, y miles in the afternoon and 5 miles in the evening, how many miles did he travel in the whole day ?

35. Madho spent $3x$ hours in reading, $2x$ hours in writing and $5x$ hours in playing, how many hours did he spend altogether ?

36. A man has four boxes, in each of which he has x rupees. How many rupees has he altogether ?

37. A purse has x rupees, x annas and x pies. Express, in pies, the value of the contents of the purse.

38. Kailash has 1 rupee, Sagar x annas and Prem y pies. How many pies have they altogether ?

39. Rama has x pounds, Krishna y shillings and Gopal z pence. How many pence have they altogether ?

40. In a cricket match, a boy scores x runs in the first over, y runs in the second, no runs in the third, $2x$ runs in the fourth and 3 runs in the fifth. How many runs did he score in all the five overs ?

Addition of Compound Expressions

8. In Compound Addition in Arithmetic when rupees, annas and pies are added to rupees, annas and pies, the amounts are so arranged in rows that rupees fall under rupees, annas under annas and pies under pies in the same vertical columns, and then the sum of pies is written down under the column of pies, the sum of annas under the column of annas, and the sum of rupees under the column of rupees. Similarly, in Algebra, if expressions of two or more terms are to be added together, they are arranged so that like terms come in the same vertical columns, and each column is then added separately (beginning with that on the left) and its sum written below it. The only difference is that in Arithmetic, if the sum of pies exceeds 12, pies are converted into annas, and the remainder is placed under pies' column, and the number of annas so obtained is added to annas in the annas' column. Similarly, if the sum of annas exceeds 16, annas are converted into rupees, and the remainder is placed under the annas' column, and the number of rupees so obtained is added to rupees in the rupees' column. The sum of rupees is then placed under rupees' column. Whereas in Algebra the sums of like terms are placed under their own columns and are not taken to the next.

For example

Rs.	as.	p.		
12	2	3	and	$12r + 2a + 3p$
6	1	2		$6r + a + 2p$
7	8	1		$7r + 8a + p$
8	3	4		$8r + 3a + 4p$
33	14	10		$33r + 14a + 10p$

EXAMPLE 1. *Find the sum of* $2x+3y+6z$, $3x+2y+7z$ *and* $4x+y$.

Here we write down the expressions so that like terms are in the same vertical columns *i.e.*, x may come under x, y under y and z under z, and then add up each column separately and write down the sum of x's under the column of x, the sum of y's under the column of y and the sum of z's under the column of z thus

$$\begin{array}{l} 2x+3y+6z \\ 3x+2y+7z \\ 4x+\ y \\ \hline 9x+6y+13z \end{array}$$

Hence the sum is $9x+6y+13z$.

EXAMPLE 2. *Find the sum of* $4yz+3xz$, $5xy+3yz+2xz$, $4xy+2yz$ *and* $yz+xz$.

$$\begin{array}{r} 4yz+3xz \\ 5xy+\ 3yz+2xz \\ 4xy+\ 2yz\qquad \\ yz+\ xz \\ \hline 9xy+10yz+6xz \end{array}$$

Hence the sum is $9xy+10yz+6xz$.

EXAMPLES III

Add together

1. $a+2b$, $2a+b$.
2. $x+3y$, $2x+5y$.
3. $x+5$, $3x+2$.
4. $a+b$, $2a+3b$, $4a+7b$.
5. $3x+y$, $x+3y$, $2x+2y$.
6. $x+y+z$, $2x+3y+4z$.
7. $2x+3y+4z$, $x+5y+9z$.
8. $2ab+3xy$, $5ab+4xy$.
9. $a+bc$, $2a+3bc$, $3a+5bc$.
10. $a+b$, $2a+3b$, $3a+5b$, $4a+7b$.
11. $a+2b+3c$, $3a+b+2c$, $2a+3b+c$.
12. $5d+6f+9$, $3d+f+4$, $d+8f+1$.
13. $5a+8b+11c$, $4a+7b+10c$, $5a+9b+12c$.
14. $5a+5b+9c$, $4a+8b+7c$, $9a+4b+6c$, $5a+7b+9c$.
15. $a+4b+7c$, $2a+5b$, $3a+c$.
16. $2l+3m+4n$, $5l+6m+7n$, $8m+9n$.
17. $a+b$, $b+c$, $c+a$.
18. $ab+bc$, $bc+ca$, $ca+ab$.
19. $\frac{3}{4}a+\frac{2}{5}b$, $\frac{1}{5}b+\frac{3}{7}c$, $\frac{4}{7}c+\frac{1}{4}a$.
20. $a+\frac{1}{2}b+\frac{1}{3}c$, $\frac{3}{4}a+b+\frac{2}{3}c$, $\frac{1}{4}a+\frac{1}{2}b+c$.
21. $ab+c$, $3ab+5c$, $7ab+2c$, $4ab+7c$, $6ab+10c$.
22. $xy+yz+zx$, $zx+xy$, $xy+yz$, $yz+zx$.
23. $pq+2qr+3rp$, $4pq+5qr$, $6pq+7rp$, $8qr+9rp$.
24. $a+b+c+1$, $2b+3c+4$, $5a+c+3$, $b+4c$, $6a$.
25. $1+x+xy$, $3x+2y+5$, $10xy+6x$, $x+y$.

Subtraction of Like Terms

9. The process of subtraction in Algebra is the same as that in Arithmetic. If, for example, 4 caps are taken away from 6 caps, we shall have 2 caps,

i.e., 6 caps − 4 caps = 2 caps.

If for the word 'caps' its first letter 'c' is written, the above statement can be written as

$$6c - 4c = 2c.$$

Hence, we see that it makes no difference whether we write 'caps' or 'c'.

Similarly, we can have

$$8a - 7a = a, \text{ etc.}$$

We have learnt in Arithmetic that things of one kind can be subtracted or taken away from the things of the same kind. For example, if 7 books are taken away from 10 books, we shall have 3 books,

i.e., 10 books − 7 books = 3 books.

If we write 'b' for 'books', the above can be written algebraically as

$$10b - 7b = 3b.$$

We thus see that in Algebra also, only like terms can be subtracted from like terms. In Arithmetic it is impossible to take away 3 pens from 5 books, similarly, in Algebra $3p$ cannot be subtracted from $5b$. Their difference is expressed as $5b - 3p$. Similarly, if y be subtracted from x, where x and y are unlike terms, the difference is expressed as $x - y$. But, if $2x$ be subtracted from $5x$, where $2x$ and $5x$ are like terms, the difference will be equal to $3x$, obtained by subtracting their numerical coefficients.

Hence, we see that *the difference of two like terms is a similar like term whose numerical coefficient is equal to the difference of the numerical coefiicients of the two terms.*

EXAMPLE 1. *Subtract* 3x *from* 5x.

If the numerical coefficient of the first term be subtracted from that of the second, we get the numerical coefficient of the difference.

The difference of the numerical coefficients $= 5 - 3 = 2$.

$$\therefore\ 5x - 3x = 2x.$$

EXAMPLE 2. *Simplify* 8xy − 5xy.

Since $8 - 5 = 3$,

$$\therefore\ 8xy - 5xy = 3xy.$$

EXAMPLES IV

(Oral)

Simplify

1. $8a-5a$.
2. $9x-x$.
3. $100l-98l$.
4. $3x-\frac{1}{2}x$.
5. $\frac{1}{3}x-\frac{1}{4}x$.
6. $xy-xy$.
7. $10ab-8ab$.
8. $5abc-abc$.
9. $12wxyz-7wxyz$.
10. Subtract y from x.
11. By how much does $6x$ exceed $3x$?
12. By how much does $7ab$ exceed $4ab$?
13. By how much does x exceed y ?
14. What must be added to $3x$ in order that the sum may be $5x$?
15. What must be added to $12lm$ in order that the sum may be $21lm$?
16. Rama has x pencils ; if he gives 5 pencils to his brother, how many pencils will he have ?
17. Kailash has x rupees ; if he gives y annas to Prem, how many rupees will he have ?

Subtraction of Compound Expressions

10. In addition, expressions which are to be added together are placed in rows in such a way that like terms come in the same vertical columns. Similarly, in subtraction the expression which is to be subtracted is placed under that from which it is to be subtracted in such a way that like terms come under one another, and then their differences are written under their own columns.

Thus if $6a+b+3c$ be subtracted from $12a+2b+5c$, the question can be written as

$$\begin{array}{r} 12a + 2b + 5c \\ 6a + b + 3c \\ \hline 6a + b + 2c \end{array}$$

Hence the difference is $6a+b+2c$.

EXAMPLES V

Subtract

1. $a+2b$ from $2a+4b$.

2. $3a+9$ from $6a+9$.

3. $2b$ from $a+3b$.

4. $2a$ from $2a+3b$.

5. $x+2y$ from $3x+2y$.

6. $x+y$ from $x+y$.

7. $\frac{1}{2}x+\frac{1}{3}y$ from $x+y$.

8. $\frac{1}{3}x+\frac{1}{5}y$ from $\frac{2}{5}x+\frac{3}{5}y$.

9. $3x+2y+z$ from $4x+3y+2z$.

10. $4a+5b+6c$ from $12a+8b+7c$.

11. $a+\frac{1}{2}b+\frac{2}{3}c$ from $2a+2b+2c$.

12. $\frac{2}{3}x+\frac{3}{4}y+\frac{4}{7}z$ from $x+y+z$.

13. $6ab+3bc+ca$ from $7ab+4bc+ca$.

14. $5xy+3zx$ from $6xy+7yz+8zx$.

15. $ab+c+3$ from $4ab+2c+9$.

16. $lm+mn+np$ from $2np+3mn+lm$.

17. $ab+2c+3d+4$ from $ab+3c+5d+7$.

18. By how much is $6x+5y$ greater than $4x+y$?

19. By how much is $5a+4b+3c$ less than $6a+5b+4c$?

20. What must be added to x in order that the sum may be $x+y$?

21. What must be added to $l+m+n$ in order that the sum may be $3l+2m+n$?

22. What must be added to $4ab+2bc+3ca$ in order that the sum may be $5ab+3bc+6ca$?

23. What must be added to $x+\frac{1}{2}y+\frac{3}{4}z$ in order that the sum may be $x+y+z$?

24. What must be added to $5xy+3yz+zx$ in order that the sum may be $8xy+3yz+\frac{3}{2}zx$?

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

11. In Arithmetic smaller numbers can be subtracted from greater numbers, or equal numbers can be subtracted from equal numbers, but greater numbers cannot be subtracted from smaller numbers. In Arithmetic we have dealt with such cases as

$$5 - 3 = 2,$$
$$5 - 4 = 1,$$
$$5 - 5 = 0,$$

but we have not dealt with the following cases :

$$5 - 6 = ?$$
$$5 - 7 = ?$$
$$4 - 5 = ?$$

From the above we see that when the minuend is 5, and the subtrahend is 3, the difference is 2, and when the minuend is 5, and the subtrahend is 4, the difference is 1, *i.e.*, as the subtrahend goes on increasing while the minuend remains the same, the difference goes on decreasing, and when the subtrahend becomes equal to the minuend, the difference becomes equal to zero. But, if the subtrahend is greater than the minuend, as in the last three examples, the difference will be smaller than zero. Thus, we see that greater numbers can only be subtracted from smaller numbers when it is supposed that there exist numbers smaller than zero. Such numbers are called **negative numbers**, while numbers greater than zero are called **positive numbers**.

12. Positive numbers are expressed by either placing '+' sign before them or by not placing any sign before them, thus

$$+1,\ +2,\ +3,\ +4,\ +5, \text{ etc.},$$
$$\text{or}\quad 1,\ 2,\ 3,\ 4,\ 5, \text{ etc.},$$

are all positive numbers, and they are read *plus one*, *plus two*, *plus three*, *plus four*, *plus five*, etc., or simply, *one*, *two*, *three*, *four*, *five*, etc.

Negative numbers are expressed by writing '−' sign before them, thus

$$-1, \ -2, \ -3, \ -4, \ -5, \text{ etc.,}$$

are negative numbers, and they are read *minus one*, *minus two*, *minus three*, *minus four*, *minus five*, etc.

Thus we see that signs '+' and '−' have two meanings : (i) the sign '+' means 'to add' and the sign '−' means 'to subtract', (ii) these signs show whether the numbers are positive or negative.

13. Suppose we start with zero and continually add one, we obtain a series of numbers which can be written as

$$0, \ 1, \ 2, \ 3, \ 4, \ 5, \ldots$$

But how are we to write the numbers if we start with zero and continually subtract one ? For example, what will be the remainder, if 1 is subtracted from 0 ?

$$0 - 1 = ?$$

Since in addition and subtraction zero has no value, hence

$$0 - 1 = -1.$$

Similarly,

$$0 - 2 = -2,$$
$$0 - 3 = -3,$$
$$0 - 4 = -4,$$
$$0 - 5 = -5,$$

etc.

Thus in Algebra the series of numbers can be written as

$$\ldots -5, \ -4, \ -3, \ -2, \ -1, \ 0, \ +1, \ +2, \ +3, \ +4, \ +5, \ldots$$

In this series we find the numbers continually increasing one by one, if we proceed from left to right, and continually decreasing one by one, if we proceed from right to left *i.e.*, $+4$ is greater than $+3$ by one, $+1$ is greater than 0 by one, 0 is greater than -1 by one,

-1 is greater than -2 by one, and -4 is greater than -5 by one; while $+5$ is greater than $+2$ by three, and -2 by seven. Similarly, $+4$ is less than $+5$ by one, $+1$ is less than $+2$ by one, -1 is less than 0 by one, -5 is less than -4 by one; while -5 is less than -3 by two, less than 0 by five and less than $+3$ by eight.

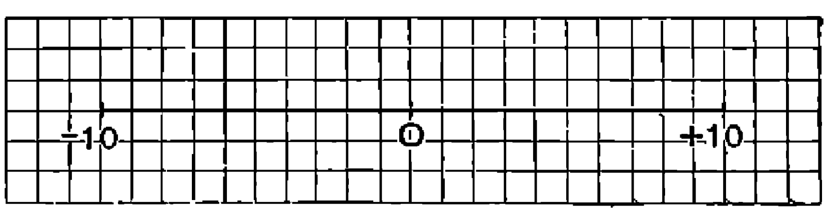
Now the examples of Article 11 can be written thus

$$5 - 3 = 2,$$
$$5 - 4 = 1,$$
$$5 - 5 = 0,$$
$$5 - 6 = -1,$$
$$5 - 7 = -2,$$
$$4 - 5 = -1.$$

Graphic Representation of Positive and Negative Numbers

14. The students can have a very clear idea of positive and negative numbers from a graphic representation on squared paper.*

Take any point O on one of the horizontal lines drawn on a squared paper and regard each division, or the side of a

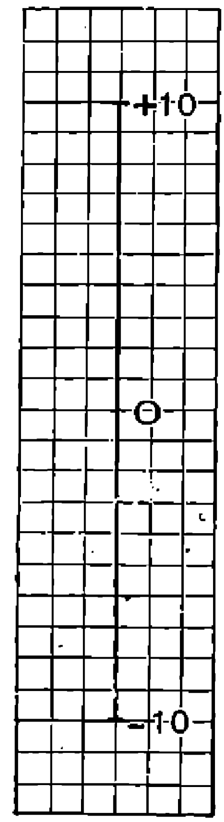


small square, as a unit. Now, if the divisions to the right of O are regarded as positive, then those to the left of

* A squared paper is one, which is divided by means of parallel straight lines into a large number of small squares, each side of which is generally one-tenth of an inch or one millimetre in length. The length of the sides of these squares is generally taken as a unit of length.

O are to be regarded as negative. Hence any positive number, + 10 for instance, may be represented by marking a point after counting 10 divisions from O to the right, and any negative number, − 10 for instance, may be represented by marking a point after counting 10 divisions from O to the left.

Similarly, if a vertical line is drawn on a squared paper and any point O is taken on it, (each division or side of a square being taken as a unit), and if the divisions measured upwards are regarded as positive, then, those measured downwards are to be regarded as negative. Hence, any positive number, + 10 for instance, may be represented by marking a point after counting 10 divisions from O upwards, and any negative number, − 10 for instance, may be represented by marking a point after counting 10 divisions from O downwards.



NOTE. It is customary to represent positive numbers by lines drawn to the right or upwards, and negative numbers by lines drawn to the left or downwards.

Concrete Illustrations of Positive and Negative Quantities

15. We shall now give some concrete illustrations to make the meaning of positive and negative numbers more clear to the students.

(i) Suppose 8 new boys are admitted to a class, while 5 old boys leave it. Then the number of boys in that class increases by three over the number at the beginning. On the contrary, suppose 5 new boys are admitted to a class, and 8 old boys leave it. Then the number of boys in that class decreases by 3 below the number at the beginning.

If the number of increase of boys be regarded positive, and decrease negative, then the above statements can be expressed algebraically thus,

$$8 \text{ boys} - 5 \text{ boys} = +3 \text{ boys},$$

$$5 \text{ boys} - 8 \text{ boys} = -3 \text{ boys}.$$

Thus, although arithmetically '− 3 boys' is an impossible expression, yet algebraically '− 3 boys' may be taken to mean a decrease of 3 boys.

(ii) Suppose a trader buys an article for Rs. 12 and sells it for Rs. 20, he gains Rs. 8 or *plus* Rs. 8. On the contrary, suppose he buys an article for Rs. 20 and sells it for Rs. 12, he loses Rs. 8. Since gain is found by subtracting what is lost from what is gained, therefore, if gain be regarded positive, and loss negative, then, a loss of Rs. 8 can be written as a gain of − Rs. 8 or *minus* Rs. 8. Hence, in the second case, we can say that he gained − Rs. 8. The above statements can now be expressed algebraically thus,

$$\text{Gain} = \text{Rs. } 20 - \text{Rs. } 12 = \text{Rs. } 8,$$

$$,, \quad = \text{Rs. } 12 - \text{Rs. } 20 = -\text{Rs. } 8.$$

Thus, although '− Rs. 8' standing by itself has no meaning whatever, yet algebraically, a gain of − Rs. 8 may be taken to mean a loss of Rs. 8.

(iii) Suppose a man first gains Rs. 100 and then loses Rs. 70, his gain is Rs. 30. On the contrary, suppose he first gains Rs. 70, then loses Rs. 100, he loses Rs. 30 or he is said to gain − Rs. 30, *i.e.*, he has to pay Rs. 30 more. Again, suppose he first loses Rs. 100, then gains Rs. 70, he loses Rs. 30 or he is said to gain − Rs. 30. While, if he first loses Rs. 100 and then again loses Rs. 70, he loses Rs. 170 altogether or he is said to gain − Rs. 170 altogether. The above statements can be expressed algebraically thus,

$$\text{Gain} = \text{Rs. } 100 - \text{Rs. } 70 = \text{Rs. } 30,$$

$$,, \quad = \text{Rs. } 70 - \text{Rs. } 100 = -\text{Rs. } 30,$$

$$,, \quad = -\text{Rs. } 100 + \text{Rs. } 70 = -\text{Rs. } 30,$$

$$,, \quad = -\text{Rs. } 100 - \text{Rs. } 70 = -\text{Rs. } 170.$$

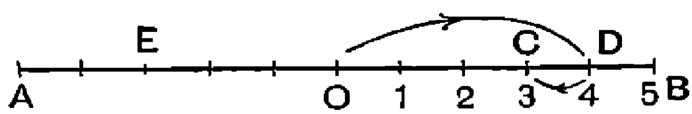
NOTE. From the above it follows that

$$
\begin{aligned}
100-70 &= 30,\\
70-100 &= -30,\\
-100+70 &= -30,\\
-100-70 &= -170.
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
-100-50 &= -150,\\
-3-4 &= -7,\\
\text{and}\quad -5-6 &= -11.
\end{aligned}
$$

(iv) Suppose AOB is a straight road 10 miles in length on which marks at intervals of one mile are made and O is its middle point. Now, if a boy goes 4 miles from O to the right and then returns 1 mile, he will be at a distance of 3 miles from O to the right, *i.e.*, he will be at the point C. (The directions of going and returning are denoted by arrowheads). If the direction of going be regarded positive and that of returning negative, he will be at a distance of



+ 3 miles from the starting point. Although the boy has travelled 4 + 1, *i.e.*, 5 miles, yet since he arrives at C, which in the direction of his going is only 3 miles from O, we say algebraically that the boy first goes + 4 miles and reaches D and then goes − 1 mile and arrives at C, *i.e.*, he has travelled + 4 miles and − 1 mile, *i.e.*, + 3 miles in all.

If the boy first goes 5 miles from O to the right and then returns 5 miles, *i.e.*, on reaching B, he travels 5 miles in the opposite direction, or to the left, then he comes back to the starting point O. Although he has travelled 5 + 5 or 10 miles in all, yet algebraically he has first travelled + 5 miles and then − 5 miles, *i.e.*, at the end of his journey he is at a distance of 0 mile from his starting point.

Again, suppose the boy first goes 5 miles from O to the right and then returns 8 miles to the left, he will pass through O and reach E, *i.e.*, he is at a distance of 3 miles to the left of O or − 3 miles to the right of O in the direction of his start. Although he has travelled 5 + 8 or 13 miles in all, yet algebraically he has first travelled + 5 miles and then − 8 miles, *i.e.*, − 3 miles in all.

NOTE. It has become a convention with regard to the meaning of negative sign in the measurement of lengths, that if lengths measured in one direction or sense along a straight line, generally to the right, are considered positive, then the lengths measured in the opposite direction, generally to the left, are to be considered negative.

16. If instead of boys, rupees and miles, we use letters, then the results of the examples of the previous article can be stated thus,

$$8b - 5b = +3b,$$
$$5b - 8b = -3b,$$
$$20r - 12r = +8r,$$
$$12r - 20r = -8r,$$
$$100r - 70r = +30r,$$
$$70r - 100r = -30r,$$
$$-100r + 70r = -30r,$$
$$-100r - 70r = -170r,$$
$$4m - m = +3m,$$
$$5m - 5m = 0,$$
$$5m - 8m = -3m.$$

EXAMPLES VI

(Oral)

1. A man has Rs. 250 but owes Rs. 150. What will he have after paying his debt ?

2. A man has Rs. 250 but owes Rs. 300. What will he have after paying his debt ?

3. A trader first gains Rs. 300 and then loses Rs. 260. What is his gain ?

4. A trader first gains Rs. 500 and then loses Rs. 750. What is his gain ?

5. A trader first loses Rs. 500 and then again loses Rs. 750. What is (i) his loss, (ii) his gain ?

6. A boy has 6 annas but has to pay 2 annas as a fine. How much will he have after paying his fine ?

7. A boy has x annas but has to pay y annas as a fine. How much will he have after paying his fine ?

In figure of Article 15 (iv), if O is the starting point and the right hand direction be regarded positive and left hand negative, find the distance between Rama and Krishna if

8. Rama goes 4 miles from O to the right and Krishna 3 miles to the left.

9. Rama goes $+4$ miles and Krishna -3 miles.

10. Rama goes $+5$ miles and Krishna $+3$ miles.

11. Rama goes -5 miles and Krishna $+2$ miles.

12. Rama goes -5 miles and Krishna -2 miles.

13. Kailash reached the school 5 minutes later and Prakash 5 minutes earlier than the right time, find how many minutes earlier Prakash reached the school before Kailash.

14. Raunaq is 5 years older than Kailash and Prem is 3 years younger than Kailash. How many years is Raunaq older than Prem ?

15. A ship first sails 25 miles to the north and then 10 miles to the south. How far is it from the starting point ?

16. A ship first sails 40 miles to the north and then 50 miles to the south. How far is it from the starting point ?

Subtract

17. 40 from 30. **18.** 20 from 0. **19.** 20 from 12.

20. 2 from -2. **21.** $10q$ from $-20q$. **22.** $40a$ from $30a$.

23. $18a$ from 0.

Add

24. 7 to -5. **25.** 2 to -1. **26.** 5 to -5.

27. $10x$ to $-10x$. **28.** $-16l$ to $20l$.

Which is greater and by how much ?

29. 1 or -1. **30.** 0 or -1. **31.** -3 or -5. **32.** -7 or -13.

Find the value of

33. $6-5$. **34.** $5-6$. **35.** $-5-6$.

36. $15x-8x$, when $x=2$. **37.** $8x-15x$, when $x=2$.

38. $-8x-15x$, when $x=2$.

Simplify

39. $ab-ab$. **40.** $-ab-ab$. **41.** $-3xyz+5xyz$.

What should be added to the following that the sum may be equal to zero?

42. -7. **43.** -15. **44.** 20. **45.** $5-7$.

46. $5-x$. **47.** $x+y$. **48.** $a-b$.

49. A train arrives at a station at 10 minutes to 7. If it was 15 minutes late, at what time would it arrive?

50. A train should have arrived at a station at 10 minutes to 9. If it were 25 minutes late, how many minutes after 9 would it arrive?

51. A student should have reached the school at 55 minutes past 9. If he arrived 8 minutes late, how many minutes after 10 did he arrive?

CHAPTER IV

BRACKETS

17. Brackets ( ) indicate that the terms enclosed within them should be considered *as one whole.*

Thus $12 - (5 + 3)$ means that the sum of 5 and 3 is to be subtracted from 12, *i.e.*,

$$12 - (5 + 3) = 12 - 8 = 4.$$

If there were no brackets then the expression $12 - 5 + 3$ would mean that 5 was to be subtracted from 12 and the difference was to be added to 3, *i.e.*,

$$12 - 5 + 3 = 7 + 3 = 10.$$

Similarly, $12 \times (5 + 3)$ means that the sum of 5 and 3 is to be multiplied by 12, *i.e.*,

$$12 \times (5 + 3) = 12 \times 8 = 96.$$

If there were no brackets then the expression $12 \times 5 + 3$ would mean that 5 was to be multiplied by 12 and the product was to be added to 3, *i.e.*,

$$12 \times 5 + 3 = 60 + 3 = 63.$$

Hence we see that *brackets should be dealt with first*, otherwise there would be a great difference in the result.

Removal of Brackets

18. From the previous Article we see that $12 + (5 + 3)$ means that 5 and 3 are to be added first, and then their sum is to be added to 12. But, if 5 is added to 12, and then their sum is added to 3, the result will be the same,

for $\quad 12 + (5 + 3) = 12 + 8 = 20,$

and $\quad 12 + \ 5 + 3 \ = 17 + 3 = 20.$

Hence, $\quad 12 + (5 + 3) = 12 + 5 + 3,$

and similarly, $\quad x + (y + z) = x + y + z.$

Again $12 + (5 - 3)$ means that 3 is to be subtracted first from 5, and then their difference is to be added to 12. But, if 5 is added to 12, and then 3 is subtracted from their sum, the result will be the same,

$$\text{for} \quad 12 + (5 - 3) = 12 + 2 = 14,$$

$$\text{and} \quad 12 + 5 - 3 = 17 - 3 = 14.$$

Hence, $$12 + (5 - 3) = 12 + 5 - 3,$$

and similarly, $$x + (y - z) = x + y - z.$$

Again, $12 - (5 + 3)$ means that 5 and 3 are to be added first and then their sum is to be subtracted from 12. But if 5 is subtracted from 12 and then 3 is subtracted from their difference, *i.e.*, if both 5 and 3 are subtracted separately from 12, the result will be the same,

$$\text{for} \quad 12 - (5 + 3) = 12 - 8 = 4,$$

$$\text{and} \quad 12 - 5 - 3 = 7 - 3 = 4.$$

If there were no brackets then the expression $12 - 5 + 3$ would mean that 5 was to be subtracted from 12 and their difference was to be added to 3, *i.e.*,

$$12 - 5 + 3 = 7 + 3 = 10.$$

But there is a great difference between the two results. Hence we see that $12 - (5 + 3)$ is *not* equal to $12 - 5 + 3$ but is equal to $12 - 5 - 3$,

and similarly, $x - (y + z) = x - y - z$.

Again $12 - (5 - 3)$ means that 3 is to be subtracted first from 5 and then their difference is to be subtracted from 12. But if 5 is subtracted from 12 and then 3 is added to the difference, the result will be the same,

$$\text{for} \quad 12 - (5 - 3) = 12 - 2 = 10,$$

$$\text{and} \quad 12 - 5 + 3 = 7 + 3 = 10.$$

Hence, $$12 - (5 - 3) = 12 - 5 + 3,$$

and similarly, $$x - (y - z) = x - y + z.$$

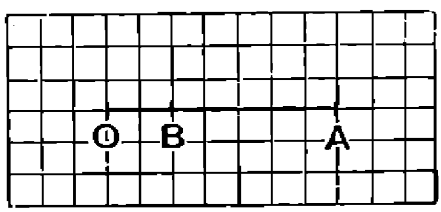
From the above examples we see that *when an expression within brackets is preceded by the positive sign, the brackets* *may be removed without changing the signs of any of its terms, but when it is preceded by the negative sign, the brackets may be removed by changing the sign of every term.*

Graphical Representation of the Removal of Brackets

19. We shall now illustrate the principles contained in the preceding article graphically on squared paper by taking the side of a small square to represent one unit.

EXAMPLE 1. *Show that* $+7+(-5)=+2$.

Take a point O on squared paper and draw a straight line OA to the right equal to 7 units.



From A measure AB to the left equal to 5 units.

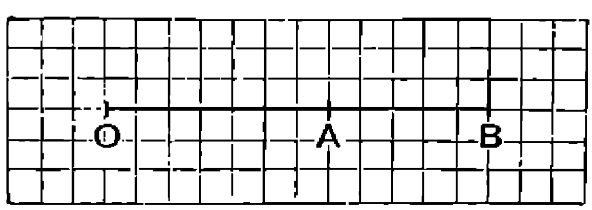
Now OB will represent $+7-5$.

But since OB is equal to $+2$,

$$\therefore \quad +7+(-5)=+7-5=+2.$$

EXAMPLE 2. *Show that* $+7-(-5)=+12$.

Take a point O on squared paper and draw a straight line OA to the right equal to 7 units.



If we added -5, we would have drawn AB to the left equal to 5 units, but since we subtract -5 we draw AB to the right equal to 5 units.

Now OB measures 7 + 5 or 12 units to the right.

$$\therefore \quad +7-(-5) = +7+5 = +12.$$

NOTE. If negative numbers are to be added to or subtracted from other numbers, they may first be placed within brackets and then the brackets may be removed, taking into consideration the rule concerning the change of signs.

20. We have already seen in Article 17 that $12 \times (5+3)$ means that first 5 and 3 are to be added together, and then their sum is to be multiplied by 12. But if 5 is multiplied by 12 and 3 is also multiplied by 12 and then their products are added, the result will be the same,

for $12 \times (5+3) = 12 \times 8 = 96$,

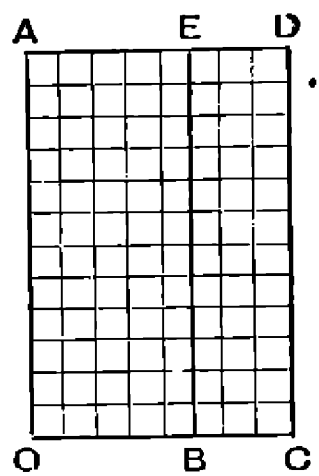
and $12 \times 5 + 12 \times 3 = 60 + 36 = 96$.

Hence, $12 \times (5+3) = 12 \times 5 + 12 \times 3$,

and similarly, $x \times (y+z) = x \times y + x \times z$

$= xy + xz$.

This can also be illustrated graphically on squared paper.



Draw OB equal to 5 units, and produce OB to C, making BC equal to 3 units.

From O draw OA perpendicular to OB equal to 12 units. Complete the rectangle OCDA.

Since OC measures $(5+3)$ units, therefore the area of the rectangle OCDA is $12 \times (5+3)$ square units.

Cut off AE from AD equal to OB or 5 units. Join EB.

Now the area of the rectangle OBEA = 12×5 sq. units,

and ,, ,, ,, ,, ,, BCDE = 12×3 sq. units.

But the rect. OCDA = rect. OBEA + rect. BCDE,

$$\therefore \quad 12 \times (5+3) = 12 \times 5 + 12 \times 3.$$

If OB represents y units, BC z units and OA x units,

$$x \times (y+z) = x \times y + x \times z$$
$$= xy + xz.$$

NOTE 1. Sometimes the sign '×' is ommitted before brackets, thus

$$x\ (y+z) = x \times (y+z) = x \times y + x \times z = xy + xz.$$

NOTE 2. Since $a \times b = b \times a$,

similarly, $x \times (y+z) = (y+z) \times x$,

or $x\ (y+z) = (y+z)\ x$.

21. Again from Article 17 we also know that $8 \times (16-4)$ means that 4 is to be subtracted first from 16 and then the difference is to be multiplied by 8. But if 16 is multiplied by 8 and 4 is also multiplied by 8 and then the product of 8 and 4 is subtracted from that of 8 and 16, the result will be the same,

for $8 \times (16-4) = 8 \times 12 = 96$,

and $8 \times 16 - 8 \times 4 = 128 - 32 = 96$.

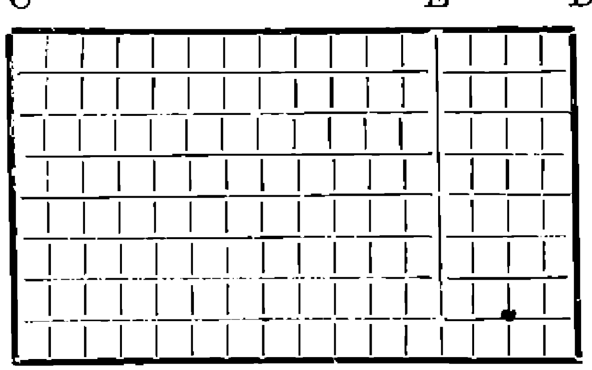
Hence, $8 \times (16-4) = 8 \times 16 - 8 \times 4$,

and similarly, $x \times (y-z) = x \times y - x \times z$

$$= xy - xz.$$

This can also be illustrated graphically on squared paper.

Draw a square OADC whose sides OA and OC measure



16 and 8 units respectively. Measure AB from A along AO and DE from D along DC equal to 4 units. Join BE.

$\therefore$ OB = CE = (16 − 4) units.

Now the area of the rect. OBEC = $8 \times (16 - 4)$ sq. units,

" " " " " OADC = 8×16 sq. units,

and " " " " " BADE = 8×4 sq. units.

But the rect. OBEC = rect. OADC − rect. BADE,

$$\therefore \quad 8 \times (16 - 4) = 8 \times 16 - 8 \times 4.$$

If OA represents y units, AB z units and OC x units,

$$x \times (y - z) = x \times y - x \times z$$
$$= xy - xz.$$

Hence we see that *when an expression within brackets is multiplied by a number, the brackets may be removed by multiplying each term of the expression within the brackets by that number.*

EXAMPLE 1. *Remove the brackets in* $8 + (6 + 2)$.

$$8 + (6 + 2) = 8 + 6 + 2$$
$$= 16.$$

EXAMPLE 2. *Find the value of* $9 - (5 + 3)$.

$$9 - (5 + 3) = 9 - 5 - 3$$
$$= 4 - 3$$
$$= 1.$$

EXAMPLE 3. *Simplify* $15x + (10x - 8x)$.

$$15x + (10x - 8x) = 15x + 10x - 8x$$
$$= 17x.$$

EXAMPLE 4. *Simplify* $2a - (3b + 5a)$.

$$2a - (3b + 5a) = 2a - 3b - 5a$$
$$= -3a - 3b.$$

EXAMPLE 5. *Remove the brackets in* $x(y + z)$.

$$x(y + z) = xy + xz.$$

EXAMPLE 6. *Find the value of* $2(x + y) - z$ *when* $x = 1$, $y = 2$ *and* $z = 3$.

$$2(x + y) - z = 2x + 2y - z$$
$$= 2 \times 1 + 2 \times 2 - 3$$
$$= 2 + 4 - 3$$
$$= 3.$$

NOTE. The rules for the insertion of brackets are the converse of those for their removal.

Thus, since $+(x-y)=+x-y$,

$+x-y=+(x-y)$.

Similarly, since $-(x-y)=-x+y$,

$\therefore\ -x+y=-(x-y)$.

And, since $x(y+z)=xy+xz$,

$xy+xz=x(y+z)$.

22. Sometimes it is necessary to use more than one set of brackets in an expression. In such cases it is better to use brackets of different shapes to avoid confusion. Those in common use are ( ), { }, [ ].

Thus $x+\{y-(z+w)\}$ means that we are to subtract from y the whole quantity within brackets ( ) and then add the result to x.

When there are more than one set of brackets, it is more convenient to remove the brackets one at a time, beginning with the innermost pair and working outwards, but it is also possible to remove the brackets beginning with the outermost pair and working inwards.

EXAMPLE 1. *Simplify, by removing brackets, the expression*

$$x+\{y-(z+w)\}.$$

Removing the brackets one by one, beginning from within, the expression $=x+\{y-z-w\}$

$=x+y-z-w$.

If we were to begin with the outermost brackets we should have

$$\text{the expression}=x+y-(z+w)$$
$$=x+y-z-w.$$

EXAMPLE 2. *Simplify* $a-[b+\{c-d(e+f)\}]$.

$$\text{The expression}=a-[b+\{c-de-df\}]$$
$$=a-[b+c-de-df]$$
$$=a-b-c+de+df.$$

23. Sometimes a line called a **vinculum** is drawn over an expression which is regarded as *a whole*. It has the same value as brackets.

EXAMPLE. *Simplify* $p+q(r-\overline{s+t})$.

The expression $=p+q(r-s-t)$

$=p+qr-qs-qt$.

EXAMPLES VII

Simplify

1. $9+(7+2)$.
2. $10+(-6+3)$.
3. $9-(7-6)$.
4. $x+(y-z)$.
5. $x-(y-z)$.
6. $16+(8-5+1)$.
7. $a-(b+c-d)$.
8. $x+3y+(2x-2y)$.
9. $x+3y-(2x-2y)$.
10. $m-3-(5-3m)$.
11. $m-3+(5-3m)$.
12. $2a-3b+(2b-3a)$.
13. $4x+3y-(2x+3y)$.
14. $b-(2a-5b)-(4b+a)$.
15. $x+(y-3x)-(x-y)$.
16. $(a+b)+(a-b)$.
17. $(a+b)-(a-b)$.
18. $a(b+c)+b(c+a)+c(a+b)$.
19. $a(b-c)+b(c-a)+c(a-b)$.
20. $n-(p-n)-(3n+2m-2p)-(n-m+2p)$.
21. $4c-(5d+3c)-(3d+5c)-(2c-7d)$.
22. $(a+b)-(a-2b)+(2a-3b)-(4a+b)$.
23. $(x-y+z)-(z-x+y)+(x+y+z)-(y-z+x)$.
24. $3x-4y-(2z-4x-2y)-(5x-3y+z)+(2x+6y+z)$.
25. $2x+3(y+z-x)+7y$.
26. $a+6(b-c)-2-(a-b+x)$.
27. $3(m-2n)+5(n-2p)-4m-(p-2m)$.
28. $16+(8-\overline{5+1})$.
29. $25-(6-\overline{9-5})$.
30. $a+(b+\overline{c-d})$.
31. $a+(b-\overline{c-d})$.
32. $7x-(6z-9y)-\overline{6x+7y-3z}$.
33. $9-\{7+(5-3)\}$.
34. $x-\{y-(z-a)\}$.
35. $8-\{7-9(5-4)\}$.
36. $8+3\{x-y(z+4)\}$.
37. $5x-\{3x+(4x-2x)\}$.
38. $7x-\{4x-(3x+6x)\}$.
39. $6a-\{3b-(2b+\overline{b-a})\}$.
40. $l-2m-(l-2n)-\{2m-l-(2n+l)\}$
41. $m-\{m+(m-\overline{m+1})\}$.
42. $[24\ (11-5)]\times 3$.
43. $18-[(8-3)-(5-2)]$.
44. $x+[y-2(x+3)+4]$.
45. $x+[y-\{z+(u-w)\}]$.
46. $b\,[a-(c-d)-(e+f)]$.
47. $a-b-\{a-b-(a+b)-\overline{a-b}\}$.
48. $x-2(y+z)-\{x+y-z-4(y-2z)\}$.
49. $4\,[a-3\{b-2(c-d)+1\}-5]$.
50. $2\,[6+4\{m-6(7-\overline{n+p})+q\}]$.

51. $a-[a-\overline{a-b}-\{a-(a-b)\}]$.

52. $4x-y-[x-(3y-z)-\{2x-2(y-z)\}]$.

53. $\{3m-(m-\overline{m+n})\}-[2n-\{3m-(m-2n)\}]$.

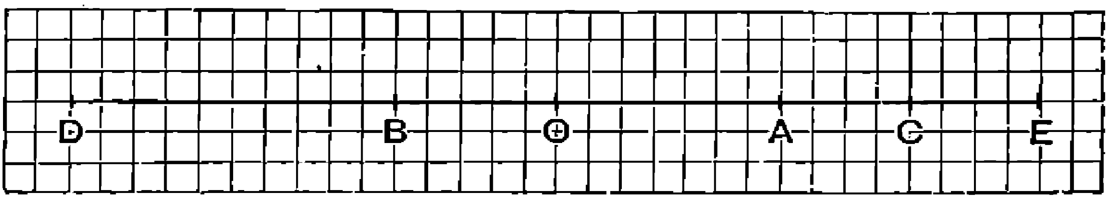
54. $a-[b-\{c-(d-\overline{e-f})\}]$. 55. $2\{a-(b+\overline{a-2b})3\}$.

56. $n-[n-\overline{m+n}-\{n-(n-\overline{m-n})\}]$.

57. $[1-\{1-(1-1-a)\}]$. 58. $x-[x-\{-x-(-x)\}]$.

EXAMPLES VIII

1. What numbers are represented by OA, OB, OC, OD and OE in the following figure ? (Take one small division as one unit).

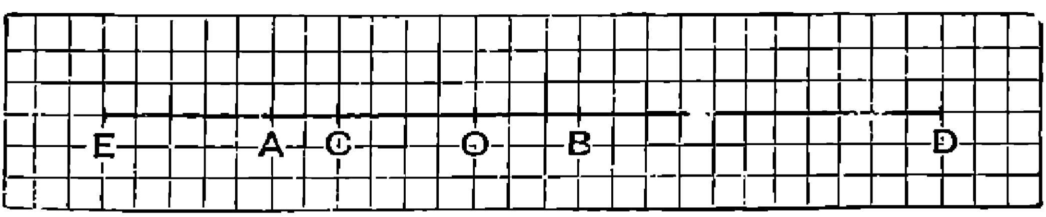


2. What numbers are represented by OA, AB, BC, CD and DE in the figure of Question 1 ?

3. In the figure of Question 1, show that

(i) $7-12=-5$. (ii) $11-26+22=7$. (iii) $15-20+5=0$.

4. What numbers are represented by OA, OB, OC, OD and OE in the following figure ? (Take one small division as one unit).



5. What numbers are represented by OA, AB, BC, CD and DE in the figure of Question 4 ?

6. In the figure of Question 4, show that

(i) $3-9=-6$. (ii) $14-25+7=-4$. (iii) $-11+5+2=-4$.

On squared paper, show that

7. $+7+5=+12$. 8. $+7+(-5)=+2$.

9. $-7+5=-2$. 10. $-7+(-5)=-12$.

11. $-7-(-5)=-2$. 12. $7\times(8+5)=7\times8+7\times5$.

13. $10(19+12)=10\times19+10\times12$. 14. $7\times(8-5)=7\times8-7\times5$.

15. $(19-12)\times10=19\times10-12\times10$.

CHAPTER V

HARDER CASES OF ADDITION AND SUBTRACTION

Addition of Expressions involving Negative Quantities

24. In finding the sum of simple or compound expressions we sometimes place them within brackets, connect them by the sign '+' and then simplify after removing brackets.

EXAMPLE 1. *Add* 8a, −2a, 5a, −4a *and* 3a.

$$\begin{aligned} \text{Sum} &= (8a)+(-2a)+(5a)+(-4a)+(3a) \\ &= 8a-2a+5a-4a+3a \\ &= 10a. \end{aligned}$$

EXAMPLE 2. *Add* 3x − 4y *and* 2x + 3y.

$$\begin{aligned} \text{Sum} &= (3x-4y)+(2x+3y) \\ &= 3x-4y+2x+3y && \text{removing brackets,} \\ &= 3x+2x-4y+3y && \text{collecting like terms,} \\ &= 5x-y. \end{aligned}$$

EXAMPLE 3. *Add* 2(x − y) *and* 3(x + y).

$$\begin{aligned} \text{Sum} &= 2(x-y)+3(x+y) \\ &= 2x-2y+3x+3y \\ &= 2x+3x-2y+3y \\ &= 5x+y. \end{aligned}$$

EXAMPLE 4. *Add* $\frac{2}{3}(a-3b)$ *and* $\frac{3}{4}(\frac{4}{9}a+\frac{8}{3}b)$.

$$\begin{aligned} \text{Sum} &= \tfrac{2}{3}(a-3b)+\tfrac{3}{4}(\tfrac{4}{9}a+\tfrac{8}{3}b) \\ &= \tfrac{2}{3}a-\tfrac{2}{3}\times 3b+\tfrac{3}{4}\times\tfrac{4}{9}a+\tfrac{3}{4}\times\tfrac{8}{3}b \\ &= \tfrac{2}{3}a-2b+\tfrac{1}{3}a+2b \\ &= \tfrac{2}{3}a+\tfrac{1}{3}a-2b+2b \\ &= a. \end{aligned}$$

EXAMPLE 5. *Add* $2a+3b-4c$, $3a-2b+4c$ *and* $a+5b+6c$.

First Method.

$$\begin{aligned}\text{Sum} &= (2a+3b-4c)+(3a-2b+4c)+(a+5b+6c)\\ &= 2a+3b-4c+3a-2b+4c+a+5b+6c\\ &= 2a+3a+a+3b-2b+5b-4c+4c+6c\\ &= 6a+6b+6c.\end{aligned}$$

Second Method.

Write each expression so that like terms fall in the same vertical columns and then add up each column separately thus,

$$\begin{array}{r} 2a+3b-4c\\ 3a-2b+4c\\ a+5b+6c\\ \hline 6a+6b+6c \end{array}$$

Hence the sum is $6a+6b+6c$.

EXAMPLE 6. *Add* $\frac{1}{3}(x+y-z)$, $\frac{2}{3}(x-y-z)$ *and* $\frac{5}{3}(-x+y+z)$.

$$\begin{aligned}\text{Sum} &= \tfrac{1}{3}(x+y-z)+\tfrac{2}{3}(x-y-z)+\tfrac{5}{3}(-x+y+z)\\ &= \tfrac{1}{3}x+\tfrac{1}{3}y-\tfrac{1}{3}z+\tfrac{2}{3}x-\tfrac{2}{3}y-\tfrac{2}{3}z-\tfrac{5}{3}x+\tfrac{5}{3}y+\tfrac{5}{3}z\\ &= \tfrac{1}{3}x+\tfrac{2}{3}x-\tfrac{5}{3}x+\tfrac{1}{3}y-\tfrac{2}{3}y+\tfrac{5}{3}y-\tfrac{1}{3}z-\tfrac{2}{3}z+\tfrac{5}{3}z\\ &= -\tfrac{2}{3}x+\tfrac{4}{3}y+\tfrac{2}{3}z.\end{aligned}$$

EXAMPLES IX

Add together

1. $3a$, a, $5a$, $7a$, $4a$.

2. x, $2x$, $4x$, $8x$, $16x$.

3. $-a$, $-2a$, $-3a$, $-4a$, $-5a$.

4. $-m$, $-3m$, $-5m$, $-7m$, $-9m$.

5. $2a$, $3a$, $-7a$, $-8a$.

6. $-3x$, $6x$, $-7x$, $5x$.

7. $-9xy$, $7xy$, $-3xy$, $4xy$.

8. $3ab$, $4ab$, $-7ab$, $4ab$, $-6ab$.

9. $\frac{5}{8}x$, $\frac{3}{4}x$, $\frac{1}{2}x$, $-\frac{3}{8}x$.

10. $2a$, $-\frac{4}{5}a$, $-\frac{3}{5}a$, $\frac{2}{5}a$.

11. a, 1, a, -1.

12. $7m$, 7, -2, $-m$.

13. k, l, $-3k$, $-2l$, $2k$, k.

14. x, y, z, $3x$, $-5y$, $-6z$, $-2z$, $4x$, $-4y$.

Simplify

15. $a+2a+3a$.

16. $a-2a+3a$.

17. $7x-x+4x-5x+3x$.

18. $ab+3ab+2ab-4ab-5ab$.

19. $7mn+3mn-2mn+8mn$.

20. $15ab-17ab+18ab+9ab-24ab$.

21. $3x+2-2x-1$.

22. $5a+4b+3b-2a-3a$.

23. $2xy-9ab+3ab-5xy-2xy+14ab+xy.$

24. $7ab-3xy-5xy+8ab-2xy-10ab-5ab.$

25. $\frac{2}{3}x-\frac{1}{3}x+x-\frac{2}{3}x.$

26. $-\frac{4}{9}x-\frac{2}{9}x+\frac{4}{9}x+\frac{7}{9}x.$

Add together

27. $x+y,\ x-y.$

28. $3a+b,\ 2a-b.$

29. $-x+y,\ x+y.$

30. $3a+b,\ 2a+b.$

31. $a-3b,\ a+4b.$

32. $2a-b,\ 3a-b.$

33. $\frac{1}{3}a+\frac{1}{2}b,\ \frac{2}{3}a-\frac{1}{3}b.$

34. $\frac{1}{2}a+\frac{1}{2}b,\ \frac{1}{2}a-\frac{3}{2}b.$

35. $\frac{3}{5}a-\frac{2}{5}b,\ \frac{2}{5}a+\frac{1}{5}b.$

36. $2(a+b),\ 3(a-b).$

37. $3(x+y),\ -2(x-y).$

38. $6(x-1),\ 5(x-2).$

39. $3(1+2x),\ 2(3-5x).$

40. $2(x+y),\ -(x+y).$

41. $3(3a-b),\ -2(2a-b).$

42. $-\frac{3}{4}(2x+3y);\ \frac{1}{2}(2x+5y).$

43. $a(x-y),\ a(x+y).$

44. $x(a+b),\ -x(a+b).$

45. $x(a+b),\ -x(a-b).$

46. $a(3-x),\ a(2+x).$

47. $-a(3x-1),\ -a(x+7).$

48. $2a+3b,\ 3a+b,\ a+2b.$

49. $3x-y,\ 3y-x,\ 3x-y.$

50. $2x+3y,\ 3x-5y,\ -5x+3y.$

51. $7a+3b,\ 5a+2b,\ -3a-4b,\ 6b-4a.$

52. $5m-6n,\ n-3m,\ 5m-7n,\ 2n-9m.$

53. $a+b-c,\ a-b+c,\ -a+b+c.$

54. $3a-4b-2c,\ 4a+2b-c,\ 2a-b-3c.$

55. $2a+4b-c,\ -a+3b+2c,\ 3a-b+c.$

56. $2x-y+3z,\ x+4y-2z,\ 4x+2y-z.$

57. $a+3b-4c,\ 3a+b-2c,\ 5a-b+c.$

58. $5l-4m-n,\ -2l+3m+2n,\ -3l+m+n.$

59. $6p+q-2r,\ -p-q-r,\ -5p+4q+r.$

60. $6x-2y+5z,\ -8x+7y+3z,\ -2x-3y+8z.$

61. $-2l+8m-6n,\ l-4m+3n,\ 5l-5m+3n.$

62. $3c+5d-e,\ -16c+5d+5e,\ 10c-10d+2e.$

63. $-3a-3b-3c,\ 5a+5b+5c,\ -2a-2b-2c.$

64. $7ax+8by-3cz,\ 2ax+by-2cz,\ 4ax-3by-5cz.$

65. $pq+qr+rp,\ -pq+qr+rp,\ pq-qr-rp.$

66. $a+b+c,\ 2(a+b-c),\ 3(a-b+c).$

67. $3(a+b-c),\ 4(2a-b-3c),\ 5(-a+2b+c).$

68. $\frac{1}{3}(x+y-z),\ \frac{2}{3}(x-y-z),\ \frac{4}{3}(-x+y+z).$

69. $\frac{1}{3}(a+b-c),\ \frac{2}{3}(a-b+c),\ \frac{7}{3}(-a+b-c).$

70. $5ab+3bc-ca,\ 2ab-4bc,\ -ab+ca.$

71. $-ab+bc+ca$, $-3ab$, $-2bc+3ca$, $ab+bc-4ca$.
72. $a-2b+3c-4d$, $b+2c$, $2a-b-d$, $3c+4d$.
73. $x+y-z+w$, $y-z+2w$, $w-6z$, $x+2z$.
74. $a+b-3$, $3+2a-3b$, $3b-5c$, $-5+a-2b$.
75. $x+y$, $x-y$, $x+z$, $x-z$, $y+z$, $y-z$.
76. $\frac{1}{2}x-\frac{1}{3}y+1$, $\frac{3}{4}x-\frac{1}{2}y+z+3$, $\frac{2}{4}x+\frac{1}{4}y-2z$.

Subtraction of Expressions involving Negative Quantities

25. In Arithmetic in finding the difference between two numbers, we place the greater number first and then the smaller number preceded by the sign '$-$', similarly in Algebra when we have to subtract one expression from another, we first place them within brackets, the expression to be subtracted being preceded by the minus sign.

EXAMPLE 1. *Subtract* $10x-3y-6z$ *from* $7x-4y+3z$.

First method.

$$\begin{aligned}\text{Difference} &= (7x-4y+3z)-(10x-3y-6z)\\ &= 7x-4y+3z-10x+3y+6z\\ &= 7x-10x-4y+3y+3z+6z\\ &= -3x-y+9z.\end{aligned}$$

Second Method.

Write the expression to be subtracted under that from which it is to be subtracted, placing like terms under one another, then change the signs of all the terms of the lower expression mentally, and proceed as in addition thus,

$$\begin{array}{r} 7x - 4y + 3z \\ 10x - 3y - 6z \\ \hline -3x - \ \ y + 9z \end{array}$$

Hence the difference is $-3x-y+9z$.

NOTE. The terms of the difference are obtained by mentally combining the like terms thus :

$7x$ *minus* $10x=-3x$, $-4y$ *plus* $3y=-y$, $3z$ *plus* $6z=9z$.

The signs need not be actually changed, but the operation of changing signs should be performed mentally.

EXAMPLE 2. *Subtract* $3(x-3y+z)$ *from* $4(x+2y-3z)$.

$$\begin{aligned}\text{Difference} &= 4(x+2y-3z)-3(x-3y+z)\\ &= 4x+8y-12z-3x+9y-3z\\ &= 4x-3x+8y+9y-12z-3z\\ &= x+17y-15z.\end{aligned}$$

EXAMPLES X

Subtract

1. b from $a+b$. **2.** $2b$ from $a-b$. **3.** $-2a$ from $a-b$.

4. $a-b$ from $a+b$. **5.** $x+y$ from $x-y$. **6.** $-3c-4d$ from $7c+9d$.

7. $a+3b-c$ from $2a+2b+2c$. **8.** $l+3m-n$ from $5l-6m+n$.

9. $3p-8q+r$ from $2p-q+4r$. **10.** $-x-y-z$ from $x+y+z$.

11. $-a-2b$ from $3a+2b-c$. **12.** $5a-3b$ from $-a-3b-4c$.

13. $-x-3y$ from $x-3y-2z$. **14.** $b-c$ from $a+b$.

15. $-2a+4b$ from $2b-3c$. **16.** $-3b+6c$ from $6a+b$.

17. $xy-yz+zx$ from $-xy-yz-zx$.

18. $ab-bc+cd-da$ from $ab+bc+cd+da$.

19. $4(a-b-c)$ from $5(a-2b+3c)$.

20. $4(x-y+3z)$ from $-3(2x+y-z)$.

21. $5(a+2b-c)$ from $-(-a+b-5c)$.

22. $6(a+b)-4(a-b)$ from $5(a-b)+3a$.

23. $7(a+b-c)+2(a+b)$ from $4(a+c)-3(a-b)$.

24. $5(a+b)-3(c+a)+(b+c)$ from $5(a+b)+4(b+c)-2(c+a)$.

25. $2(x+y)-(y-z)+3(x+z)$ from $-7(x-y)+5(y+z)-3(z-x)$.

26. $a-b-2c$ from the sum of $a+b+c$ and $a-b+c$.

27. $x+y+z$ from the sum of $x-y-z$ and $3x+2y-z$.

28. $c+11d-e$ from the sum of $4c+5d-3e$ and $-3c-9d+3e$.

29. $x+y+z$ from $x+2y+5z$ and the difference from $3x-2y+z$.

30. $3a+2b$ from $4a-3b+5c$ and the difference from $a-2b+3c$.

31. the sum of $5a-3b+6c$ and $3a+3b-5c$ from the sum of $2a+3b+4c$ and $a-4b-2c$.

32. $3x+5y-7z$ students appear for an examination. If $2x+3y+z$ of these pass, how many fail? If $x=10$, $y=5$ and $z=1$, find the number of passes and failures.

33. There are $3x+4y-z$ passengers in a train, of whom $x-y+z$ are in first class, and $2x+3y-z$ in third. Find the number of passengers in the second class. If $x=45$, $y=36$ and $z=5$, find the number of passengers in each class.

34. The population of a village is $7x-3y+5z$, of whom $4x+2y-2z$ are men, and $2x+y+z$ women. Find the number of children in the village. If $x=80$, $y=10$ and $z=50$, find the number of men, women and children in the village.

EXAMPLES XI

If $a=5$, $b=3$, $c=1$, $x=-1$ and $y=-3$, find the value of

1. $a+2b+3c$. **2.** $2a+3b-4c$. **3.** $-2a+4b+3x$.

4. $6a-3b-2y$. **5.** $3x+4y-5b$. **6.** $3c-5x+2y$.

7. $5a-4x+4y+2c-3b$. **8.** $7c-4x+a-2b+9y$.

9. $3ab+2xy$. **10.** $ac+bx+xy-bc$.

11. $(a+b)c$. **12.** $3(a+b)-4(x-y)$.

13. $aby+bcy+axy$. **14.** $3c-\frac{1}{2}bc+\frac{1}{3}xy$.

15. $\frac{ab}{5}-\frac{2xy}{3}+\frac{ay}{9}$.

If $l=6$, $m=2$, $n=0$ and $p=-3$, find the value of

16. $16lm-4mn+3np$. **17.** $4(l+m+n)-3p$.

18. $2(l-m)+3(m-n)-5(n-p)$. **19.** $3(l+m)+7(2m-n)-8(n-5p)$.

20. $\frac{1}{4}lm-\frac{1}{3}mp+n$.

CHAPTER VI

MULTIPLICATION

26. In Arithmetic we know that multiplication of two numbers means taking one of the numbers as many times as there are units in the other. Thus 5×4 means '*to add 4 five times*,' *i.e.*,

$$5 \times 4 = 4 + 4 + 4 + 4 + 4$$
$$= 20$$

Similarly, in Algebra $5 \times a$ means 'to add a five times,' *i.e.*,

$$5 \times a = a + a + a + a + a$$
$$= 5a.$$

Similarly, $100 \times a = a + a + a + a + \ldots\ldots 100$ times

$$= 100a,$$

and $b \times a = a + a + a + a + \ldots\ldots b$ times

$$= ba. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again, in Arithmetic we know that $5 \times 4 = 4 \times 5$, similarly, in Algebra $a \times b = b \times a$,

and $a \times b \times c = a \times c \times b = b \times a \times c = b \times c \times a = c \times a \times b = c \times b \times a$.

Law of Signs

27. We have seen in the previous article if numbers are positive, their multiplication means *repeated addition*, similarly, if numbers are negative their multiplication means *repeated subtraction*. Thus -5×4 means '*to subtract 4 five times*,' *i.e.*,

$$-5 \times 4 = -(4) - (4) - (4) - (4) - (4)$$
$$= -20,$$

and $-5 \times -4 = -(-4) - (-4) - (-4) - (-4) - (-4)$

$$= 4 + 4 + 4 + 4 + 4$$
$$= 20.$$

$$\text{Similarly, } -b \times a = -(a)-(a)-(a)\ldots\ldots b \text{ times}$$
$$= -ba. \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$-a \times b = -(b)-(b)-(b)\ldots\ldots a \text{ times}$$
$$= -ba. \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$\text{and} \quad -b \times -a = -(-a)-(-a)-(-a)\ldots b \text{ times}$$
$$= a+a+a+\ldots\ldots\ldots\ldots b \text{ times}$$
$$= ba. \quad \ldots\ldots\ldots\ldots \quad (4)$$

Now writing the results numbered (1), (2), (3) and (4) we have

$$(+b) \times (+a) = +ba,$$
$$(-b) \times (+a) = -ba,$$
$$(+b) \times (-a) = -ba,$$
$$(-b) \times (-a) = \quad ba.$$

Hence, we see that *the product of two factors of like signs is positive and of unlike signs is negative.*

The above can also be written more briefly thus $+\times+=+$, $-\times+=-$, $+\times-=-$, $-\times-=+$.

Multiplication of Powers

28. If a number or quantity is multiplied by itself more than once, the product is called the **power** of that number or quantity, and is written by placing a small figure or letter above and to the right of the given number indicating the number of times that number or quantity is multiplied by itself.

Thus

$x \times x$, which is written as x^2, is called the **second power** of x, and is read as *x squared* or *x to the power* 2,

$x \times x \times x$, which is written as x^3, is called the **third power** of x, and is read as *x cubed* or *x to the power* 3,

$x \times x \times x \times x$, which is written as x^4, is called the **fourth power** of x, and is read as *x to the power* 4, and $x \times x \times x$......100 times, which is written as x^{100}, is called the **hundredth power** of x, and is read as *x to the power* 100.

Similarly, $x \times x \times x$......n times, which is written as x^n, is called the **n^th^ power** of x, and is read as *x to the power n.*

The small figure or letter, placed above the given number and indicating the number of times that number has been multiplied by itself, is called the **index** of that number.

Thus, the index of x^4 is 4.

NOTE. When a quantity x is taken only once, it is generally written x and not x^1. Thus the power of a number when no figure is written above it may be taken to be one.

Since $x^2 = x \times x$,

and $x^3 = x \times x \times x$.

$$\therefore \quad x^2 \times x^3 = x \times x \times x \times x \times x$$
$$= x^5.$$

Similarly, since $x^3 = x \times x \times x$,

and $x^5 = x \times x \times x \times x \times x$.

$$x^3 \times x^5 = x \times x \times x \times x \times x \times x \times x \times x$$
$$= x^8.$$

Hence, we see that *the index of the product of any two powers of the same number or quantity is equal to the sum of the indices of the factors.*

EXAMPLE 1. *Multiply* 3a *by* 4.

$$3a \times 4 = 3 \times a \times 4$$
$$= 3 \times 4 \times a$$
$$= 12a.$$

EXAMPLE 2. *Simplify* $a \times 3 \times a \times 5$.

$$a \times 3 \times a \times 5 = 3 \times 5 \times a \times a$$
$$= 15 \times a^2$$
$$= 15a^2.$$

EXAMPLE 3. *Multiply* 2x *by* -3.

$$-3 \times 2x = -3 \times 2 \times x$$
$$= -6 \times x$$
$$= -6x.$$

EXAMPLE 4. *Simplify* $b \times (-2) \times (-3) \times 6b$.

$$\text{The expression} = (-2) \times (-3) \times 6 \times b \times b$$
$$= 6 \times 6 \times b^2$$
$$= 36b^2.$$

EXAMPLE 5. *Multiply* a^4 *by* a^3.

$$a^3 \times a^4 = a \times a \times a \times a \times a \times a \times a$$
$$= a^7.$$

Or more briefly thus

$$a^3 \times a^4 = a^{3+4} = a^7.$$

EXAMPLE 6. *Simplify* $a \times (-3b) \times 5c$.

$$\text{The expression} = (-3) \times 5 \times a \times b \times c$$
$$= -15abc.$$

EXAMPLE 7. *Simplify* $l^2 \times l^3 \times l^4$.

$$l^2 \times l^3 \times l^4 = l^{2+3+4} = l^9.$$

EXAMPLE 8. *Simplify* $3a \times (-2a) \times a$.

$$\text{The expression} = 3 \times (-2) \times a \times a \times a$$
$$= -6a^3.$$

EXAMPLE 9. *Simplify* $(-2x) \times 3x^2 \times (-4x^3)$.

$$\text{The expression} = (-2) \times 3 \times (-4) \times x \times x^2 \times x^3$$
$$= 24x^{1+2+3}$$
$$= 24x^6.$$

EXAMPLE 10. *Find the product of* a^4b^2 *and* a^2b^3.

$$a^4b^2 \times a^2b^3 = a^4 \times a^2 \times b^2 \times b^3$$
$$= a^{4+2} \times b^{2+3}$$
$$= a^6 \times b^5$$
$$= a^6b^5.$$

From the above examples we see that *to find the product of two or more simple expressions (expressions containing only one term), first write down the sign of the result, then multiply the numerical coefficients to find the numerical coefficient of the product and then give each letter which occurs an index equal to the sum of the idices of that letter in the factors.*

EXAMPLES XII

(Oral)

Multiply

1. $2x$ by 3. **2.** $3a$ by 5. **3.** $3a$ by -3. **4.** 6 by $2x$.

5. 7 by $-3a$. **6.** $-2a$ by 5. **7.** $-3b$ by -2. **8.** $2x$ by $3y$.

9. $2l$ by $5m$. **10.** $9b$ by $4c$. **11.** $3p$ by $-2q$. **12.** $-2b$ by $3b$.

13. $-3a$ by $-3b$. **14.** $-x$ by $-8y$. **15.** $2ab$ by c.

16. $2c$ by $-bd$. **17.** cxy by $5b$. **18.** $-6ax$ by $2by$.

19. $-2pq$ by $-3rs$. **20.** $-ab$ by $-cd$. **21.** a by a.

22. x^2 by x^2. **23.** x^3 by x. **24.** x^3 by x^5.

25. $-x$ by x^2. **26.** $-a^3$ by $-a^5$. **27.** b^2 by $-b^3$.

28. $-b^{10}$ by $-b^7$. **29.** xy by x. **30.** ab by ab.

31. $3ab$ by $-3ab$. **32.** $-3xy$ by $2xy$. **33.** $-2a^2$ by $-3ab$.

34. $3p^2q^2$ by $7pq^4$. **35.** $4x^2$ by $3x^3$. **36.** $4a^2$ by $-2a^3$.

37. $-p^5$ by $-p^{10}$. **38.** xyz by xyz.

39. a^3bc^2 by ab^2c^3. **40.** $a^2b^3c^4$ by $-ab^2c^3$.

41. $\frac{3}{4}x^3$ by $-\frac{4}{3}x^2$. **42.** $\frac{5}{3}a^2b$ by $\frac{6}{7}ab^2$.

43. $-\frac{2}{3}a^2b$ by $-\frac{3}{4}b^2c$. **44.** $-\frac{4}{7}x^2y$ by $\frac{21}{8}yz^2$.

45. Find the square of -2, x^2, xy^3, $-\frac{1}{2}x^2y^2$, $\frac{2}{3}xy^2z^3$.

46. Find the cube of -3, -5, $2x^2$, $-3x^3y$, $-\frac{1}{2}xy^2z^4$.

47. Find the fourth power of -1, $3x$, $\frac{1}{3}x^2$, x^2y^2, $-x^2y^3$.

Express as a power

48. $2^2 \cdot 2^3$. **49.** $x \cdot x^2 \cdot x^3$. **50.** $a^2 \cdot a^5 \cdot a^8 \cdot a^{10}$.

Simplify

51. $a^2 \times b^2 \times c$. **52.** $2a \times 3b^2 \times 5c^3$. **53.** $a^2 \times ab \times b^2$.

54. $a^2x \times x^2 \times (-y)$. **55.** $ax^2 \times by \times 2y^3$. **56.** $x^2 \times x^3 \times x^5$.

57. $3y^2 \times 4y^2 \times 5y^3$. **58.** $p^3 \times (-3p^2) \times 5p^3$.

59. $a^3b \times (-bc^2) \times cb$. **60.** $\frac{1}{2}x^2 \times (-\frac{3}{2}xy) \times \frac{4}{3}yz$.

Multiplication of Simple Expressions

29. Let us now consider the following examples :

(i) $(-2) \times (-3) \times (-5) = 6 \times (-5) = -30$.

Here, first -2 is multiplied by -3, the product is 6, then this product is multiplied by -5, the final product is -30.

(ii) $(-2)\times(-3)\times(-5)\times(-7)$

$$=6\times(-5)\times(-7)=(-30)\times(-7)=210.$$

Here, first -2 is multiplied by -3, the product is 6, then this product is multiplied by -5, the next product is -30, then -30 is multiplied by -7, the final product is 210.

(iii) $(-2)\times(-3)\times(-5)\times(-7)\times(-8)$

$$=6\times(-5)\times(-7)\times(-8)=(-30)\times(-7)\times(-8)$$
$$=210\times(-8)=-1680.$$

30. From the above we see that *the product of negative quantities is positive when the number of negative quantities is even, and negative when the number of negative quantities is odd.*

EXAMPLE 1. *Find the value of* $(-x)^2$.

$$(-x)^2=(-x)\times(-x)=x^2.$$

EXAMPLE 2. *Find the value of* $(-x)^3$.

$$(-x)^3=(-x)\times(-x)\times(-x)$$
$$=x^2\times(-x)$$
$$=-x^3.$$

EXAMPLE 3. *Find the value of* $(-x)^4$.

$$(-x)^4=(-x)\times(-x)\times(-x)\times(-x)$$
$$=x^2\times(-x)\times(-x)$$
$$=(-x^3)\times(-x)$$
$$=x^4.$$

EXAMPLE 4. *Simplify* $(-1)^5$.

$$(-1)^5=(-1)\times(-1)\times(-1)\times(-1)\times(-1)$$
$$=1\times(-1)\times(-1)\times(-1)$$
$$=(-1)\times(-1)\times(-1)$$
$$=1\times(-1)$$
$$=-1.$$

EXAMPLE 5. *Simplify* $(2a)^3$.

$$(2a)^3=2a\times2a\times2a$$
$$=4a^2\times2a$$
$$=8a^3.$$

EXAMPLE 6. *Simplify* $(-3x)^4$.

$$(-3x)^4 = (-3x)\times(-3x)\times(-3x)\times(-3x)$$
$$= 9x^2 \times (-3x)\times(-3x)$$
$$= (-27x^3)\times(-3x)$$
$$= 81x^4.$$

EXAMPLE 7. *Simplify* $(a^2)^3$.

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6.$$

EXAMPLE 8. *Simplify* $(-x^3)^5$.

$$(-x^3)^5 = (-x^3)\times(-x^3)\times(-x^3)\times(-x^3)\times(-x^3).$$

Since the number of negative factors is odd, the sign of the product is negative.

$$\therefore \text{ the product} = -x^3 \times x^3 \times x^3 \times x^3 \times x^3$$
$$= -x^{3+3+3+3+3}$$
$$= -x^{15}.$$

EXAMPLE 9. *Simplify* $(-2a^2b)^3$.

$$(-2a^2b)^3 = (-2a^2b)\times(-2a^2b)\times(-2a^2b)$$
$$= (-2)\times(-2)\times(-2)\times a^2 \times a^2 \times a^2 \times b \times b \times b$$
$$= -8a^{2+2+2}\,b^{1+1+1}$$
$$= -8a^6b^3.$$

EXAMPLE 10. *Simplify* $(-\frac{1}{2}x^2)\times(-\frac{2}{3}xy)\times(-\frac{4}{5}y^2)$.

$$\text{The expression} = (-\tfrac{1}{2})\times(-\tfrac{2}{3})\times(-\tfrac{4}{5})\times x^2 \times xy \times y^2$$
$$= -\tfrac{4}{15}x^{2+1}\,y^{1+2}$$
$$= -\tfrac{4}{15}x^3y^3.$$

EXAMPLE 11. *Simplify* $(-x^2)^3 \times x^3$.

$$(-x^2)^3 \times x^3 = (-x^2)\times(-x^2)\times(-x^2)\times x^3$$
$$= -x^2 \times x^2 \times x^2 \times x^3$$
$$= -x^{2+2+2+3}$$
$$= -x^9.$$

EXAMPLE 12. *Simplify* $(3x^2)^2 \times (-xy)^3 \times (-2y)^2$.

$$\text{The expression} = (3x^2)\times(3x^2)\times(-xy)\times(-xy)\times(-xy)\times(-2y)\times(-2y).$$

Since the number of negative factors is odd, the sign of the product is negative.

$$\therefore \text{ the product} = -3\times3\times2\times2\times x^2\times x^2\times xy\times xy\times xy\times y\times y$$
$$= -36x^{2+2+1+1+1}\,y^{1+1+1+1+1}$$
$$= -36x^7y^5.$$

EXAMPLES XIII

(*Examples* 1—15 *may be attempted orally*)

Find the value of

1. $(x^2)^2$. 2. $(-x^2)^2$. 3. $(-1)^3$. 4. $(-p^2)^5$.
5. $(-p^3)^4$. 6. $(-1)^6$. 7. $(-1)^{11}$. 8. $(-a^3)^7$.
9. $(-2a)^3$. 10. $(-2xy)^3$. 11. $(-3ab)^4$. 12. $(-2x^2)^6$.
13. $(-2ab^2)^3$. 14. $(-3x^2y^2)^3$. 15. $(-3xy^2)^4$. 16. $(a^2)^3\times(a^3)^4$.
17. $(-a^2)^3\times(-a^3)^4$. 18. $(-1)^5-(-x)^3$.
19. $(abc)^2\times(-abc)^2$. 20. $(-xy^2z^3)^2\times(-x^3y^2z)^3$.
21. $(-2)\times(-3)\times 4$. 22. $(-a)\times(-b)\times c$.
23. $(-x)\times(-x)\times(-x)$. 24. $x\times(-x)\times x$.
25. $(-x^3)\times x^5\times(-x^7)$. 26. $(-a)\times(-a^2)\times a^4$.
27. $(-2a^2)\times(-2a^2)\times(-2a^2)$. 28. $(-xy)\times(-xy)\times(-xy)$.
29. $abx\times(-bcx)\times(-cax)$. 30. $l^2m\times(-m^2n)\times(-n^2l)$.
31. $(-4a^2b)\times(-3b^2c)\times(-c^3b^3)$. 32. $(-x)^2\times(-y)^2\times(-z)^2$.
33. $(-x)^3\times(-y)^3\times(-z)^3$. 34. $(-a^2)^2\times(b^2)^3\times(-c^2)^4$.
35. $(-1)^3\times(ax)^2\times(-by)^3\times(cz)^2$.
36. $\frac{1}{2}lm^2n^3\times(-\frac{2}{3}m^2n^3p^4)\times(-\frac{3}{4}n^4pl^2)$.

EXAMPLES XIV

1. If $a=-1$ and $b=2$, find the value of
(i) a^2, a^3, b^2, b^3, a^2b^2, a^3b^2, a^3b^3, a^4b^2 and a^5b^3.
(ii) a^2-1, a^2+b^2, $2a^2+b^2$, a^3-b, $2a^3-3b$, a^3+b^3 and a^4-b^4.

2. If $a=-1$ and $b=-2$, find the value of
$4a+b^2$, $3a^3-2b$, $3a^3-2b^2$, a^2+b^2, a^2-b^2 and $-2ab+3$.

3. If $l=4$ and $m=-7$, find the value of
$7l-3m$, $8l+5m-lm$, $l^2-3lm-m^2$ and $l^3-2m^2+8l-4m$.

4. If $a=1$, $b=5$ and $c=0$, find the value of
a^2-b^2+2c, $c^2-c\,(2a+3b)$, $b^3-4ac+21$ and $3(a+b)-2c(a+c)$.

5. If $x=6$, $y=2$ and $z=0$, find the value of
$x^2-y^2+z^2$, x^3-zy^2-7xy, $2(x+y)+3z^2$ and $4(x+y)(x-y)-5xyz$.

6. If $a=2$ and $b=-3$, find the value of
$(a+b)(a-b)$, a^2-ab+b^2, a^3-b^3 and a^4+b^4.

7. If $a=-1$ and $b=2$, find the value of
$(ab)^2$, $(-ab)^3$, $(-3a^2b)^3$, $(-\frac{1}{4}a^3b^2)^2$, $(-3ab^2)^3-(-2a^3b^2)^3$ and
$(-\frac{1}{4}ab^3)^2 \times (\frac{1}{2}a^3b)^3$.

8. Find the value of x^2-3x+2 when $x=1, 2, 5, 10, 0, -2, -7$.

Solution.

When $x=$	1	2	5	10	0	-2	-7
$x^2=$	1	4	25	100	0	4	49
$-3x=$	-3	-6	-15	-30	0	6	21
and $2=$	2	2	2	2	2	2	2
$\therefore x^2-3x+2=$	0	0	12	72	2	12	72

Hence the values of x^2-3x+2 are 0, 0, 12, 72, 2, 12 and 72.

9. Find the value of x^2-5x+6 when $x=0, 1, 2, 3, -1, -2, -3$.
10. Find the value of $x^2-8x+16$ when $x=0, 1, 3, 5, -2, -4$.
11. Find the value of $2a^2-5a-9$ when $a=1, 4, 7, -2, -5, -8$.
12. Find the value of x^3-x+1 when $x=0, 1, -1, 2, -2, 5, -5$.

Multiplication of a Compound Expression by a Simple Expression

31. We have seen in Article 20 if an expression within brackets is multiplied by a number, the brackets may be removed by multiplying each term of the expression within the brackets by that number, thus

$$x \times (y+z) = x \times y + x \times z.$$

EXAMPLE 1. *Multiply* (x − y) *by* 2.

$$2 \times (x - y) = 2 \times x - 2 \times y$$
$$= 2x - 2y.$$

EXAMPLE 2. *Multiply* (a + b) by − 2c.

$$-2c \times (a + b) = (-2c \times a) + (-2c \times b)$$
$$= -2ca - 2cb.$$

EXAMPLE 3. *Multiply* (b + c − d) *by* a.

$$a \times (b + c - d) = a \times b + a \times c - a \times d$$
$$= ab + ac - ad.$$

EXAMPLE 4. *Multiply* ($x^2 - 2yz + z^2$) *by* $-x^2y^2$.

$$-x^2y^2 \times (x^2 - 2yz + z^2)$$
$$= (-x^2y^2 \times x^2) - (-x^2y^2 \times 2yz) + (-x^2y^2 \times z^2)$$
$$= -x^4y^2 + 2x^2y^3z - x^2y^2z^2.$$

EXAMPLE 5. *Simplify* $(\frac{2}{3}a^2 - \frac{1}{2}ab + \frac{5}{6}b^2) \times 12ab$.

$$(\tfrac{2}{3}a^2 - \tfrac{1}{2}ab + \tfrac{5}{6}b^2) \times 12ab$$
$$= \tfrac{2}{3}a^2 \times 12ab - \tfrac{1}{2}ab \times 12ab + \tfrac{5}{6}b^2 \times 12ab$$
$$= 8a^3b - 6a^2b^2 + 10ab^3.$$

EXAMPLES XV

(Examples 1—18 *may be attempted orally)*

Multiply

1. $a+1$ by 2. **2.** $x-1$ by -3. **3.** $a-4$ by 4.

4. $x-y$ by -7. **5.** $a-2b$ by -7. **6.** $x+y$ by $2x$.

7. $x-y$ by $2x$. **8.** $3x-2y$ by $-5y$. **9.** $a+5b-3c$ by 5.

10. $a-3b+zc$ by -3. **11.** $a+b+c$ by $2a$.

12. $p-q-r$ by p^2. **13.** $-p+q-r$ by $-p^2$.

14. x^2+x+1 by $3x$. **15.** x^2-2x+1 by $-2x$.

16. $3a^2-2a+5$ by $-3a$. **17.** $5m^2-3m+2$ by $-3m$.

18. a^3-2a^2-3a-4 by $5a^2$. **19.** $xy+yz+zx$ by xyz.

20. $ab-bc+ca$ by $-abc$. **21.** $2ab-3bc+4ca$ by $-5abc$.

22. $x^2+2xy+y^2$ by x^3. **23.** $a^2+b^2+c^2$ by abc.

24. $1-2x-3x^2+x^3$ by $-3x$. **25.** x^3-3x^2+3x+1 by $2x$.

26. x^4-2x^2+3 by $-3x^3$. **27.** $-l^3-m^3-n^3$ by $-2lmn$.

28. $-5x^3-xy^4z^3+9y^5z^2$ by $-6x^3y^2z$.

29. $\frac{1}{6}a-\frac{2}{3}b-c$ by $-\frac{3}{5}ab^2c$.

30. $\frac{1}{2}p^2q-\frac{3}{5}q^2r+\frac{4}{5}r^2p$ by $-20p^2q^3r^4$.

32. Addition and Subtraction of Expressions involving terms of different powers of the same letter. We have already seen that only like terms can be added together and also only like terms can be subtracted from one another. Just as $2x+3x=5x$, similarly $2x^2+3x^2=5x^2$. But the expression $2x+3x^2$ cannot be simplified further as x and x^2 are not terms of the same kind. Similarly, the difference of $2x$ and $3x^2$ is written $2x-3x^2$.

EXAMPLE 1. *Find the sum of* $2x^2+3x+4$, $3x^2-5x+7$, $4x^2-7x-10$ *and* $5x^2+9x-12$.

$$\begin{aligned}\text{Sum} &= (2x^2+3x+4)+(3x^2-5x+7)+(4x^2-7x-10)+(5x^2+9x-12)\\ &= 2x^2+3x+4+3x^2-5x+7+4x^2-7x-10+5x^2+9x-12\\ &= 2x^2+3x^2+4x^2+5x^2+3x-5x-7x+9x+4+7-10-12\\ &= 14x^2-11.\end{aligned}$$

Or arranging the expressions in rows so that like terms (terms having the same powers of x) may come in the same vertical columns, and then adding thus

$$\begin{array}{rrr} 2x^2 & +\ 3x & +\ 4\\ 3x^2 & -\ 5x & +\ 7\\ 4x^2 & -\ 7x & -10\\ 5x^2 & +\ 9x & -12\\ \hline 14x^2 & & -11 \end{array}$$

Hence, the sum is $14x^2-11$.

NOTE. Here the terms are arranged in descending powers of x.

EXAMPLE 2. *Subtract* $2a^2-5ab+3b^2$ *from* $3a^2+2ab-5b^2$.

$$\begin{aligned}\text{The difference} &= (3a^2+2ab-5b^2)-(2a^2-5ab+3b^2)\\ &= 3a^2+2ab-5b^2-2a^2+5ab-3b^2\\ &= 3a^2-2a^2+2ab+5ab-5b^2-3b^2\\ &= a^2+7ab-8b^2.\end{aligned}$$

Or arranging the expressions in rows and proceeding with subtraction as in Article 25, thus

$$\begin{array}{r} 3a^2+2ab-5b^2\\ 2a^2-5ab+3b^2\\ \hline a^2+7ab-8b^2. \end{array}$$

Hence, the difference is $a^2+7ab-8b^2$.

EXAMPLE 3. *Simplify* $a(a+b)-3a^2$.

$$a(a+b)-3a^2=a\times a+a\times b-3a^2$$
$$=a^2+ab-3a^2$$
$$=ab-2a^2.$$

EXAMPLE 4. *Simplify* $a^3-a\{a^2-(a^2-\overline{a^2-1})+1\}$.

$$\text{The expression }=a^3-a\{a^2-(a^2-a^2+1)+1\}$$
$$=a^3-a\{a^2-a^2+a^2-1+1\}$$
$$=a^3-a^3+a^3-a^3+a-a$$
$$=0.$$

EXAMPLES XVI

Find the sum of

1. $2a^2+5a$, $11a^2+3a$, $14a^2+17a$.
2. $4a^2-3b^2$, $6a^2+5b^2$, a^2-7b^2.
3. $-6x^2+3x$, $-2x^2-8x$, $8x^2+5x$.
4. $4a-3a^2$, $-6a+6a^2$, $8a-7a^2$, $10a-9a^2$.
5. $x+2x^2$, $3x-4x^2$, $5x+6x^2$, $-7x+8x^2$.
6. x^2+2x+3, $4x^2+5x+6$, $7x^2+8x+9$.
7. $7a^2+6a-3$, $2a^2-4a+6$, $-3a^2+a+5$.
8. $7a^2-4a+1$, $6a^2+3a-5$, $-12a^2+4$.
9. $2x^2-2xy+3y^2$, $-2x^2+5xy+4y^2$, $x^2-3xy-5y^2$.
10. $2x^2+7$, $-x^2-x+5$, $3x-8$, x^2-2x-4.
11. $2a^2+4ab-4b^2$, $-a^2+ab-2b^2$, $-4a^2-3ab+b^2$, $5a^2-2ab+3b^2$.
12. a^2-9a-3, $3a^2+9a$, $-5a^2+6a-7$, $a+1$, $5a^2+3$.
13. a^3+2a^2+3a+4, $5a^3-6a^2+7a-8$, $-9a^3+a^2-2a-3$.
14. x^3+x^2+x, $2x^3-3x^2+4x-4$, x^3+2x^2-3x, $5x^2-7x+1$.
15. $5+3x-4x^2$, $2-5x+6x^2+2x^3$, $1-2x^2+3x^4$, $7+4x-3x^2-6x^4$, $-4-2x+5x^2-2x^3+x^4$.

Subtract

16. a^2-2a-1 from $3a^2+5a-3$.
17. x^2-x+2 from $6x^2-x-1$.
18. $4x^2+5x-3$ from $7x^2+5x+6$.
19. $4a^2-5ab+3b^2$ from $5a^2-7ab-b^2$.
20. $2a^2+ab-2b^2$ from $a^2+2ab-3b^2$.
21. $a^2+3ab+5b^2$ from a^2-b^2.

22. $-2a^3+3a^2-4a+5$ from $2a^3-3a^2+4a-5$.

23. a^3-4a^2+5a+9 from a^3+2a^2+3a+4.

24. a^3-5a+1 from $2a^3-3a^2+5$.

25. $x^4-2x^3+3x^2-4x+5$ from $-6x^4+7x^3-8x^2+9x-1$.

26. $7x^3-5x^2y+3xy^2-y^3$ from $2x^3+4x^2y-6xy^2-8y^3$.

27. $x^3-3xy^2+y^3$ from $3x^3+2x^2y+y^3$.

28. Subtract the sum of $2x^2+x-5$ and $3x^2-x+4$ from $5x^2+x-1$.

29. Take a^2+a+3 from $-a^2+2a+5$ and the result from a^2+a+2.

30. From $3x^2+5xy+9y^2$ take the sum of x^2-xy+y^2 and $2x^2-6xy+8y^2$.

Simplify

31. $x(y-z)+x(z-y)$.

32. $2(x^2-y^2)\times xy$.

33. $x(x+y)-y(x+y)$.

34. $x(x-y)+y(x-y)$.

35. $x^2+x(x+1)$.

36. $3x^2-x(2x-1)-x$.

37. $-3(1-a^2)-2\{a^2-(3-2a^2)\}$.

38. $5\{x^2-(x+1)\}-3x(2-3x)-8\{4-x(1-x)\}$.

39. $3x^2-2(y^2-\overline{x^2-z^2})-3\{(x^2-y^2+z^2)-\overline{z^2-y^2}\}$.

40. $1-a-(1-\overline{a+a^2})-\{1-(a-a^2+a^3)\}$.

Multiplication of Compound Expressions

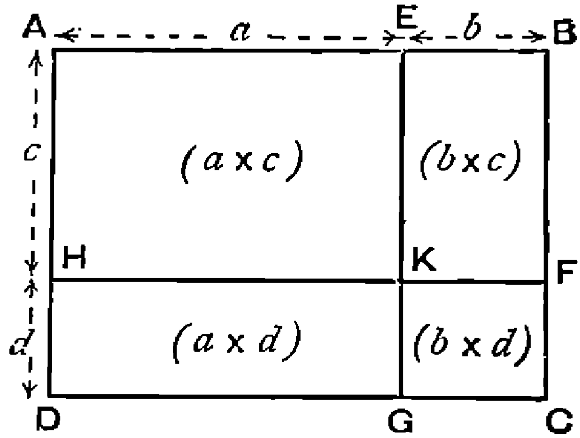
33. Let us consider the product of $(a+b)$ and $(c+d)$. Suppose $(a+b)$ is equal to any quantity x,

$$\therefore\ (a+b)\times(c+d)=x\times(c+d)$$
$$=x\times c+x\times d$$
$$=xc+xd$$
$$=(a+b)c+(a+b)d \quad \text{replacing } x \text{ by } (a+b),$$
$$=ac+bc+ad+bd.$$

The above result may be illustrated graphically thus :

Take a straight line AE equal to a units in length. Produce AE to B, making EB equal to b units. From A draw AH perpendicular to AB equal to c units. Produce

AH to D making HD equal to d units. Complete the rectangle ABCD. From E draw EG parallel to AD and



from H draw HF parallel to AB meeting DC and BC in points G and F respectively and intersecting at K.

Now the rectangle ABCD has its adjacent sides AB and AD equal to $(a+b)$ and $(c+d)$ units respectively,

$$\therefore \text{ its area} = (a+b) \times (c+d) \text{ sq. units.}$$

From the figure it is clear that the rectangle ABCD is made up of four smaller rectangles AEKH, EBFK, HKGD and KFCG, whose areas are $a \times c$, $b \times c$, $a \times d$ and $b \times d$ sq. units respectively.

$$\therefore \quad (a+b)(c+d) = ac + bc + ad + bd.$$

EXAMPLE 1. *Multiply* (x+2) *by* (x+3).

Suppose $(x+2)$ is equal to any quantity 'a'.

$$\begin{aligned} \therefore \quad (x+2) \times (x+3) &= a \times (x+3) \\ &= a \times x + a \times 3 \\ &= (x+2) \times x + (x+2) \times 3 \\ &= x^2 + 2x + 3x + 6 \\ &= x^2 + 5x + 6. \end{aligned}$$

EXAMPLE 2. *Multiply* (x − 2) *by* (x − 5).

Suppose $(x-2)$ is equal to any quantity 'a'.

$$\begin{aligned} \therefore \quad (x-2) \times (x-5) &= a \times (x-5) \\ &= a \times x - a \times 5 \\ &= (x-2) \times x - (x-2) \times 5 \\ &= x^2 - 2x - 5x + 10 \\ &= x^2 - 7x + 10. \end{aligned}$$

From the above examples, we see that *the product of two compound expressions is equal to the algebraic sum of the products obtained by multiplying each term of the one expression by each term of the other, the terms including the accompanying signs.*

NOTE. Since $x \times 0 = x(a-a)$
$= xa - xa$
$= 0,$

and similarly $0 \times x = (a-a)x$
$= ax - ax$
$= 0.$

Hence, we see that *if a number or quantity is multiplied by zero, the product is equal to zero.*

34. The working of examples 1 and 2 of the previous article can also be arranged thus

$$\begin{array}{ll} x+2 & \\ x+3 & \\ \hline x^2+2x & \text{... multiplying } (x+2) \text{ by } x, \\ \quad\;\; 3x+6 & \text{... ,, ,, ,, } 3, \\ \hline x^2+5x+6 & \text{adding the partial products.} \end{array}$$

Hence, the product of $(x+2)$ and $(x+3)$ is x^2+5x+6.

Similarly

$$\begin{array}{l} x-2 \\ x-5 \\ \hline x^2-2x \\ \quad\;\; -5x+10 \\ \hline x^2-7x+10 \end{array}$$

Hence, the product of $(x-2)$ and $(x-5)$ is $x^2-7x+10$.

EXAMPLE 1. *Multiply* $(x+y)$ *by* $(a+b)$.

$$\begin{array}{l} x+y \\ a+b \\ \hline ax+ay \\ \qquad\quad +bx+by \\ \hline ax+ay+bx+by \end{array}$$

Hence, the product is $ax+ay+bx+by$.

EXAMPLE 2. *Multiply* (3 − x) *by* (7 + 2x).

$$\begin{array}{l} 3 - x \\ 7 + 2x \\ \hline 21 - 7x \\ \quad\ + 6x - 2x^2 \\ \hline 21 - x - 2x^2 \end{array}$$

Hence, the product is $21 - x - 2x^2$.

EXAMPLE 3. *Multiply* (a + b) *by* (a − b).

$$\begin{array}{l} a + b \\ a - b \\ \hline a^2 + ab \\ \quad\ - ab - b^2 \\ \hline a^2 \qquad - b^2 \end{array}$$

Hence, the product is $a^2 - b^2$.

35. Arrangement of an expression according to the powers of a letter. If in an expression containing several terms of different powers of the same letter, the term containing the highest power of that letter be placed first, the term containing the next highest power be placed next, and so on, the term containing the lowest power of that letter or the term which does not contain that letter be placed last, the expression is said to be **arranged according to descending powers** of that letter. Similarly, if the term which does not contain that letter or if the term which contains the lowest power of that letter be placed first, the term containing the next highest power of that letter be placed next, and so on, the term containing the highest power of that letter be placed last, the expression is said to be **arranged according to ascending powers** of that letter.

Thus the expression $3x^2 - 2x - 1$ is arranged according to descending powers of x while $1 + 2x - 3x^2$ is arranged according to ascending powers of x. Again, the expression $d^2 + dc + c^2$ is arranged according to descending powers of d and ascending powers of c.

36. When finding the product of two expressions, it is often desirable to re-arrange the terms, if necessary, of both the expressions either according to descending powers or ascending powers of the same letter.

EXAMPLE 1. *Multiply* $3x^2-2x-1$ *by* $2x-3$.

$$\begin{array}{r} 3x^2-2x-1 \\ 2x-3 \\ \hline 6x^3-4x^2-2x \\ -9x^2+6x+3 \\ \hline 6x^3-13x^2+4x+3 \end{array}$$

Hence, the product is $6x^3-13x^2+4x+3$.

EXAMPLE 2. *Multiply* d^2+dc+c^2 *by* d^2-dc+c^2.

$$\begin{array}{l} d^2+dc+c^2 \\ d^2-dc+c^2 \\ \hline d^4+d^3c+d^2c^2 \\ \quad -d^3c-d^2c^2-dc^3 \\ \quad\quad +d^2c^2+dc^3+c^4 \\ \hline d^4 \quad +d^2c^2 \quad +c^4 \end{array}$$

Hence, the product is $d^4+d^2c^2+c^4$.

Hence, *to multiply two compound expressions, place one under the other and draw a line below the second. Then multiply the successive terms of the first expression by the first term on the left of the second expression, and place the products thus obtained under the line in a horizontal row. Again, multiply the terms of the first expression by the second term of the second expression, and place the products in a second horizontal row, arranging the terms so that like terms fall under one another in the same vertical columns. Continue this process of multiplication till all the terms of the second expression are dealt with, and then draw a line below the last row of products. The required product will be obtained by adding the rows of the partial products.*

EXAMPLES XVII

Multiply

1. $x+1$ by $x+2$.
2. $x-2$ by $x-3$.
3. $x+2$ by $x-3$.
4. $x-2$ by $x+3$.
5. $x+3$ by $x+8$.
6. $x+6$ by $x-7$.

7. $1+x$ by $x+3$. 8. $5+x$ by $x+6$. 9. $x+1$ by $x-1$.
10. $1-7x$ by $1+3x$. 11. $x-3$ by $x+3$. 12. $7-x$ by $7+x$.
13. $x+y$ by $x+y$. 14. $x+2a$ by $x+3a$.
15. $x-2y$ by $x+2y$. 16. $2x-y$ by $x-2y$.
17. $a-3b$ by $a+2b$. 18. $-4a+b$ by $5a-b$.
19. $2x-3$ by $3x+4$. 20. $4x+5$ by $4x-5$.
21. $px+q$ by $px-q$. 22. $ca-b$ by $ca+b$.
23. a^2-b^2 by a^2+b^2. 24. p^2-3q^2 by p^2+4q^2.
25. $a-b^3$ by a^3-b. 26. $py+1$ by $qy-1$.
27. $ax+1$ by $bx-1$. 28. p^2+q by p^2-3r.
29. x^2+a^2 by $x+a$. 30. x^2-a^2 by $x-a$.
31. $a-b+c$ by $b-c$. 32. x^2-2x+1 by $x-1$.
33. $2x^2-3x+1$ by $2x-1$. 34. x^2-ax+a^2 by $x+a$.
35. x^2+6x+9 by $x+3$. 36. x^2-3x-5 by $2x-5$.
37. $c^2+2cd+d^2$ by $-c-2d$. 38. $x^4+x^2y^2+y^4$ by x^2-y^2.
39. a^2+b-2 by a^2-b+2. 40. a^2-3a+1 by a^2-3a+1.
41. $a+b+c$ by $a+b+c$. 42. $x-y+z$ by $x+y-z$.
43. $2x-y+3z$ by $2x+y-3z$.

37. EXAMPLE 1. *If the price of one horse is Rs.* 125, *what is the price of* 50 *horses?*

Since the price of 1 horse is Rs. 125,

∴ ,, ,, ,, 50 horses ,, Rs. $50\times125=$ Rs. 6250.

EXAMPLE 2. *If the price of one horse is Rs.* x, *what is the price of* y *horses?*

Since the price of 1 horse is Rs. x,

∴ ,, ,, ,, y horses ,, Rs. $x\times y=$ Rs. xy.

EXAMPLE 3. *If the price of* 1 *seer of rice is* 4 *annas, what is the price of* x *maunds of rice?*

Since x md. $=40\times x$ sr., and $4a=$ Re. $4\times\frac{1}{16}=$ Re. $\frac{1}{4}$,

∴ if the price of 1 sr. of rice is Re. $\frac{1}{4}$,

∴ ,, ,, ,, $40x$,, ,, Rs. $\frac{1}{4}\times40x=$ Rs. $10x$.

EXAMPLE 4. *If the price of* 5 *books is Rs.* 5x, *what is the price of* y *books?*

Since the price of 5 books is Rs. $5x$,

,, 1 book ,, Rs. $\frac{5x}{5}=$ Rs. x,

,, y books ,, Rs. $x\times y=$ Rs. xy.

EXAMPLES XVIII

1. If the price of one ox is Rs. 80, what is the price of x oxen ?

2. If the price of one book is Rs. 5, what is the price of a books ?

3. If the price of one cap is Rs. x, what is the price of 10 caps ?

4. If the price of one chair is Rs. x, what is the price of y chairs ?

5. A man goes $2x$ miles in one hour, how far will he go in x hours ?

6. If the price of one box is Rs. a, what is the price of y boxes ?

7. If the price of 3 inkpots is $3x$ annas, what is the price of x inkpots ?

8. If the price of one table is Rs. x, what is the price of $5x$ tables ?

9. If the interest on a certain sum of money for 1 year is Rs. x, what will be the interest on the same sum for x^2 years ?

10. If a man gains Rs. $2x^2$ in one year, what will he gain in $3x^2$ years ?

11. A man has x sons and y daughters. He gives Rs. 5 to each of his sons and Rs. 4 to each of his daughters, how much does he give to them ?

12. One room has x almirahs and another room has y almirahs. If each almirah in the first room contains 50 books and each in the second contains 60 books, how many books are there in all the almirahs ?

13. A man buys x boxes, y tables and z chairs. If each box costs Rs. x, each table Rs. y and each chair Rs. z, how much do they cost altogether ?

14. A class-room contains x desks and $2x$ chairs. If each desk costs Rs. 6 and each chair Rs. 4, how much do all the desks and chairs cost ?

15. A boy buys x^2 books and x^3 pens. If each book costs x^2 annas and each pen x annas, how much do they cost altogether ?

16. A man first runs for half an hour at the rate of $2x^2$ miles an hour, then walks for x hours at the rate of $2x$ miles an hour and then rides for $2\frac{1}{2}$ hours at the rate of $8x^2$ miles an hour. How far does he go in all ?

17. The length of a rectangle is x^3 ft. and breadth x^2 ft., what is its area ?

18. The length of a rectangle is $(a+b)$ ft. and breadth $(a-b)$ ft., what is its area ?

19. If one foot of cloth costs a rupees, what will x yards and y feet of cloth cost ?

CHAPTER VII

DIVISION

38. In Arithmetic when we divide 12 by 4, we find 'how many times 4 is contained in 12', or 'what number multiplied by 4 gives 12'. The quotient

$$12 \div 4 \text{ or } \tfrac{12}{4} = 3.$$

But in Algebra if we divide a by b, since we do not know what numbers are represented by a and b, we simply write down the quotient as $a \div b$ or $\frac{a}{b}$.

In Arithmetic $\frac{3 \times 4}{4} = 3$,

similarly in Algebra $\frac{x \times y}{y} = x$.

And $$\frac{+ab}{+a} = \frac{+a \times +b}{+a} = +b \qquad (1)$$

$$\frac{-ab}{-a} = \frac{(-a) \times +b}{-a} = +b \qquad (2)$$

$$\frac{-ab}{+a} = \frac{+a \times (-b)}{+a} = -b \qquad (3)$$

$$\frac{ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b \qquad (4)$$

Hence, we see that *in division, if the signs of the numerator and denominator are alike the quotient is positive, if they are unlike the quotient is negative.*

The above can also be written more briefly thus

$$\frac{+}{+} = +, \quad \frac{-}{-} = +, \quad \frac{-}{+} = -, \quad \frac{+}{-} = -.$$

Thus it is clear that **the rule of signs** is the same in division as in multiplication, *i.e., like signs give plus and unlike signs give minus.*

Division of Simple Expressions

39. Since $a^5 = a \times a \times a \times a \times a$,

and $a^3 = a \times a \times a$,

$$a^5 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a \times a = a^2.$$

Similarly $$\frac{a^6}{a^3} = \frac{a \times a \times a \times a \times a \times a}{a \times a \times a} = a \times a \times a = a^3.$$

Hence, we see that *when one power of a quantity is divided by another power of the same quantity, the index of the quotient is obtained by subtracting the index of the divisor from the index of the dividend.*

NOTE. Since $a^0 = a^{4-4} = a^4 \div a^4 = \frac{a^4}{a^4} = 1$,

hence, we see that *if the index of a quantity is zero, the quantity is equal to one.*

EXAMPLE 1. *Divide* $4x^2y$ *by* $2xy$.

$$4x^2y \div 2xy = \frac{4x^2y}{2xy} = \frac{2 \times 2 \times x \times x \times y}{2 \times x \times y} = 2x.$$

EXAMPLE 2. *Divide* $15x^7$ *by* $-5x^2$.

$$15x^7 \div -5x^2 = \frac{15x^7}{-5x^2} = -\frac{15x^7}{5x^2} \text{ [unlike signs give minus]}$$
$$= -3x^{7-2} = -3x^5.$$

EXAMPLE 3. *Divide* $15a^4b^3c^2$ *by* $3a^2bc$.

$$\frac{15a^4b^3c^2}{3a^2bc} = \frac{3 \times 5 \times a^4 \times b^3 \times c^2}{3 \times a^2 \times b \times c} = 5a^{4-2}b^{3-1}c^{2-1} = 5a^2b^2c.$$

EXAMPLE 4. *Divide* $-15a^3b^2c$ *by* $-5abc$.

$$\frac{-15a^3b^2c}{-5abc} = +\frac{15a^3b^2c}{5abc} \text{ [like signs give plus]}$$
$$= 3a^{3-1}b^{2-1}c^{1-1} = 3a^2b.$$

EXAMPLE 5. *Divide* $-18a^5b^3c^2$ *by* $6a^2b^3c^2$.

$$\frac{-18a^5b^3c^2}{6a^2b^3c^2} = -\frac{18a^5b^3c^2}{6a^2b^3c^2} = -3a^{5-2}b^{3-3}c^{2-2} = -3a^3.$$

Hence, *to divide one simple expression by another, first divide the numerical coefficient of the dividend by the numerical coefficient, if any, of the divisor, then obtain the index of each letter in the quotient, by subtracting the index of that letter in the divisor, from that in the dividend and then prefix the proper sign.*

EXAMPLES XIX

(*Examples* 1 – 33 *may be taken orally*)

Divide

1. $5x$ by 5. **2.** $5x$ by x. **3.** $-5x$ by 5.

4. $-5x$ by -5. **5.** $-5x$ by $-x$. **6.** $-a^2$ by $-a^2$.

7. $-a^2$ by a^2. **8.** $-a^2$ by a. **9.** $-x^2$ by $-x$.

10. a^5 by a^2. **11.** $6a^3$ by $2a^2$. **12.** $-24a^4$ by $6a^3$.

13. $-12x^2y$ by $3x$. **14.** $-6ax^2$ by $-2x^2$. **15.** $8a^2b^2$ by $-4ab$.

16. $-a^4b^6$ by $-a^2b^2$. **17.** $-3m^3n^3$ by $-3m^3n^3$.

18. $-11p^2q^4$ by $11p^2q^4$. **19.** $-21a^3x^4$ by $-7a^3x$.

20. $-a^4b^2$ by $-ab$. **21.** $-4x^4y^4$ by $-4x^2y^2$.

22. m^2np^2 by $-mnp$. **23.** $10m^4n^3$ by $-2m^4$.

24. $63a^5b^4c^6$ by $-7ab^2c^4$. **25.** $-91a^7b^3c^5$ by $-13a^6b^3c^4$.

Simplify

26. $\dfrac{96a^7b^6}{8a^2b^2}$. **27.** $\dfrac{-27p^6q^7r^2}{-9p^3q^3r}$. **28.** $\dfrac{-56a^{11}b^9c^7}{8a^8b^6c^4}$.

29. $\dfrac{63pq^2r^2}{-7pq}$. **30.** $\dfrac{-64lmn}{8lm}$. **31.** $\dfrac{-81x^6y^4z^2}{-9x^3y^2z^2}$.

32. $\dfrac{49a^5b^4c^2d^5}{7a^2b^3c^2d^4}$. **33.** $\dfrac{-77a^7b^3cd^8}{14a^3b^2cd^5}$. **34.** $5x^3 \div (-x)^2$.

35. $8a^3b \div (-2a)^2$. **36.** $-x^6 \div (-x)^5$. **37.** $(2x)^3 \div (-3x)^2$.

38. $x^8 \div (-x)^8$. **39.** $(-x^4y^5) \div (-x)^3$.

If $a = -1$, $b = -2$, $c = -3$, find the value of

40. $\dfrac{a}{bc}$. **41.** $\dfrac{a^2}{-2bc^2}$. **42.** $\dfrac{4a^5c}{4b}$.

43. $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2}$. **44.** $\dfrac{a^2+b^2}{b^2+c^2}$. **45.** $\dfrac{ab+c}{ab-c}$.

46. $\dfrac{a^2+3b^2c}{2a^3-5bc}$. **47.** $\dfrac{a(b+c)}{c(a-b)}$.

If $a=1$, $b=-2$, $c=3$, $x=-4$, $y=5$, $z=0$, find the value of

48. $\frac{(a+y)^2}{(x-z)^2}-\frac{(c^2-a)}{(b^2-y)}$.

49. $\frac{a^2-b^2}{a^2b^2}-\frac{(a+b+z)}{(a+b-c)}$.

50. $\frac{(a+b+c)^2}{(x+y+z)^2}-\frac{4(a-x)}{5(b-z)}$.

51. $\frac{a}{b}+\frac{4c}{3x}-\frac{3ac}{10}+\frac{b^3+z^3}{b^2-z^2}$.

52. $\frac{8y}{a}-\frac{2b^2}{y}+\frac{a-x+y-z}{ax-bz}$.

Division of Compound Expression by Simple Expression

40. In multiplication

$$(a+b)\times c=a\times c+b\times c=ac+bc,$$

similarly, in division

$$(a+b)\div c=a\div c+b\div c,$$

$$i.e.,\ \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}.$$

Hence, *to divide a compound expression by a simple expression, divide each term of the dividend by the divisor and add the partial quotients.*

EXAMPLE 1. *Divide* ab+ac *by* a.

$$\frac{ab+ac}{a}=\frac{ab}{a}+\frac{ac}{a}=b+c.$$

EXAMPLE 2. *Divide* 6x−12y+3z *by* −3.

$$\frac{6x-12y+3z}{-3}=\left(\frac{6x}{-3}\right)-\left(\frac{12y}{-3}\right)+\left(\frac{3z}{-3}\right)$$
$$=(-2x)-(-4y)+(-z)$$
$$=-2x+4y-z.$$

EXAMPLE 3. *Divide* $a^3b^2c-a^2bc^3-ab^3c^2$ *by* −abc.

$$\frac{a^3b^2c-a^2bc^3-ab^3c^2}{-abc}=\left(\frac{a^3b^2c}{-abc}\right)-\left(\frac{a^2bc^3}{-abc}\right)-\left(\frac{ab^3c^2}{-abc}\right)$$
$$=(-a^2b)-(-ac^2)-(-b^2c)=-a^2b+ac^2+b^2c.$$

EXAMPLE 4. *Simplify* $\frac{45x^6y^5z^4+27x^4y^6z^5-18x^5y^4z^6}{9x^3y^3z^3}$.

The expression $=\frac{45x^6y^5z^4}{9x^3y^3z^3}+\frac{27x^4y^6z^5}{9x^3y^3z^3}-\frac{18x^5y^4z^6}{9x^3y^3z^3}$

$$=5x^3y^2z+3xy^3z^2-2x^2yz^3.$$

EXAMPLE 5. *Simplify* $\frac{2x+3}{3}+\frac{x-2}{4}-\frac{5x+3}{6}$.

The expression $=\left(\frac{2x}{3}+\frac{3}{3}\right)+\left(\frac{x}{4}-\frac{2}{4}\right)-\left(\frac{5x}{6}+\frac{3}{6}\right)$

$$=\frac{2x}{3}+1+\frac{x}{4}-\frac{1}{2}-\frac{5x}{6}-\frac{1}{2}$$

$$=\frac{2x}{3}+\frac{x}{4}-\frac{5x}{6}+1-\frac{1}{2}-\frac{1}{2}$$

$$=\frac{1}{12}x.$$

NOTE. The straight line of a fraction is also regarded as a bracket. Thus, $\frac{5x+3}{6}$ means the same thing as $(5x+3)\div 6$ or $\frac{1}{6}(5x+3)$. When it is preceded by the sign '$-$', the signs of the terms over it are changed when it is removed, as in the above example

$$-\frac{5x+3}{6}=-\left(\frac{5x}{6}+\frac{3}{6}\right)=-\frac{5x}{6}-\frac{3}{6}.$$

EXAMPLE 6. *Simplify* $\frac{5x^2-4x}{20x}-\frac{x^3-x}{2x}-\frac{1}{2}$.

The expression $=\frac{5x^2}{20x}-\frac{4x}{20x}-\frac{x^3}{2x}+\frac{x}{2x}-\frac{1}{2}$

$$=\frac{x}{4}-\frac{1}{5}-\frac{x^2}{2}+\frac{1}{2}-\frac{1}{2}$$

$$=-\frac{x^2}{2}+\frac{x}{4}-\frac{1}{5}.$$

NOTE. Since $\frac{0}{b}=\frac{a-a}{b}=\frac{a}{b}-\frac{a}{b}=0$,

hence, we see that *if zero is divided by a quantity, the quotient is zero, or if the numerator of a fraction is zero, the fraction is equal to zero.*

EXAMPLES XX

(*Examples* 1—14 *may be taken orally*)

Divide

1. $2a+2$ by 2.
2. $3a^5-9$ by -3
3. a^2+a by a.
4. $-b^2+ab$ by $-b$.
5. $5x^2-15x$ by $-5x$.
6. $9a^2b+9ab^2$ by $3ab$.
7. $4x^2y-12xy^2$ by $-2xy$.
8. $a^4b^3-a^3b^4$ by a^2b^2.
9. $-2a^2bc+3ab^2c$ by $-abc$.
10. $12a^4-18a^2b^2$ by $-6a^2$.

11. $ax+ay+az$ by a. **12.** $6ac-9bc+15cd$ by $-3c$.

13. $5x^6+3x^4+2x^2$ by x^2. **14.** $-xy+ay-by$ by $-y$.

15. $-4a^3+20a^5-12a^7$ by $-4a^3$.

16. $ab^4-ab^2+a^3b^2$ by $-ab^2$.

17. $a^2b+ab^2+a^2b^2$ by $-ab$.

18. $6x^2y^2-12x^3y^3+18x^4y^4$ by $6x^2y^2$.

19. $l^4m^3n^3-l^2m^2n^2+lm^2n$ by $-lmn$.

20. $x^2y^2z^2-xy^2z^2-xyz^2$ by xyz.

Simplify

21. $\dfrac{-a^2x-ax^2-a^2x^2}{ax}$. **22.** $\dfrac{7a^3b^2+35a^4b^3-21a^3b^3}{-7a^3b^2}$.

23. $\dfrac{4x^4y^6-8x^5y^5-28x^6y^4}{-4x^3y^3}$. **24.** $\dfrac{70p^2q^4r^6+50p^4q^6r^2-20p^6q^2r^4}{-10p^2q^2r^2}$.

25. $\dfrac{2x+1}{3}+\dfrac{x+2}{5}$. **26.** $\dfrac{x^2+2x}{4x}-\dfrac{x^2-x}{8x}$.

27. $\dfrac{3x-13}{7}+\dfrac{11-4x}{3}$. **28.** $2x-\dfrac{1}{3x}(x^2+27x)-16$.

29. $\dfrac{7a^3+2a^2}{5a^2}-\dfrac{4a-1}{2}$. **30.** $\dfrac{2x^3-7x^2}{4x}-\dfrac{x^2-2x}{7x}+\dfrac{5x}{7}$.

31. $\dfrac{x}{4}-\dfrac{5x^2+8x}{6x}-\dfrac{2x^3-9x^2}{3x}$. **32.** $\dfrac{3x-1}{3}+\dfrac{5}{12}-\dfrac{x}{4}-\dfrac{2x^2+x}{5x}$.

33. $6-\dfrac{x-1}{2}-\dfrac{x-2}{3}-\dfrac{3-x}{4}$.

34. $\dfrac{1}{3x}(x-2x^2)-\dfrac{1}{6x^2}(4x^2-5x^3)+\dfrac{13}{42}$.

35. $\dfrac{3a-4ax}{5a}-\dfrac{4a^2+5a^2x}{9a^2}+\dfrac{7x^2+11x}{15x}$.

Division by a Compound Expression

41. The method of division of compound expressions in Algebra is similar to the method of Long Division in Arithmetic. Suppose we have to divide 651 by 31, we work out the division thus

$$
\begin{array}{r|l}
31)\ 651\ (& 21 \\
62 & \\
\hline
31 & \\
31 & \\
\hline
\end{array}
$$

Now, since 651 can be written as $600+50+1$ or $6\times10^2+5\times10+1$ and 31 as $30+1$ or $3\times10+1$, the above working can be arranged as

$$\begin{array}{r|l|l} 3.10+1 & 6.10^2+5.10+1 & 2.10+1 \\ & \underline{6.10^2+2.10} & \\ & \quad 3.10+1 & \\ & \quad \underline{3.10+1} & \end{array}$$

Now, if in the above we replace 10 by x, we have

$$\begin{array}{r|l|l} 3.x+1 & 6.x^2+5.x+1 & 2.x+1 \\ & \underline{6.x^2+2.x} & \\ & \quad 3.x+1 & \\ & \quad \underline{3.x+1} & \end{array}$$

To divide $6x^2+5x+1$ by $3x+1$, we first divide $6x^2$, the first term of the dividend by $3x$, the first term of the divisor. The quotient $2x$, thus obtained, is the first term of the required quotient. Now all the terms of the divisor are multiplied by $2x$ and the result subtracted from the dividend. Thus we have the remainder $3x+1$. We now treat this remainder as a new dividend and divide its first term $3x$ by the first term $3x$ of the divisor. The quotient is 1. This is the second term of the required quotient. On multiplying the divisor by 1, and subtracting the result from $3x+1$, we find there is no remainder. Hence, the complete quotient is the sum of the partial quotients, *i.e.*, $2x+1$.

NOTE. As in multiplication, so in division it is also desirable to re-arrange the terms, if necessary, of both dividend and divisor either according to descending powers or ascending powers of the same letter.

EXAMPLE 1. *Divide* $x^2-7x+12$ *by* $x-3$.

$$\begin{array}{r|l|l} x-3 & x^2-7x+12 & x-4 \\ & \underline{x^2-3x} & \\ & \quad -4x+12 & \\ & \quad \underline{-4x+12} & \end{array}$$

Hence, the quotient is $x-4$.

EXAMPLE 2. *Divide* $x^3+a^3+a^2x+ax^2$ *by* x^2+a^2.

Arranging the dividend according to the descending powers of x, we have $x^3+ax^2+a^2x+a^3$.

$$\begin{array}{r|l|l} x^2+a^2 & x^3+ax^2+a^2x+a^3 & x+a \\ & \underline{x^3 \qquad\quad +a^2x} & \\ & \quad ax^2 \qquad\quad +a^3 & \\ & \quad \underline{ax^2 \qquad\quad +a^3} & \end{array}$$

Hence, the quotient is $x+a$.

EXAMPLE 3. *Divide* $8a^3 - b^3$ *by* $2ab + 4a^2 + b^2$.

Arranging the divisor according to the descending powers of a, we have $4a^2 + 2ab + b^2$.

$$\begin{array}{r|l|l} 4a^2+2ab+b^2 & 8a^3 \qquad\qquad\qquad\quad -b^3 & 2a-b \\ & \underline{8a^3+4a^2b+2ab^2\qquad} & \\ & -4a^2b-2ab^2-b^3 & \\ & \underline{-4a^2b-2ab^2-b^3} & \end{array}$$

Hence, the quotient is $2a - b$.

EXAMPLE 4. *Divide* $27x^3 - 27x^2y + 9xy^2 - y^3$ *by* $3x - y$.

$$\begin{array}{r|l|l} 3x-y & 27x^3-27x^2y+9xy^2-y^3 & 9x^2-6xy+y^2 \\ & \underline{27x^3-\ \ 9x^2y} & \\ & -18x^2y+9xy^2 & \\ & \underline{-18x^2y+6xy^2} & \\ & 3xy^2-y^3 & \\ & \underline{3xy^2-y^3} & \end{array}$$

Hence, the quotient is $9x^2 - 6xy + y^2$.

EXAMPLE 5. *Divide* $x^4 + x^3 - 3x^2 + 2x + 1$ *by* $x^2 - 2x + 1$.

$$\begin{array}{r|l|l} x^2-2x+1 & x^4+x^3-3x^2+2x+1 & x^2+3x+2 \\ & \underline{x^4-2x^3+x^2} & \\ & 3x^3-4x^2+2x & \\ & \underline{3x^3-6x^2+3x} & \\ & 2x^2-x+1 & \\ & \underline{2x^2-4x+2} & \\ & 3x-1 & \end{array}$$

Now the division cannot be carried on any further, since the highest power of x in the remainder cannot be divided by the highest power of x in the divisor. Hence, the quotient is $x^2 + 3x + 2$, and the remainder is $3x - 1$. In such a case the division is **inexact**.

From the above examples it will be clear that in order to divide one compound expression by another, we proceed as follows :

(i) *Arrange both dividend and divisor according to descending or ascending powers of some common letter.*

(ii) *Divide the first term of the dividend by the first term of the divisor. The result is the first term of the quotient.*

(iii) *Multiply each term of the divisor by the first term of the quotient thus obtained and subtract the product from the dividend.*

(iv) *Treat the remainder as a new dividend and proceed as before.*

(v) *Continue the process till there is no remainder or the highest power of the letter in the remainder is lower than the highest power of the same letter in the divisor.*

(vi) *The complete quotient is the sum of the partial quotients obtained during the course of division.*

EXAMPLES XXI

Divide

1. x^2+3x+2 by $x+1$.
2. $x^2+9x+18$ by $x+6$.
3. $x^2-7x+12$ by $x-3$.
4. x^2-2x+1 by $x-1$.
5. $x^2-14x+49$ by $x-7$.
6. $3x^2+5x+2$ by $x+1$.
7. $2x^2+3x-2$ by $x+2$.
8. $8x^2+14x+3$ by $2x+3$.
9. $5x^2-7xy-6y^2$ by $x-2y$.
10. $9x^2-3xy-2y^2$ by $3x-2y$.
11. $18x^2+27xa-5a^2$ by $6x-a$.
12. $1+6x+9x^2$ by $1+3x$.
13. $25-30a+9a^2$ by $5-3a$.
14. $25-x^2$ by $5-x$.
15. x^3+x^2-3x+9 by $x+3$.
16. $2x^3+3x^2-8x-12$ by $2x+3$.
17. $2+4x^2+3x+6x^3$ by $3x+2$.
18. $6-a^3-11a+6a^2$ by $2-a$.
19. $12-3x^2+x^3-4x$ by $3-x$.
20. $x^3-39x+18$ by $x-6$.
21. $x^3-134x+143$ by $x-11$.
22. $28x^3-45x^2+25x-56$ by $4x-7$.
23. $49x^3-29x-6$ by $7x-6$.
24. $27x^3-8y^3$ by $3x-2y$.
25. $81-x^6$ by $9+x^3$.
26. $8a^3-12a^2b+6ab^2-b^3$ by $2a-b$.
27. $a^3+3a^2b+3ab^2+b^3$ by $a+b$.
28. $x^3-3x^2y+3xy^2-y^3$ by $x-y$.
29. $14x^3+47x^2y-25xy^2-63y^3$ by $2x+7y$.
30. a^3+1 by $a+1$.
31. x^3-1 by $x-1$.
32. $px^2-p^2x+2x-2p$ by $x-p$.
33. $a^2x^2+abx+acx+bc$ by $ax+b$.
34. $27x^2+3bcx-9ax-abc$ by $3x-a$.
35. $27k^3+9k^2-3k-10$ by $3k-2$.
36. $4a^3-16a^2b+21ab^2-9b^3$ by $a-b$.
37. $a^2+b^2+c^2+2ab+2bc+2ca$ by $a+b+c$.
38. $a^2-b^2-c^2+2bc$ by $a-b+c$.
39. x^4+x^2+1 by x^2-x+1.
40. a^6+1 by a^2+1.
41. x^3+4x^2+7x+9 by x^2+3x+2.

42. $x^3-8x^2+18x-15$ by x^2-3x+3.
43. $12x^3+38x-35-23x^2$ by $3x^2+7-2x$.
44. $x^3+2x^2y-2xy^2+3y^3$ by x^2-xy+y^2.
45. $3x^3-23x^2y+17xy^2-2y^3$ by $x^2-7xy+y^2$.
46. a^6-1 by $a-1$. 47. $21k^3-5k^2-3k-2$ by $7k^2+3k+1$.
48. $12p^3-19p^2-2p+8$ by $4p^2-p-2$.
49. $a^4-4a^3-18a^2-9a+3$ by a^2-7a+1.
50. $3m^5+3m^4+2m^3+1$ by $3m^3-m+1$.
51. $x^4-4+34x-25x^2$ by $3-5x+x^2$.
52. $8x^6+y^6$ by $2x^2+y^2$.
53. $81x^4-1$ by $3x-1$. 54. x^9-y^9 by x^3-y^3.
55. $25a^6-44a^4+4a^2-9$ by $5a^3-2a^2-4a-3$.
56. $7x(x^2-4)+96x^2$ by $7x-2$.
57. $3(5+x)-7x^2(1+2x)+10x^3$ by $5-4x$.
58. $10(x^2-2ax)-3(ax-4a^2)$ by $2x-3a$.
59. the sum of $2a^2(3a-10)$ and $2(13a-10)$ by $2a-4$.
60. the sum of (x^3+6b^3) and $-bx(2x+3b)$ by $x-2b$.

42. EXAMPLE 1. *A man has* Rs. 20. *He distributes this sum equally among his* 4 *sons. What does each son get ?*

Since 4 sons get Rs. 20,

∴ 1 son gets Rs. $\frac{20}{4}$=Rs. 5.

EXAMPLE 2. *A man has* Rs. x. *He distributes this sum equally among his* 4 *sons. What does each son get ?*

Since 4 sons get Rs. x,

1 son gets Rs. $\frac{x}{4}$.

EXAMPLE 3. *A man has* Rs. x. *He distributes this sum equally among his* y *sons. What does each son get ?*

Since y sons get Rs. x,

1 son gets Rs. $\frac{x}{y}$.

EXAMPLE 4. *If the price of* 10 *books is* Rs. 10x, *what is the price of* a *books ?*

Since the price of 10 books is Rs. $10x$,

,, ,, 1 book ,, Rs. $\frac{10x}{10}$=Rs. x,

,, ,, a books ,, Rs. $x\times a$=Rs. xa.

EXAMPLE 5. *A man has* Rs. x. *He spends* Rs. y, *and with the rest he buys* 5 *inkpots. What is the price of each inkpot ?*

Of Rs. x, the man spends Rs. y ; therefore, he has Rs. $(x-y)$ left.

∴ Now the price of 5 inkpots is Rs. $(x-y)$,

,, ,, 1 inkpot is Rs. $\frac{x-y}{5}$.

EXAMPLE 6. *Magan has* $2x^3$ *nuts. He gives* x^2 *nuts to Prem and distributes the rest equally among* x^2 *boys. How many nuts does each boy get ?*

After giving x^2 nuts to Prem, Magan has $(2x^3-x^2)$ nuts left.

Now x^2 boys get $2x^3-x^2$ nuts,

1 boy gets $\frac{2x^3-x^2}{x^2}$ nuts $=(2x-1)$ nuts.

EXAMPLES XXII

1. A man has x oranges. He distributes these equally among 5 boys. How many oranges does each boy get ?

2. A man has Rs. 20. He distributes these equally among x boys. How much does each boy get ?

3. A boy has 100 almonds. He distributes these equally among a friends. How many almonds does each friend get ?

4. A teacher has x guavas. He distributes these equally among y boys. How many guavas does each boy get ?

5. A trader gains Rs. a in x years, how much does he gain in 1 year ?

6. If the price of x maunds of gram is Rs. a, what is the price of 1 maund of gram ?

7. If the price of a maunds of rice is Rs. b, what is the price of 1 seer of rice ?

8. If the price of x chataks of sugar is 1 anna, what is the price of y seers of sugar ?

9. If the interest on Rs. x for 1 year is Rs. y, what is the interest on Rs. 100 for 1 year ?

10. If Krishna scored x^2 runs in x overs in a cricket match, how many runs did he score in one over if he scored equal runs in each over ?

11. If Rs. x is the interest on Rs. x^6 for 1 year, on what sum will Re. 1 be the interest for 1 year ?

12. A man has Rs. x. He spends Rs. y and distributes the rest equally among his three sons. What does each son get ?

13. Kailash has a apples. He gives b apples to Prem and distributes the rest equally among 35 friends. How many does each friend get ?

14. A man has Rs. x. He spends Rs. y in buying tea and the rest in buying z bags of cotton. What is the price of each bag of cotton ?

15. In a cricket match Vidya played a overs and scored equal runs in each over. If he scored $(x+y)$ runs in all, how many runs did he score in each over ?

16. A man has $(x+y+z)$ rupees. He spends this sum in purchasing a books. What is the price of each book ?

17. A man has £x^2. He spends £$2x$ in purchasing one cow and the rest in purchasing x goats. What does he pay for each goat ?

18. Rahim has $5x^3$ pencils. He leaves x pencils at home and distributes the rest equally among x friends. How many pencils does each friend get ?

19. A man spends y dozen rupees in buying goats in one place and $6y^2$ rupees in buying goats in another place. If the price of the goats is the same and the number of goats bought is $6y$, what is the price of one goat ?

20. A man plucks x^3 mangoes from one tree, $2x^2$ mangoes from a second and $3x$ from a third and distributes them equally among $2x$ boys. How many mangoes does each boy get ?

MISCELLANEOUS EXAMPLES I

A

1. Three numbers x, y and z are to be added together. What is their sum ?

2. What number is greater than 6 by 9 ? What number is greater than p by q ?

3. What number is less than 15 by 6 ? What number is less than p by q ?

4. A number is represented by a. Write down in Algebra, 'twice the number', 'four times the number', 'four times the number increased by five', and 'four times the number diminished by five'.

5. If $a=0$, what are the values of $a\times 2$, $a\times b$ and ab ?

6. If $x=1$, $y=2$ and $z=3$, find the value of

(i) $x+y+z$, $x+y-z$, $x-y+z$, $x-y-z$.

(ii) $xy+yz+zx$, $xy+yz-zx$, $xy-yz+zx$, $xy-yz-zx$.

(iii) $\frac{x+y}{z}$, $\frac{x+z}{y}$, $\frac{y+z}{x}$, $\frac{x-y}{z}$, $\frac{x-z}{y}$, $\frac{y-z}{x}$.

(iv) xyz, $\frac{1}{xyz}$, $\frac{xy}{z}$, $\frac{yz}{x}$, $\frac{zx}{y}$.

B

1. Write down in as many ways as possible the sum of a, b and c.

2. What are the meanings of $-x$, when $+x$ has the following meanings?

(i) $+x$ means 'Rama goes x miles east.'

(ii) $+x$ means 'Rama has x rupees.'

(iii) $+x$ means 'Rama owes x rupees.'

3. Suppose a represents 12 and b represents 4, what are the values of (i) $a+b$, (ii) $a-b$, (iii) ab, and (iv) $\frac{a}{b}$?

4. Find the value of

(i) $a(-b)$, (ii) $-a(-b)$, (iii) $\frac{a}{-b}$, (iv) $\frac{-a}{-b}$, (v) $\frac{-a}{b}$,

when $a=2$ and $b=3$.

5. A boy has x pens; he buys y more and sells z. How many has he left?

6. A boy has x pens which cost 2 annas each and y pens which cost 3 annas each. How many pens has he? What do they cost?

C

1. Distinguish between *like* and *unlike terms*. Pick out the like terms in the expression

$$x^2+2xy+y^2-3x^2-2y^2+5xy-xy+4x^2.$$

2. Explain clearly why $a-(b+c)=a-b-c$?

3. If $x=5$ and $y=3x$, what is the value of y?

4. By how much does a exceed $-b$?

5. By how much does $-b$ exceed a?

6. If the side of a square is x ft., what is its perimeter?

D

1. Distinguish between *co-efficient* and *index*. Express 5 as (i) the co-efficient of x, (ii) the index of x. What is the value of each when $x=2$?

2. What is the difference between $2a$ and a^2 when $a=5$?

3. Write in a shorter way $aaaaa$ and $3xxxxxxx$.

4. Write in full, without indices, x^3 and x^5. How many x's are there when x^3 and x^5 are multiplied together?

5. What is the value of 1^{20}?

6. What number is equal to a^b when $a=2$ and $b=3$?

E

1. What number must be subtracted from a that the result may be $a+b$?

2. What is the value of a° ?

3. What is the product when $\frac{a}{b}$ is multiplied by b ?

4. What is the quotient when (i) x is divided by y, (ii) when x is divided by x ?

5. Find the quotients in the following :

(i) $(xy+x^2y)\div x$.

(ii) $(x^3y-x^2)\div x$.

(iii) $(ax+bx-cx)\div x$.

(iv) $(2x^3y+4x^2y^2-6xy^3)\div 2xy$.

6. If a rectangle is l inches long and b inches broad, find its area.

F

1. Show that $a(b+c)=ab+ac$, by taking

(i) $a=1$, $b=2$, $c=3$. (ii) $a=2$, $b=-3$, $c=4$.

2. What is the value of $(x-1)^3$ when

(i) $x=0$, (ii) $x=1$, (iii) $x=-1$?

3. Suppose $+x$ means the journey from Jubbulpore to Allahabad, what does $-x$ mean ? What does $-(-x)$ mean ?

4. The sides of a triangle measure a inches, b inches and c inches. What is its perimeter ?

5. The top of a table is l feet long and b feet broad. If an ant crawls alround the table, how far has it travelled ?

6. Differentiate between the following expressions :

(i) $a+b\times c+d$. (ii) $(a+b)\times c+d$.

(iii) $a+b\times(c+d)$. (iv) $(a+b)\times(c+d)$.

Find the value of each expression when $a=1$, $b=2$, $c=3$ and $d=4$.

G

1. A boy has a pen, a book and a pencil. Can he write his possessions in one figure ?

2. From the square of a take the square of b, and subtract $ab+b^2$ from the result.

3. Express algebraically the product of the sum of a and b by their difference.

4. Draw on squared paper a figure similar to the second figure given on page 26. Take 1 in = 10 units. Put your pencil point on O and make the journeys $+12$ and then -5, one after the other without taking the pencil off the paper. Where is your pencil point at the end of the journey ?

5. If a train goes 120 miles in 4 hours, how far does it go (i) in 1 hour, and (ii) in 3 hours at the same rate ?

If a train goes x miles in h hours, how far does it go (i) in 1 hour and (ii) in y hours at the same rate ?

6. There are 6 rooms in a house. Each room contains 3 almirahs, and each almirah contains 40 books. How many books are there in the house ?

There are x rooms in a house. Each room contains y almirahs, and each almirah contains z books. How many books are there in the house ?

H

1. Distinguish between x^4 and $4x$. What is the value of each when $x=3$?

2. Express in words the difference in the meanings of $(x+y)^2$ and x^2+y^2.

3. I had x annas and lost y of them. How many pies had I left ? How many rupees ?

4. If '$+5$' means go 5 miles east, what does '-5' mean ? What does $2\times(+5)$ or $2(+5)$ mean ? What does $2(-5)$ mean ?

5. In the expressions $3x^3+2x^4$, x^3+5x^2, $x+3x^3$, find (i) the sum of the indices, (ii) the sum of the coefficients.

6. A square room measures x feet each way. How many yards of carpet $\frac{1}{2}x$ feet wide will be necessary to cover it ?

I

1. In the following addition sums, first add the expressions and put the result in the space at the bottom. If $a=1$, $b=2$ and $c=3$, write the value of each expression in the space to the right of it. Add these values and write their sum in the space marked S (for sum). Then put the values of a, b and c in your answer and check your results.

(i)

$a+b+c$	
$2a+3b+4c$	
$5a+7b+2c$	
	S

(ii)

$a+2b+3c$	
$4a+5b$	
$6b+7c$	
$8a+9b+10c$	
	S

(iii)

$5a+b-2c$	
$a-3b+4c$	
$2a-4b-5c$	
$3a+2b+c$	
	S

(iv)

$7a+8b+3c$	
$5b-12c$	
$17a-14b$	
$13b-10c$	
	S

2. Work out the following subtraction sums and put the difference in the space at the bottom. If $a=4$, $b=2$ and $c=5$, write the value of each expression in the space to the right of it. Subtract the value of the lower expression from the upper and write their difference in the space marked D (for difference). Then put the values of a, b and c in your answer and check your results.

(i)

$2a+5b+8c$	
$a+2b+3c$	
	D

(ii)

$7a+11b+9c$	
$10b+8c$	
	D

(iii)

$a+b+c$	
$a+2b+3c$	
	D

(iv)

$4a+6b-9c$	
$3a-5b-c$	
	D

3. Add together $\frac{1}{2}(2a-3b+2c)$, $a-\frac{1}{2}(b-2c)$ and $\frac{1}{3}a+b+\frac{1}{6}c$, and take from the result $\frac{1}{3}a-2b-c$.

4. Multiply $x^3+2a^2x+a^3$ by $x^3-2a^2x+a^3$.

5. Divide x^4+4y^4 by $x^2-2xy+2y^2$.

6. Use squared paper to illustrate the following :

(i) $8-5=3$. (ii) $8-5-6=-3$.

J

1. Write down $(2x+3y+4z)$ three times in the form of an addition sum. Show by addition, that

$$3(2x+3y+4z)=(3\times 2x)+(3\times 3y)+(3\times 4z).$$

2. Add the product $(a+b+1)(a-b+2)$ to the product $(a+b-1)(a-b-2)$, and verify the result when $a=5$, $b=2$.

3. Multiply $1+3x-x^3-x^4$ by $1-x+2x^2$.

4. Divide $2x^4+2x^3-11x^2+13x-3$ by x^2+3x-1.

5. Simplify $2a-3\{b-c(2+a)\}$ and $3c(a-1)-(b-2a)$, and subtract the first expression from the second.

6. A man walks m miles in h hours. How many miles does he walk in one hour ? How many minutes does he take to walk one mile ?

CHAPTER VIII

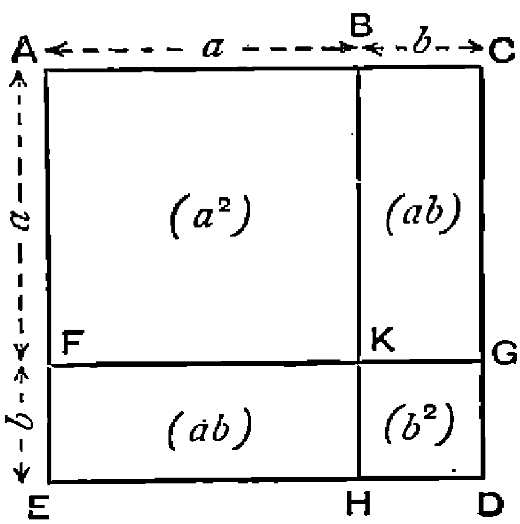
SOME IMPORTANT FORMULÆ

43. We have seen that

$$\begin{aligned}(a+b)^2 &= (a+b)\,(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

This can also be illustrated by means of a diagram.

Let AB be a straight line equal to a units in length. Produce AB to C, making BC equal to b units in length. Now AC contains $(a+b)$ units in length.



On AC and AB draw the squares ACDE and ABKF. Produce BK and FK to meet ED and CD in H and G.

Now from the figure it is clear that

sq. ACDE = sq. ABKF + rect. BCGK + rect. FKHE + sq. KGDH.

$$\begin{aligned}(a+b)^2 &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

Hence, we see that *the square of the sum of two quantities is equal to the sum of their squares plus twice their product.*

44. Since $(a+b)^2 = a^2 + 2ab + b^2$,
writing 3 for b, we have

$$(a+3)^2 = a^2 + 2.a.3 + 3^2$$
$$= a^2 + 6a + 9.$$

Similarly
$$(x+3y)^2 = (x)^2 + 2(x)(3y) + (3y)^2$$
$$= x^2 + 6xy + 9y^2 \;;$$

and
$$(2a+5b)^2 = (2a)^2 + 2(2a)(5b) + (5b)^2$$
$$= 4a^2 + 20ab + 25b^2.$$

EXAMPLE 1. *Show, on squared paper, that*

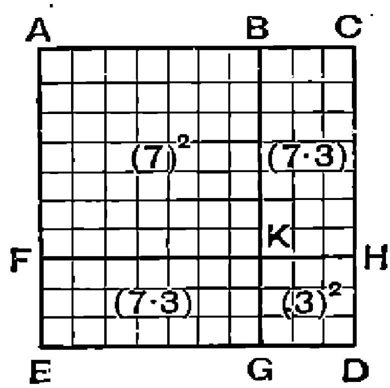
$$(7+3)^2 = 7^2 + 2.7.3 + 3^2.$$

Suppose a square ACDE is drawn on a straight line AC equal to (7+3) units in length. Cut off AB from AC equal to 7 units, then BC will be equal to 3 units in length.

On AB draw a square ABKF.

Produce BK and FK to meet ED and CD in G and H respectively.

Now from the figure it is clear that



sq. ACDE = sq. ABKF + rect. BCHK + rect. FKGE + sq. KHDG.

$$\therefore \quad (7+3)^2 = 7^2 + 7.3 + 7.3 + 3^2$$
$$= 7^2 + 2.7.3 + 3^2.$$

EXAMPLE 2. *Without actual multiplication, find the value of* 1001^2.

$$1001^2 = (1000+1)^2$$
$$= 1000^2 + 2.1000.1 + 1^2$$
$$= 1{,}000{,}000 + 2{,}000 + 1$$
$$= 1{,}002{,}001.$$

EXAMPLE 3. *Without actual multiplication, find the value of* $100{\cdot}5^2$.

$$100{\cdot}5^2 = (100 + {\cdot}5)^2$$
$$= 100^2 + 2 \times 100 \times {\cdot}5 + ({\cdot}5)^2$$
$$= 10{,}000 + 100 + {\cdot}25$$
$$= 10{,}100{\cdot}25.$$

EXAMPLES XXIII

Remove the brackets (*Oral*)

1. $(x+y)^2$.	**2.** $(a+x)^2$.	**3.** $(l+m)^2$.
4. $(x+5)^2$.	**5.** $(3+a)^2$.	**6.** $(p+7)^2$.
7. $(5+c)^2$.	**8.** $(2x+3)^2$.	**9.** $(3x+2)^2$.
10. $(3p+5)^2$.	**11.** $(1+3x)^2$.	**12.** $(5x+2y)^2$.
13. $(6a+5b)^2$.	**14.** $(7l+2m)^2$.	**15.** $(4p+3q)^2$.
16. $(a^2+1)^2$.	**17.** $(a^2+b^2)^2$.	**18.** $(p^2+q)^2$.
19. $(2x^3+y^3)^2$.	**20.** $(3x^3+2y^3)^2$.	

On squared paper show that

21. $10^2=(8+2)^2=8^2+2.8.2+2^2$.

22. $10^2=(6+4)^2=6^2+2.6.4+4^2$.

23. $12^2=(9+3)^2=9^2+2.9.3+3^2$.

24. $15^2=(10+5)^2=10^2+2.10.5+5^2$.

25. $18^2=(6+12)^2=6^2+2.6.12+12^2$.

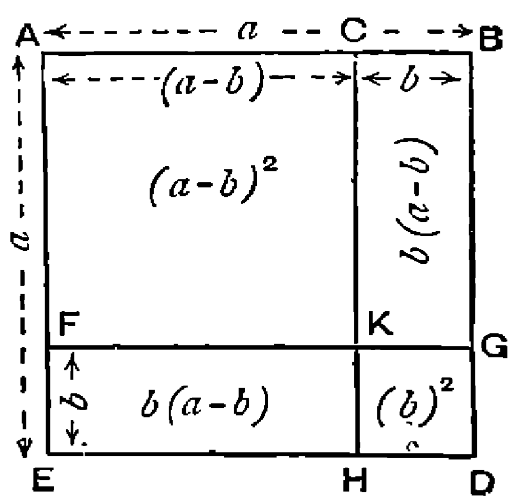
Without actual multiplication, find the value of

26. 102^2.	**27.** 201^2.	**28.** 105^2.
29. 203^2.	**30.** 1001^2.	**31.** 1002^2.
32. 10003^2.	**33.** 20010^2.	**34.** $100{\cdot}3^2$
35. $50{\cdot}2^2$.	**36.** $800{\cdot}3^2$.	

45. We know that

$$(a-b)^2=(a-b)\,(a-b)$$
$$=a^2-ab-ab+b^2$$
$$=a^2-2ab+b^2.$$

This can also be illustrated by means of a diagram.



Let AB be a straight line equal to a units in length. From BA cut off BC equal to b units in length. Now AC contains $(a-b)$ units in length.

On AC and AB draw the squares ACKF and ABDE. Produce CK and FK to meet ED and BD in H and G.

Now, from the figure it is clear that

sq. ACKF = sq. ABDE − sq. KGDH − rect. CBGK − rect. FKHE.

$$\therefore \quad (a-b)^2 = a^2 - b^2 - b(a-b) - b(a-b)$$
$$= a^2 - b^2 - ab + b^2 - ab + b^2$$
$$= a^2 - 2ab + b^2.$$

Hence, we see that *the square of the difference of two quantities is equal to the sum of their squares minus twice their product.*

46. Since $(a-b)^2 = a^2 - 2ab + b^2$,

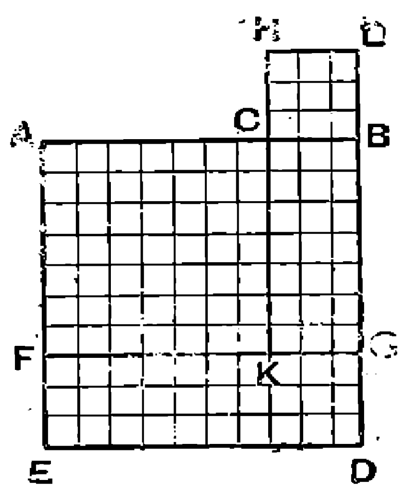
$$\therefore \quad (a-3)^2 = a^2 - 2.a.3 + 3^2$$
$$= a^2 - 6a + 9.$$

Similarly $(x-3y)^2 = (x)^2 - 2(x)(3y) + (3y)^2$
$$= x^2 - 6xy + 9y^2.$$

and $(2a-5b)^2 = (2a)^2 - 2(2a)(5b) + (5b)^2$
$$= 4a^2 - 20ab + 25b^2.$$

EXAMPLE 1. *Show, on squared paper, that*
$$(10-3)^2 = 10^2 - 2.10.3 + 3^2.$$

Suppose a straight line AB is drawn equal to 10 units in length. From BA cut off BC equal to 3 units. On AB and BC draw squares ABDE and BCHL on opposite sides. From EA cut off EF equal to 3 units. Draw FG parallel to ED. Produce HC to meet FG in K.



Now from the figure it is clear that

sq. ACKF = sq. ABDE − rect. FGDE − rect. HKGL + sq. CBLH.

$$\therefore \quad (10-3)^2 = 10^2 - 10.3 - 10.3 + 3^2$$
$$= 10^2 - 2.10.3 + 3^2.$$

EXAMPLE 2. *Without actual multiplication, find the value of* 999^2.

$$999^2 = (1000-1)^2$$
$$= 1000^2 - 2.1000.1 + 1.$$
$$= 1,000,000 - 2,000 + 1$$
$$= 998,001.$$

EXAMPLE 3. *Without actual multiplication, find the value of* $99{\cdot}5^2$.

$$\begin{aligned}99{\cdot}5^2 &= (100 - {\cdot}5)^2\\ &= 100^2 - 2 \times 100 \times {\cdot}5 + ({\cdot}5)^2\\ &= 10{,}000 - 100 + {\cdot}25\\ &= 9{,}900{\cdot}25.\end{aligned}$$

EXAMPLES XXIV

Remove the brackets (*Oral*)

1. $(x-y)^2$.
2. $(x-a)^2$.
3. $(l-m)^2$.
4. $(x-5)^2$.
5. $(p-6)^2$.
6. $(5-b)^2$.
7. $(2x-3)^2$.
8. $(3a-2)^2$.
9. $(5p-3)^2$.
10. $(1-2x)^2$.
11. $(2x-3y)^2$.
12. $(5x-2y)^2$.
13. $(6a-5b)^2$.
14. $(7l-2m)^2$.
15. $(4k-3l)^2$.
16. $(x^2-1)^2$.
17. $(x^2-y^2)^2$.
18. $(x-y^2)^2$.
19. $(3x^2-4y^2)^2$.
20. $(2x^3-y^3)^2$.
21. $(5x^3-2y^3)^2$.
22. $(7a^3-3b^3)^2$.

On squared paper show that

23. $10^2=(12-2)^2=12^2-2.12.2+2^2$.
24. $10^2=(15-5)^2=15^2-2.15.5+5^2$.
25. $9^2=(12-3)^2=12^2-2.12.3+3^2$.
26. $12^2=(18-6)^2=18^2-2.18.6+6^2$.
27. $8^2=(11-3)^2=11^2-2.11.3+3^2$.

Without actual multiplication, find the value of

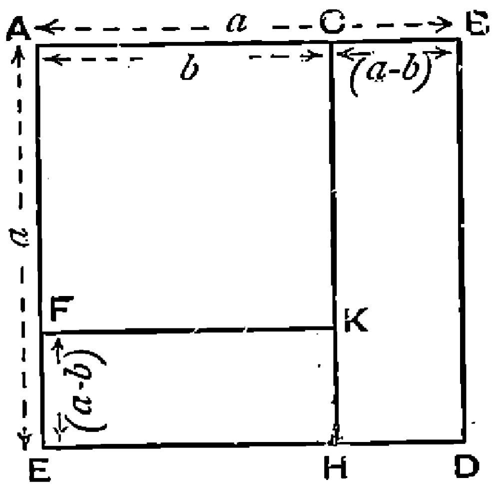
28. 98^2.
29. 999^2.
30. 499^2.
31. 9999^2.
32. $9{\cdot}9^2$.
33. $999{\cdot}8^2$.
34. $899{\cdot}6^2$
35. $99{\cdot}97^2$.

47. We know that

$$(a+b)(a-b) = (a+b)x \qquad \text{[putting } x \text{ for } (a-b)]$$
$$= ax + bx$$
$$= a(a-b) + b(a-b) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$= a^2 - ab + ab - b^2$$
$$= a^2 - b^2.$$

This can be illustrated by means of a diagram.

Let AB be a straight line equal to a units in length. From AB cut off AC equal to b units. Now CB contains $(a-b)$ units in length.



On AB and AC draw squares ABDE and ACKF. Produce CK to meet ED in H.

Now from the figure it is clear that

sq. ABDE − sq. ACKF = rect. CBDH + rect. FKHE.

$$\therefore \quad a^2 - b^2 = a(a-b) + b(a-b)$$

$$= (a+b)(a-b). \qquad \text{[from (1)]}$$

i.e., $(a+b)(a-b) = a^2 - b^2$.

Hence, we see that *the difference of the squares of two quantities is equal to the product of their sum and difference.*

48. Since $(a+b)(a-b) = a^2 - b^2$,

$\therefore \quad (a+3)(a-3) = a^2 - 3^2 = a^2 - 9.$

Similarly $(a+3b)(a-3b) = (a)^2 - (3b)^2 = a^2 - 9b^2$,

and $(2x+3y)(2x-3y) = (2x)^2 - (3y)^2 = 4x^2 - 9y^2$.

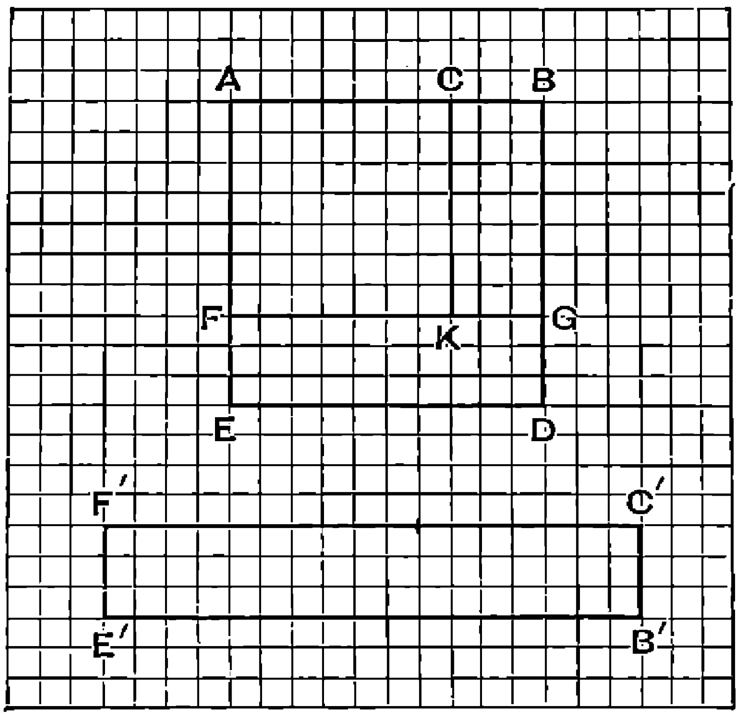
EXAMPLE 1. *Show, on squared paper, that*

$$10^2 - 7^2 = (10+7)(10-7).$$

Suppose AB is a straight line 10 units in length. From AB cut off AC equal to 7 units. On AB and AC draw squares ABDE and ACKF. Produce FK to meet BD in G.

Now, from the figure it is clear that

$$10^2 - 7^2 = \text{sq. } ABDE - \text{sq. } ACKF$$
$$= \text{rect. } FGDE + \text{rect. } KGBC.$$



Since the breadth of both the rectangles is $(10-7)$ units, these two rectangles can be joined into one by placing their ends together, and the length of this new rectangle will be $(10+7)$ units and breadth $(10-7)$ units. Hence, its area will contain $(10+7)(10-7)$ sq. units.

$$10^2 - 7^2 = (10+7)(10-7).$$

EXAMPLE 2. *Without actual multiplication, find the value of* $1000^2 - 999^2$.

$$1000^2 - 999^2 = (1000+999)(1000-999)$$
$$= 1999 \times 1$$
$$= 1999.$$

EXAMPLE 3. *Without actual multiplication, find the value of* 101×99.

$$101 \times 99 = (100+1)(100-1)$$
$$= (100)^2 - (1)^2$$
$$= 10000 - 1$$
$$= 9999.$$

EXAMPLE 4. *Find the value of* $99{\cdot}6 \times 100{\cdot}4$.

$$\begin{aligned}99{\cdot}6 \times 100{\cdot}4 &= (100 - {\cdot}4)(100 + {\cdot}4)\\ &= (100)^2 - ({\cdot}4)^2\\ &= 10000 - {\cdot}16\\ &= 9999{\cdot}84.\end{aligned}$$

EXAMPLES XXV

Multiply (*orally*)

1. $(x+1)(x-1)$. **2.** $(x-2)(x+2)$. **3.** $(1+a)(1-a)$.

4. $(a+4)(a-4)$. **5.** $(5-y)(5+y)$. **6.** $(7-a)(7+a)$.

7. $(x+y)(x-y)$. **8.** $(p-q)(p+q)$. **9.** $(a-3b)(a+3b)$.

10. $(3l+2m)(3l-2m)$. **11.** $(5a+6x)(5a-6x)$.

12. $(-a-b)(-a+b)$. **13.** $(-x+2a)(-x-2a)$.

14. $(a^2+b^2)(a^2-b^2)$. **15.** $(x^2+3y^2)(x^2-3y^2)$.

16. $(lx-m)(lx+m)$. **17.** $(-x^2-y)(-x^2+y)$.

18. $(1-a^3)(1+a^3)$. **19.** $(1+ax^2)(1-ax^2)$.

20. $(9-8x^2)(9+8x^2)$.

Show, on squared paper, that

21. $10^2-8^2=(10+8)(10-8)$. **22.** $12^2-5^2=(12+5)(12-5)$.

23. $15^2-10^2=(15+10)(15-10)$. **24.** $20^2-18^2=(20+18)(20-18)$.

25. $25^2-20^2=(25+20)(25-20)$.

Without actual multiplication, find the value of

26. 31^2-29^2. **27.** 100^2-98^2. **28.** 1005^2-995^2.

29. $100{\cdot}7^2-99{\cdot}3^2$. **30.** 1002×998. **31.** 203×197.

32. 83×77. **33.** $11{\cdot}5\times10{\cdot}5$. **34.** $20{\cdot}04\times19{\cdot}96$.

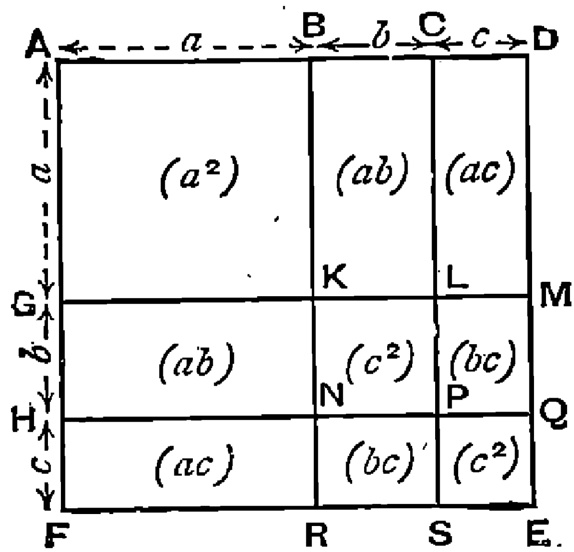
35. $1{\cdot}96\times2{\cdot}04$.

49. We know that

$$\begin{aligned}(a+b+c)^2 &= (a+b+c)(a+b+c)\\ &= a^2+ab+ac+ab+b^2+bc+ca+cb+c^2\\ &= a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

This can be illustrated by means of a diagram.

Let ADEF be a square drawn on a straight line AD equal to $(a+b+c)$ units in length.



From AD and AF cut off AB and AG equal to a units. Again from BD and GF cut off BC and GH equal to b units. Then CD and HF each will contain c units of length.

From B and C draw BR and CS parallel to AF meeting FE in R and S. Again from G and H draw GM and HQ parallel to AD meeting BR, CS and DE in points K, N ; L, P and M, Q respectively.

Now from the figure it is clear that

sq. ADEF = sq. ABKG + sq. KLPN + sq. PQES + rect. BCLK + rect. CDML + rect. LMQP + rect. GKNH + rect. HNRF + rect. NPSR.

$$(a+b+c)^2 = a^2+b^2+c^2+ab+ac+bc+ab+ac+bc$$
$$= a^2+b^2+c^2+2ab+2ac+2bc.$$

Hence, we see that *the square of the sum of three quantities is equal to the sum of their squares plus twice the sum of the products of two quantities at a time.*

50. Since $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$,

$\therefore \quad (x+y+2z)^2$
$$= (x)^2+(y)^2+(2z)^2+2(x)(y)+2(x)(2z)+2(y)(2z)$$
$$= x^2+y^2+4z^2+2xy+4xz+4yz.$$

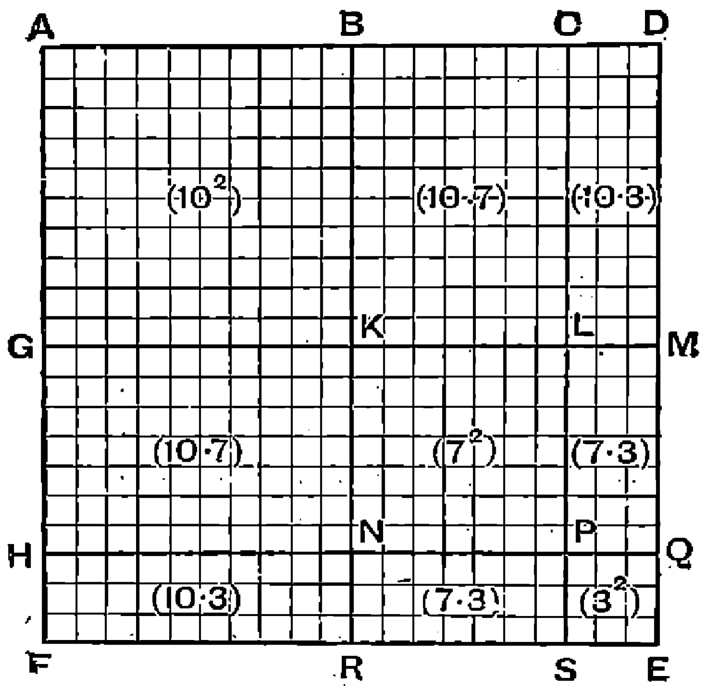
Similarly

$(a+2b+3c)^2$

$$= (a)^2 + (2b)^2 + (3c)^2 + 2(a)(2b) + 2(2b)(3c) + 2(a)(3c)$$
$$= a^2 + 4b^2 + 9c^2 + 4ab + 12bc + 6ac.$$

EXAMPLE. *Show, on squared paper, that*

$$(10+7+3)^2 = 10^2 + 7^2 + 3^2 + 2.10.7 + 2.10.3 + 2.7.3.$$



Take a straight line AD equal to 20 units in length and describe a square ADEF on it. Take two points B and C in AD, and G and H in AF such that

AB = AG = 10 units,
BC = GH = 7 units,
and CD = HF = 3 units.

Through B and C draw two straight lines BR and CS parallel to AF and through G and H draw two straight lines GM and HQ parallel to AD.

Now it is clear from the figure that

sq. ADEF $= (10+7+3)^2$

$$= 10^2 + 7^2 + 3^2 + 10.7 + 10.3 + 7.3 + 10.7 + 10.3 + 7.3$$
$$= 10^2 + 7^2 + 3^2 + 2.10.7 + 2.10.3 + 2.7.3.$$

51. Since $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$,

$$\therefore\ [a-b+c]^2 = [a+(-b)+c]^2$$
$$= a^2+(-b)^2+c^2+2a(-b)+2ac+2(-b)c$$
$$= a^2+b^2+c^2-2ab+2ac-2bc.$$

Similarly

$$[a-b-2c]^2 = [a+(-b)+(-2c)]^2$$
$$= a^2+(-b)^2+(-2c)^2+2a(-b)+2a(-2c)+2(-b)(-2c)$$
$$= a^2+b^2+4c^2-2ab-4ac+4bc.$$

EXAMPLES XXVI

Without actual multiplication, remove the brackets (*orally*).

1. $(x+y+z)^2$. **2.** $(x+2y+z)^2$. **3.** $(x+2y+4z)^2$.
4. $(1+2x+3y)^2$. **5.** $(2+3x+5a)^2$. **6.** $(2l+3m+7n)^2$.
7. $(x-2y+z)^2$. **8.** $(x+2y-3z)^2$. **9.** $(2a-b-4c)^2$.
10. $(3a-2x-1)^2$. **11.** $(a^2+b^2+c^2)^2$. **12.** $(ax+y^2-z^2)^2$.
13. $(ax+by+cz)^2$. **14.** $(2l+mn-3p)^2$.

Show, on squared paper, that

15. $(9+7+2)^2 = 9^2+7^2+2^2+2.9.7+2.9.2+2.7.2$.
16. $(11+6+4)^2 = 11^2+6^2+4^2+2.11.6+2.11.4+2.6.4$.

CHAPTER IX

EVOLUTION AND SUBSTITUTION

Evolution

52. The **square root** of a quantity is that quantity whose square or second power is equal to the original quantity.

Thus the square root of 16 is 4 or -4,

$$\text{for } 4^2 = 16,$$

and $(-4)^2 = (-4) \times (-4) = 16.$

Similarly, the square root of $36a^2$ is $+6a$ or $-6a$,

for $(+6a)^2 = (+6a) \times (+6a) = 36a^2$,

and $(-6a)^2 = (-6a) \times (-6a) = 36a^2$.

NOTE. For the present we shall deal with the positive values only.

The sign '$\sqrt{\ }$' is called the **root sign**. The square root is denoted by $\sqrt[2]{\ }$ or simply $\sqrt{\ }$.

53. The **cube root** of a quantity is that quantity whose cube or third power is equal to the original quantity.

Thus the cube root of 8 is $+2$,

$$\text{for } (+2) \times (+2) \times (+2) = 8.$$

The cube root of 8 is *not* -2,

$$\text{for } (-2) \times (-2) \times (-2) = -8 \text{ and } \textit{not } 8.$$

Similarly, the cube root of $-125x^3$ is $-5x$,

$$\text{for } (-5x)(-5x)(-5x) = -125x^3.$$

The cube root is denoted by $\sqrt[3]{\ }$.

Similarly, the **fourth root** of a quantity is that quantity whose fourth power is equal to the original quantity, the **fifth root** of a quantity is that quantity whose fifth power is equal to the original quantity, and so on.

The fourth root, fifth root,...are denoted by $\sqrt[4]{\ }$, $\sqrt[5]{\ }$,...

The process of finding a root of a quantity is called **Evolution**.

EXAMPLE 1. *Find the value of* $\sqrt{225}$.

$$\sqrt{225}=\sqrt{3\times3\times5\times5}=3\times5=15.$$

NOTE. The square root of $3\times3\times5\times5$ which can be written in the form of $\sqrt{(3\times3\times5\times5)}$ is often written as $\sqrt{3\times3\times5\times5}$, the line above used as a vinculum is equivalent to a bracket.

EXAMPLE 2. *Find the square root of* $9x^2$.

Since $9x^2=3\times3\times x\times x$,

$$\therefore \qquad \sqrt{9x^2}=3\times x=3x.$$

Hence, the square root of $9x^2$ is $3x$.

EXAMPLE 3. *Simplify* $\sqrt[3]{27a^3x^6}$.

$$\sqrt[3]{27a^3x^6}=\sqrt[3]{3.3.3.a.a.a.x.x.x.x.x.x}=3ax^2.$$

EXAMPLE 4. *Find the value of* $\sqrt[3]{-8p^3}$.

$$\sqrt[3]{-8p^3}=\sqrt[3]{(-2).(-2).(-2).p.p.p}=-2p.$$

From the above examples it is clear that *to find the root of a power of a quantity we must divide the index of the original power by the index of the root.*

EXAMPLE 5. *Simplify* $\sqrt[5]{32x^5y^{10}}$.

$$\sqrt[5]{32x^5y^{10}}=\sqrt[5]{2^5.x^5.y^{10}}=2^{\frac{5}{5}}.x^{\frac{5}{5}}.y^{\frac{10}{5}}=2xy^2.$$

EXAMPLE 6. *Simplify* $\sqrt[5]{-\dfrac{32a^{10}b^5}{243c^{15}}}$.

$$\sqrt[5]{-\frac{32a^{10}b^5}{243c^{15}}}=\sqrt[5]{\frac{(-2)^5a^{10}b^5}{3^5c^{15}}}=\frac{(-2)a^2b}{3c^3}=-\frac{2a^2b}{3c^3}.$$

From the above examples we see that

(i) *The root of a product of a number of quantities is equal to the product of the roots of the factors of the quantities.*

(ii) *The root of a fraction is obtained by taking the roots of the numerator and denominator separately.*

(iii) *An odd root of a positive quantity is positive and an odd root of a negative quantity is negative.*

EXAMPLES XXVII

(Examples 1—30 may be taken orally)

Find the square root of

1.	a^2.	2.	a^6.	3.	$\frac{9}{16}$.	4.	$\frac{9}{4}x^4$.
5.	a^4b^2.	6.	$4a^2x^4y^2$.	7.	$9x^2y^6$.	8.	$25c^2a^8$.
9.	$a^{24}b^{16}c^8$.	10.	$16a^6y^{20}$.	11.	$81y^{30}$.	12.	$\frac{16}{25}a^4$.
13.	$\frac{4}{9}a^4b^2c^4$.	14.	$\cdot25x^4y^4z^4$.	15.	$\frac{169a^4}{25b^2}$.	16.	$100x^8y^6z^4$.
17.	$\frac{144p^2q^4}{121r^6}$.	18.	$\frac{49l^2m^4n^6}{196p^6q^4r^2}$.				

Find the cube root of

19.	a^3.	20.	x^6.	21.	a^3b^9.	22.	$8x^{12}$.
23.	$-b^6$.	24.	$64a^3b^{12}$.	25.	$-27a^9b^{12}$.	26.	$-125a^3b^3$.
27.	$\frac{8l^3}{27m^3}$.	28.	$-\frac{27p^3q^6}{64r^9}$.	29.	$\frac{125x^6}{216y^3z^9}$.	30.	$-\frac{216a^{15}}{343p^9q^{18}}$.

Find the value of

31.	$\sqrt{144b^2c^4}$.	32.	$\sqrt{49x^4y^6z^2}$.	33.	$\sqrt[3]{125a^3b^9}$.
34.	$\sqrt[3]{-343a^{12}b^{15}}$.	35.	$\sqrt[3]{-64l^3m^3}$.	36.	$\sqrt[4]{a^4b^8}$.
37.	$\sqrt[4]{\frac{x^4y^8}{81}}$.	38.	$\sqrt[5]{32x^{10}}$.	39.	$\sqrt[5]{x^{10}y^{15}}$.
40.	$\sqrt[6]{64b^6a^{12}}$.				

41. The area of a square is x^2 square feet, what is the length of its side?

42. The area of a square is $49x^2y^2$ square yards, what is the length of its side?

43. The length and breadth of a rectangle are $9x$ yards and $4x$ yards respectively. Find the length of a side of a square whose area is equal to the area of the rectangle.

44. The length and breadth of a rectangle are $16x^2$ yards and y^2 yards respectively. Find the length of a side of a square whose area is equal to the area of the rectangle.

Substitution

54. EXAMPLE 1. *Find the value of* $\sqrt[3]{4a}$ *when* $a=2$.

$$\sqrt[3]{4a}=\sqrt[3]{4\times2}=\sqrt[3]{2\times2\times2}=2.$$

EXAMPLE 2. *Find the value of* $\sqrt{3ab}$ *when* a=2 *and* b=6.

$$\sqrt{3ab}=\sqrt{3\times2\times6}=\sqrt{3\times2\times3\times2}=3\times2=6.$$

EXAMPLE 3. *Find the value of* $\sqrt[3]{a^2b^2c^2}$ *when* a=−4, b=2 *and* c=−1.

$$\sqrt[3]{a^2b^2c^2}=\sqrt[3]{(-4)^2(2)^2(-1)^2}=\sqrt[3]{16\times4\times1}=\sqrt[3]{4\times4\times4}=4.$$

EXAMPLE 4. *Find the value of* $\sqrt[3]{\frac{x}{2z}}-\sqrt[4]{\frac{3y}{4z}}+\sqrt{\frac{x}{4z}}$ *when* x=16, y=27 *and* z=64.

$$\sqrt[3]{\frac{x}{2z}}-\sqrt[4]{\frac{3y}{4z}}+\sqrt{\frac{x}{4z}}=\sqrt[3]{\frac{16}{2\times64}}-\sqrt[4]{\frac{3\times27}{4\times64}}+\sqrt{\frac{16}{4\times64}}$$

$$=\sqrt[3]{\frac{1}{2\times2\times2}}-\sqrt[4]{\frac{3\times3\times3\times3}{4\times4\times4\times4}}+\sqrt{\frac{1}{4\times4}}=\frac{1}{2}-\frac{3}{4}+\frac{1}{4}=0.$$

EXAMPLE 5. *Find the value of* $4\sqrt{a^2f}-3\sqrt{b^4c^2}+5\sqrt{d^2}$ *when* a=4, b=−3, c=−1, d=0 *and* f=4.

$$\text{The expression}=4\sqrt{(4)^2.4}-3\sqrt{(-3)^4.(-1)^2}+5\sqrt{(0)^2}$$
$$=4\sqrt{16.4}-3\sqrt{81}+0$$
$$=4\times8-3\times9$$
$$=32-27$$
$$=5.$$

EXAMPLES XXVIII

If $a=1$, $b=2$ and $c=3$, find the value of

1. $\sqrt{a}$. 2. $\sqrt{2b}$. 3. $\sqrt{3c}$. 4. $\sqrt[3]{4b}$.
5. $\sqrt[3]{a}$. 6. $\sqrt{6bc}$. 7. $\sqrt[3]{4ab}$. 8. $a\sqrt{12c}$.
9. $\sqrt[3]{2b^2}$. 10. $\sqrt[4]{6b^3c^3}$.

If $a=9$, $b=4$, $c=1$ and $d=25$, find the value of

11. $\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$. 12. $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{d}}$.

13. $\sqrt{abc}+\sqrt{bcd}+\sqrt{acd}+\sqrt{abd}$.

14. $\sqrt[3]{2bc}+\sqrt[3]{15ad}+\sqrt[3]{6ac}$.

If $w=-1$, $x=-4$, $y=-3$ and $z=2$, find the value of

15. $\sqrt{wxz^2}$. 16. $\sqrt{9w^3x}$. 17. $\sqrt{wx^3z^2}$.
18. $\sqrt{w^4y^2z^2}$. 19. $w.\sqrt{x^2y^2z^4}$. 20. $\sqrt[3]{6xy^2}$.

21. $\sqrt[3]{27x^3y^3}$. 22. $-y\sqrt[3]{x^4z^4}$. 23. $\sqrt[3]{2w^2z}$.

24. $\sqrt[3]{wx^3y^3}$.

If $l=1$, $m=0$, $n=-1$ and $p=2$, find the value of

25. $\sqrt{3(l+p)}$. 26. $\sqrt{l+m+n}$. 27. $\sqrt{5(l^2+p^2)}$.

28. $\sqrt[4]{l^2+m^2}$. 29. $\sqrt[3]{l^2+lm+m^2}$.

30. $\sqrt{l^2+m^2+n^2+2ln}$. 31. $\sqrt{l^3}+\sqrt{m^2+n^2}$.

32. $\sqrt{l^2+m^2+n^2+2lm+2ln+2mn}$.

If $a=-1$, $b=2$, $x=-3$ and $y=4$, find the value of

33. $\sqrt[3]{x^3-y^3+17ab}$. 34. $\sqrt{-4xy-13ay}$.

35. $\sqrt[4]{3x^2b-x^2}\times\sqrt{b^2+y^2+2by}$.

36. $\sqrt{\dfrac{2a^3-2bx-x^2}{y^2+4x}}$. 37. $\dfrac{\sqrt{b^2+a^2+y}}{\sqrt[3]{5xy-y}}$.

If $x=-4$, $y=-3$, $z=-1$, $a=4$, $b=1$ and $c=0$, find the value of

38. $2\sqrt{xz}-3\sqrt{ab}+\sqrt{cy}$. 39. $5z\sqrt{3yz}-4\sqrt{4a^2}-2z\sqrt{3yz}$.

40. $8\sqrt{y^2a}-5\sqrt{b^3z^2}+7\sqrt{c^2}$. 41. $\sqrt[3]{ax^2}+\sqrt[3]{bz^2}-\sqrt[3]{3y^2}-\sqrt[3]{c^2z}$.

If $a=1$ and $b=9$, find the value of

42. $\dfrac{a+2\sqrt{ab}+b}{\sqrt{a}+\sqrt{b}}+\dfrac{a-2\sqrt{ab}+b}{\sqrt{a}-\sqrt{b}}$.

43. $\sqrt{a}(a+b)+\sqrt{b}(a-b)-\sqrt{ab}(a+\sqrt{ab}+b)$.

44. $(\sqrt{a}+\sqrt{b})(a-\sqrt{ab}+b)+(\sqrt{a}-\sqrt{b})(a+\sqrt{ab}+b)$.

CHAPTER X

SIMPLE EQUATIONS

55. When we express algebraically the relation between two expressions which are equal, the statement is called an **equation**. The expressions that are equal are connected by the sign '='.

Thus $$x+5=x+5 \quad \text{..........................} \quad (1)$$
is an equation. The two expressions on either side of the sign of equality are called its **sides**.

Equation such as (1), in which both the sides are *always* equal whatever the value of x may be, is called an **identical equation** or simply an **identity**. While the equation, $x+5=8$, which is not true for all values of x but is true only when x has a particular value (3 in this case), is called a **conditional equation** or simply an **equation**.

The letter whose value is to be found in an equation is called the **unknown quantity**, and the process of finding its value is called **solving the equation**.

The number, which when substituted for the letter in an equation, makes both the sides equal is said to **satisfy** the equation, and the value of the unknown quantity which satisfies the equation is called the **root** of the equation.

Thus in the equation

$$x+5=8,$$

the letter 'x' whose value is to be found is the *unknown quantity* and the value 3 of x which satisfies the equation is its *root*.

NOTE. It should be remembered that quantities expressed by letters which are supposed to be known, are generally represented by the first letters of the alphabet $a, b, c, \ldots$; while quantities which are unknown and whose values are to be found, are represented by the last letters of the alphabet x, y, z.

56. Let us now consider the identity

$$8=8.$$

Adding 2 to both sides,

$$8+2=8+2,$$

i.e., $10=10.$

Subtracting 2 from both sides,

$$8-2=8-2,$$

i.e., $6=6.$

Multiplying both sides by 2,

$$8\times 2=8\times 2,$$

i.e., $16=16.$

Dividing both sides by 2,

$$8\div 2=8\div 2,$$

i.e., $4=4.$

Hence, we see that

(i) *If equals be added to equals, the sums are equal.*

(ii) *If equals be subtracted from equals, the differences are equal.*

(iii) *If equals be multiplied by equals, the products are equal.*

(iv) *If equals be divided by equals, the quotients are equal.*

The process of solving equations greatly depends on the foregoing axioms as will be clear from the following examples :

EXAMPLE 1. *Solve the equation* $x-2=3$.

Addding 2 to both sides,

$x-2+2=3+2$, ...[Axiom (i).]

$\therefore \quad x=5.$

EXAMPLE 2. *Solve the equation* $x+2=3$.

Subtracting 2 from both sides,

$x+2-2=3-2$, ...[Axiom (ii).]

$\therefore \quad x=1.$

EXAMPLE 3. *Solve the equation* $\frac{x}{5}=-4$.

Multiplying both sides by 5,

$$5\times\frac{x}{5}=5\times(-4), \quad \text{..............[Axiom (iii).]}$$

$$x=-20.$$

EXAMPLE 4. *Solve the equation* $-3x=-9$.

Dividing both sides by -3,

$$\frac{-3x}{-3}=\frac{-9}{-3}, \quad \text{.........................[Axiom (iv).]}$$

$$\therefore \quad x=3.$$

EXAMPLES XXIX

(Oral)

Solve the following equations :

1. $x-1=3$.
2. $x-3=5$.
3. $x-4=7$.
4. $x-9=11$.
5. $x+7=10$.
6. $x+2=7$.
7. $x-6=-3$.
8. $x+3=-9$.
9. $x+2=0$.
10. $x-6=0$.
11. $x-a=2a$.
12. $x+3a=5a$.
13. $5x=15$.
14. $3x=9$.
15. $2x=0$.
16. $7x=49$.
17. $5x=1$.
18. $2x=-108$.
19. $10x=5$.
20. $33x=11$.
21. $6x=-42$.
22. $-66x=-11$.
23. $-x=6$.
24. $ax=a^2$.
25. $ax=b$.
26. $\frac{x}{2}=1$.
27. $-\frac{x}{2}=3$.
28. $\frac{x}{3}=6$.
29. $-\frac{x}{4}=-7$.
30. $-\frac{x}{3}=0$.
31. $\frac{2x}{3}=4$.
32. $\frac{x}{6}=\frac{1}{3}$.
33. $\frac{3x}{4}=-\frac{3}{4}$.
34. $\frac{x}{3}-1=0$.
35. $\frac{x}{4}-5=0$.
36. $\frac{2x}{5}+\frac{1}{15}=0$.
37. $\frac{x-2}{5}=0$.
38. $\frac{3x-4}{7}=2$.
39. $2(x-1)=4$.
40. $6(x-5)=0$.

57. EXAMPLE 1. *Solve the equation*

$$5x-6=3x+12 \quad \ldots\ldots(1)$$

Adding 6 to both sides,

$$5x-6+6=3x+12+6,$$

$$\therefore \quad 5x=3x+12+6,$$

Subtracting $3x$ from both sides,

$$5x-3x=3x-3x+12+6,$$

$$\therefore \quad 5x-3x=12+6, \quad \ldots\ldots(2)$$

$$\therefore \quad 2x=18.$$

Dividing both sides by 2,

$$\frac{2x}{2}=\frac{18}{2},$$

$$x=9.$$

Comparing (1) and (2), we find that $3x$ disappears on the right side but appears on the left side with its sign changed, and similarly -6 disappears from the left side but appears on the right side with its sign changed.

Hence, we see that *any term may be transferred from one side of an equation to the other by changing its sign.*

Such a transfer of terms is called **transposition.**

While solving an equation, it is desirable to transpose all the terms containing the unknown quantity to one side of the equation (preferably to the left), and other terms not containing the unknown quantity to the other.

NOTE. If the value of the unknown quantity obtained from the equation be substituted for the unknown quantity in the equation and the equation is satisfied, *i.e.*, both the sides become equal, we say that the value obtained is correct. Thus, if 9 be written for x in (1),

the left side $=5\times9-6=45-6=39$,

and the right side $=3\times9+12=27+12=39$.

Since both sides are equal, the value 9 obtained for x is correct. This process is called **verification.**

EXAMPLE 2. *Solve the equation*

$$4x+3=13-3x+8+x,$$

and verify your answer.

Transposing terms involving x to the left side and other terms to the right,

$$4x+3x-x=13+8-3,$$

$$\therefore \quad 6x=18.$$

Dividing both sides by 6,

$$x=3.$$

Verification. When $x=3$,

the right side $=13-3\times3+8+3=13-9+8+3=15$,

and the left side $=4\times3+3=12+3=15$.

Since both the sides are equal, the solution is correct.

EXAMPLE 3. *Solve the equation*

$$4(x+2)=3-3(2x-5).$$

Removing brackets,

$$4x+8=3-6x+15.$$

Transposing, $4x+6x=3+15-8$.

$$\therefore \quad 10x=10.$$

Dividing both sides by 10,

$$x=1.$$

EXAMPLE 4. *Solve the equation*

$$6=5x-3-4x+5. \quad \text{.................................(1)}$$

Multiplying both sides by -1,

$$-6=-5x+3+4x-5. \quad \text{.................................(2)}$$

Transposing the terms on the right side to the left and the terms on the left side to the right.

$$5x-3-4x+5=6 \quad \text{...(3)}$$

Transposing -3 and $+5$ to the right side,

$$5x-4x=6+3-5,$$

$$\therefore \quad x=4.$$

Comparing (1), (2) and (3) we see that *the equality is not affected if* (i) *the signs of all the terms on both sides of an equation be changed*, (ii) *the entire sides of an equation be interchanged.*

EXAMPLE 5. *For what value of* x *will*

$$3x+2(x-5)-10$$

be equal to zero ?

Suppose $3x+2(x-5)-10=0$.

Removing brackets,

$$3x+2x-10-10=0,$$
$$\therefore \quad 5x=20,$$
$$\therefore \quad x=4.$$

Hence, if x is equal to 4, the given expression will be equal to zero.

NOTE. The above examples have been fully worked out in every detail. Students should note that each step should be written on a separate line and the process of deriving it from the preceding one should be clearly indicated. They are also cautioned against putting a meaningless sign of equality at the beginning of a line.

EXAMPLES XXX

Solve the following equations and verify your answers :

1. $3x-5=13$. **2.** $3x=20+x$. **3.** $x=8-x$.

4. $2x=1-4x$. **5.** $-x=12+x$. **6.** $3x+4=16$.

7. $3y+4=-17$. **8.** $4y-3=2y+1$. **9.** $2x+15=27-4x$.

10. $6x+11=4x+27$. **11.** $4x+7=x+7$. **12.** $\frac{x}{2}+\frac{x}{3}=5$.

13. $y=4+\frac{y}{2}$. **14.** $y-\frac{y}{2}=14$. **15.** $\frac{2x}{3}=\frac{x}{3}+6$.

16. $\frac{x}{3}=10-\frac{x}{3}$. **17.** $x+a=3a-x$. **18.** $x+b=a-2x$.

19. $5x+7+3x=24-4x-11$. **20.** $0=10-6x-20+10x$.

21. $1=7x-5x-3x$. **22.** $8-3x=6x+10-x+6$.

23. $x-15-9x-26=2+3x-10$. **24.** $0=5-2x+50-9x$.

25. $4(x-2)=5(x-1)$. **26.** $16x-11=19(1+x)$.

27. $2(7-x)=7x-31$. **28.** $14(x-18)=6(x-14)$.

29. $3x-(2x-7)=14$. **30.** $0=8(23+2x)+13(x-3)$.

31. $y-3-8=-5(13-2y)$. **32.** $6(x-1)-(3x+13)+9=0$.

33. $2(4-x)-(1+16x)=3(x-7)$. **34.** $5-18(3-x)+7(21x-9)=3x$.

35. $\cdot5(x+1)=\cdot3x+1\cdot12$. **36.** $4(x-1)=2(x-\cdot3)$.

37. $4(x-1\cdot5)=2\cdot34$. **38.** $3(2x-\cdot5)=1\cdot4$.

39. $7-\{3x-(4x-5)-3\}=0$.

40. $2x-\{7-2(x-3)+4x\}-27=40$.

41. $4(x-3)-5(x+11)=3(3-x)-9(13-x)+20$.

42. $3(5-x)+4(x-4)=9(x+3)-6(1-x)-1$.

43. $6(6x-5)-5(7x-8)=12(4-x)+1$.

44. $(x-5)(x+3)=(x-7)(x+4)$.

45. $(x-2)(x+3)=(x-1)(x+1)$.

46. $(x-5)^2-(x+2)^2=-2$.

47. $(x-1)(x+6)-(x-2)(x-3)=3$.

48. $(x+1)(x+2)+6=x(x+7)$.

For what values of x will the following expressions be equal to zero ?

49. $3(x+5)-(x+7)-x$.

50. $13(2x-1)-5(5x+7)$.

51. $(x-2)(x-3)-(x-4)(x+1)$.

52. $(x-a)+(x-b)+(x-c)$.

58. If fractions occur in equations, these can be cleared if we multiply both sides of the equation by the least common multiple of the denominators.

EXAMPLE 1. *Solve the equation*

$$6-\frac{x-1}{2}-\frac{x-2}{3}=\frac{3-x}{4},$$

and verify your answer.

Multiply each term of both sides by 12, the L. C. M. of the denominators 2, 3 and 4,

$$12\times6-\frac{12(x-1)}{2}-\frac{12(x-2)}{3}=\frac{12(3-x)}{4},$$

$$72-6(x-1)-4(x-2)=3(3-x),$$

$$\therefore\ 72-6x+6-4x+8=9-3x.$$

Transposing,

$$-6x-4x+3x=9-72-6-8,$$

$$\therefore\quad -7x=-77.$$

Dividing by -7,

$$x=11.$$

Verification. When $x=11$,

$$\text{the right side}=\frac{3-11}{4}=\frac{-8}{4}=-2,$$

$$\text{and the left side}=6-\frac{11-1}{2}-\frac{11-2}{3}=6-5-3=-2.$$

Since both sides are equal, the solution is correct.

EXAMPLE 2. *Solve the equation*

$$\frac{1}{7}(3x+5)-\frac{1}{15}(9x-35)=\frac{2x}{3}-\frac{16}{3}.$$

Multiplying both sides by 105, the L. C. M. of 7, 15 and 3,

$$105\times\frac{1}{7}(3x+5)-105\times\frac{1}{15}(9x-35)=105\times\frac{2x}{3}-105\times\frac{16}{3},$$

$$15(3x+5)-7(9x-35)=35\times 2x-35\times 16,$$

$$45x+75-63x+245=70x-560,$$

$$45x-63x-70x=-560-75-245,$$

$$\therefore\quad -88x=-880.$$

Dividing both sides by -88,

$$x=10.$$

EXAMPLES XXXI

Solve the following equations :

1. $\frac{8}{9}=-\frac{1}{3}x.$
2. $-\frac{2}{3}x=\frac{3}{7}.$
3. $\frac{1}{2}x+\frac{1}{3}x=10.$
4. $\frac{1}{3}x-\frac{1}{4}x=1.$
5. $2\left(\frac{1}{3}x-\frac{1}{5}x\right)=3.$
6. $\frac{1}{2}\left(\frac{x}{4}-\frac{x}{5}\right)=\frac{1}{5}.$
7. $\frac{1}{4}\left(\frac{2x}{3}-\frac{x}{6}\right)=\frac{1}{3}.$
8. $\frac{1}{39}\left(4x+\frac{x}{3}\right)=\frac{2}{5}.$
9. $\frac{x}{2}+x=\frac{7}{4}-\frac{x}{4}.$
10. $\frac{x}{3}-\frac{x}{8}=\frac{11}{2}-\frac{x}{4}.$
11. $\frac{x}{3}+\frac{x-8}{4}=5.$
12. $\frac{x+1}{4}-\frac{x-1}{5}=1.$
13. $\frac{x+3}{5}-8=\frac{1-x}{4}.$
14. $6\frac{1}{3}-\frac{x-7}{3}=\frac{4x-2}{5}.$
15. $\frac{x-3}{5}-\frac{x-5}{4}=\frac{2}{3}.$
16. $\frac{2x}{5}+\frac{x-2}{3}=2x-7.$
17. $x+\frac{1}{6}=\frac{2x}{3}+\frac{2}{3}-\frac{3x}{4}.$
18. $\frac{2x-1}{4}+\frac{1}{2}=\frac{3}{2}-\frac{1-x}{5}.$
19. $\frac{x-\frac{1}{2}}{6}-\frac{7x-3}{6}+\frac{3}{7}=0.$
20. $\frac{x}{4}-\frac{5x+8}{6}=\frac{2x-9}{3}.$
21. $\frac{2(2x-1)}{9}-\frac{3x-2}{13}=1.$
22. $\frac{3x}{4}-\frac{2x-9}{3}=\frac{5x+8}{3}+\frac{x}{2}.$

23. $\frac{1}{2}(x-1)-\frac{1}{4}(x-3)=\frac{1}{3}(x-2)$.

24. $\frac{1}{3}(x+1)+\frac{1}{2}(x-1)-\frac{1}{4}(3x-7)=2$.

25. $\frac{1}{3}(x-2)-\frac{1}{7}(x-4)=\frac{1}{12}(2x-3)-2\frac{3}{4}$.

26. $\frac{1}{2}(x+1)-\frac{1}{4}=x-\frac{1}{3}(2x-1)$.

27. $\frac{1}{3}(\frac{1}{2}x-2)-2(x-30)=\frac{1}{7}(x-6)-7$.

28. $4x-\frac{1}{2}(x-1)=4+\frac{1}{3}(2x-2)+24$.

29. $\frac{1}{5}(7x+1)-\frac{1}{3}(17-2x)=\frac{1}{4}(5x+1)$.

30. $\frac{1}{4}(x-5)-\frac{1}{12}(x-6)=\frac{1}{3}(x+7)+3(x+9)$.

31. $\frac{x}{7}-x-\frac{1-x}{2}+1=0$.

32. $\frac{x+3}{3}-(1-x)-\frac{x-2}{2}=0$.

33. $6-\frac{3-x}{4}=\frac{x-1}{2}+\frac{x-2}{3}$.

34. $\frac{2x-1}{3}-\frac{5x-4}{6}+\frac{13}{42}=0$.

35. $\frac{x+1}{3}-\frac{x+4}{5}+\frac{x+3}{4}=16$.

36. $\frac{2x+7}{3}-\frac{3x+5}{7}=10-\frac{3x}{5}$.

37. $\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3}$.

38. $\frac{3}{2}(x-1)-\frac{2(x+2)}{3}=4+\frac{3-x}{4}$.

39. $\frac{x+3}{4}-\frac{x+4}{5}=\frac{x+5}{6}-\frac{x+6}{7}$.

40. $\frac{15-\frac{2}{3}x}{5}-\frac{2x+5}{2\frac{1}{2}}=\frac{17-\frac{14}{3}x}{3}$.

41. $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}$.

42. $x-\frac{3-x}{5}=3.\frac{x-1}{2}+\frac{x+1}{5}-\frac{3}{10}$.

43. $\frac{1}{5}(x-8)+\frac{4+x}{4}+\frac{x-1}{7}=7-\frac{23-x}{5}$.

44. $\frac{3-x}{2}-\frac{1}{3}\left(\frac{3-2x}{4}\right)=\frac{2x+3}{7}+\left(\frac{1}{4}-\frac{1+3x}{2}\right)$.

45. $\frac{2x}{3}-\frac{1}{2}(x-1)=\frac{x-1}{3}-\frac{1}{4}(x-2)+\frac{x-3}{5}$.

46. $\frac{3x-2}{5}+\frac{4x-1}{7}-\frac{10x}{9}=5(x-9)+3-\frac{x}{3}$.

47. $\frac{x+5}{6}+\frac{1}{9}\left(\frac{x}{2}+\frac{2}{5}\right)-\frac{2}{3}(3+2x)=\frac{4x-14}{3}+\frac{x+10}{10}$.

48. $\cdot 5x+\frac{\cdot 02x+\cdot 07}{\cdot 03}-\frac{x+2}{9}=9\cdot 5.$

49. $\frac{5\cdot 2x}{13}-\frac{1-\cdot 2x}{5}\left(\frac{3}{5}-\cdot 1\right)=\cdot 1x-\frac{5x-2}{4}+\cdot 028.$

50. $120x-4[5x-2\{6x+7(x-8)\}]=16-4[3x-2\{x-6(x-1)\}].$

For what values of x will the following expressions be equal to zero ?

51. $\frac{x-3}{4}-\frac{2(x-7)}{3}+\frac{x+9}{12}.$

52. $\frac{2x+1}{3}-\frac{x+6}{5}-(x-3).$

53. $\frac{2(x+2)}{3}-2(x-1)+2x-7\frac{1}{3}.$

54. $\frac{7x-1}{4}-\frac{1}{3}\left(2x-\frac{1-x}{2}\right)-6\frac{1}{2}.$

55. $\frac{2x-3}{6}+\frac{3x-8}{11}-\frac{4x+15}{33}-\frac{1}{2}.$

CHAPTER XI

SYMBOLICAL EXPRESSIONS

59. In examples in foregoing chapters we have already given many questions on symbolic representation in which letters have been used as symbols for numbers. In this chapter we shall give more examples where letters will be used to represent numbers.

If a student finds it difficult to express a statement in algebraic symbols, he may mentally replace the letters by small arithmetical numbers and make it an easy arithmetical sum, and then apply the method to the algebraic problem.

EXAMPLE 1. *By how much does* 8 *exceed* 6 ? *By how much does* x *exceed* y ?

8 exceeds 6 by $8-6$ *i.e.*, 2.

Similarly x exceeds y by $x-y$.

EXAMPLE 2. *What must be added to* 4 *to obtain* 7 ? *What must be added to* x *to obtain* y ?

To 4 must be added $7-4$ *i.e.*, 3 to obtain 7.

Similarly, to x must be added $y-x$ to obtain y.

EXAMPLE 3. *The sum of two numbers is* 12, *and one of them is* 8 ; *what is the other ? The sum of two numbers is* x, *and one of them is* 3 ; *what is the other ?*

The second number is $12-8$ *i.e.*, 4.

Similarly, the second number is $x-3$.

EXAMPLE 4. *How many inches are there in* 5 *yards ? How many inches are there in* x *yards ?*

5 yd. $=36\times 5$ in. $=180$ in.

Similarly x yd. $=36\times x$ in. $=36x$ in.

EXAMPLE 5. *If the price of one book is* 2 *rupees, what is the price of* 5 *books ? If the price of one book is* x *rupees, what is the price of* y *books ?*

The price of 1 book = Rs. 2,

∴ ,, ,, ,, 5 books = Rs. $5\times 2=$ Rs. 10.

Similarly, if ,, ,, ,, 1 book = Rs. x,

∴ ,, ,, ,, y books = Rs. yx.

NOTE. Students should remember that since letters are used as symbols to represent numbers, if the work deals with concrete quantities, the units should always be mentioned. Thus, it is *not* correct to say 'the price of a book is x' but that 'the price of a book is x rupees or x pounds'.

EXAMPLE 6. *A boy walks at the rate of* 5 *miles an hour, how far will he go in* 3 *hours? A boy walks at the rate of* x *miles an hour, how far will he go in* 3 *hours?*

In 1 hr., the boy walks 5 miles,

∴ ,, 3 hr., ,, ,, ,, 3×5 ,, $=15$ miles.

Similarly, if ,, 1 hr., ,, ,, ,, x miles,

∴ ,, 3 hr., ,, ,, ,, $3x$ miles.

EXAMPLE 7. *Krishna is* 30 *years old and Gopal* 18 *years old, how many years is Krishna older than Gopal? Krishna is* x *years old and Gopal* y *years old, how many years is Krishna older than Gopal?*

Krishna is older than Gopal by $(30-18)$ years *i.e.*, 12 years.

Similarly, Krishna is older than Gopal by $(x-y)$ years.

EXAMPLE 8. *Mohammad has* 5 *rupees and he spends* 48 *annas, how many annas has he left? Mohammad has* x *rupees and he spends* y *annas, how many annas has he left?*

Since Rs. $5=16\times5$ annas $=80$ annas, therefore Mohammad has $(80-48)$ annas *i.e.*, 32 annas left.

Again, since Rs. $x=16x$ annas, therefore Mohammad has $(16x-y)$ annas left.

EXAMPLES XXXII

(*Many of these examples may be taken orally*)

1. By how much does x exceed 4?
2. By how much does a exceed 5?
3. What number is greater than x by 3?
4. What number is greater than a by b?
5. What must be added to x to obtain 7?
6. What must be added to a to obtain b?
7. What is the sum of x and y?
8. If x is greater than y, what is the difference between x and y?
9. What is the product of x and y?

10. The sum of two numbers is x and one of the numbers is 5, what is the other number ?

11. The sum of two numbers is 12 and one of the numbers is a, what is the other number ?

12. What number is greater than x by 30 ?

13. What number is less than x by 30 ?

14. What number is greater than x by y ?

15. What number is less than x by y ?

16. Express a rupees in annas. **17.** Express a rupees in pies.

18. Express x pounds in shillings.

19. Express x pounds in pence. **20.** Express x yards in feet.

21. Express x yards in inches. **22.** Express x yards in miles.

23. Express y metres in centimetres.

24. Express y metres in millimetres.

25. Express y metres in decimetres.

26. What is the double of x ?

27. What is five times x ? **28.** What is half x ?

29. What is the sum of 7 times x and 3 ?

30. What is the difference if 13 be subtracted from 10 times x ?

31. A man's present age is a years. What was his age 5 years ago ? What will be his age 5 years hence ?

32. Rama is x years old and Krishna y years old. By how many years is Rama older than Krishna ?

33. A boy will be x years old 6 years hence. What is his present age ?

34. How many times is 3 contained in x ?

35. How many times is x contained in $5x$?

36. How many times is x contained in x^2 ?

37. What number divided by x will give 5 ?

38. What number divided by x will give y ?

39. I have a rupees and spend 25 annas, how many annas have I left ?

40. I have x rupees and spend y annas, how many annas have I left ?

41. I have x maunds of rice and give 15 seers of rice to my friend, how many seers of rice have I left ?

42. I have x maunds of rice and give y seers of rice to my friend, how many seers of rice have I left ?

43. If the price of one orange is x pies, what is the price of (i) 5 oranges in pies ? (ii) x oranges in pies ? (iii) x oranges in annas ?

44. If the price of one dozen pencils is 18 annas, what is the price of x dozen pencils (i) in annas ? (ii) in rupees ?

45. A boy walks x miles in one hour. (i) How many miles will he walk in 5 hours ? (ii) How many miles will he walk in x hours ? (iii) How many yards will he walk in x hours ?

46. A horse runs a miles in 5 hours. How many miles will he run in (i) a hours ? (ii) x hours ?

47. The price of x seers of ghee is y rupees. (i) What is the price of 1 seer of ghee ? (ii) What is the price of a seers of ghee ? (iii) How much ghee can be purchased for 1 rupee ? (iv) How much ghee can be purchased for b rupees ?

48. The speed of a train is x miles per hour. What is its speed in feet per second ?

49. What is the number whose fifth part is equal to 1 ?

50. If four times a certain length is 100 yards, what is the length ?

51. By how much does $x-3$ exceed $x-4$?

52. By how much does $2x-3$ exceed $x+6$?

53. What is the area of a rectangle whose length is l feet and breadth b feet ?

54. What is the area of a rectangle whose breadth is x feet and whose length is three times its breadth ?

55. What is the area of a square whose side is x feet ?

56. A father is five times as old as his son, who is x years old. (i) What is the father's present age ? (ii) What was the father's age 10 years ago ? (iii) What will be the father's age 10 years hence ?

57. x mangoes are distributed among y boys. How many mangoes does each boy get ?

58. Mussoorie is 15 miles from Dehra-Dun. A man leaves Dehra-Dun for Mussoorie at the rate of x miles per hour, and, another leaves Mussoorie for Dehra-Dun at the rate of y miles per hour. If both start at the same time, find when and where they will meet.

59. Hardwar is 32 miles from Dehra-Dun. A man leaves Dehra-Dun for Hardwar and goes x miles on the first day and y miles on the second. How far is he from Hardwar at the end of his second day's journey ?

60. A man's annual income is x rupees. If he pays income-tax at the rate of 5 pies in the rupee, how much does he pay as income-tax ?

61. What is 4% of $100x$?

62. What is x% of 500 ?

63. What is 5% of x ?

64. What is 4% of a ?

65. Kailash buys a thing for x rupees and sells it at a gain of y%, what is his selling price ?

66. If one man can reap a field in x hours, in how many hours can a men reap the same field ?

67. A man, after giving x rupees to each of his three sons, has x rupees left. How many rupees had he at first ?

68. A man's annual income is x rupees and annual expenditure y rupees. In how many months can he save 1000 rupees ?

60. The numbers 1, 2, 3, 4,...are called **integers**. The integers which differ by *unity* are called *consecutive numbers.* Thus 4 and 5, 80 and 81, x and $(x+1)$ are pairs of consecutive numbers.

Numbers which are divisible by 2 are called **even numbers**. Thus 2, 4, 6,..., and $2x$, $4x$, $6x$,... are even numbers.

Numbers which are not divisible by 2 are called **odd numbers**. Thus 1, 3, 5,..., and $(2x-1)$, $(2x+1)$, $(2x+3)$,... are odd numbers.

NOTE. Consecutive even numbers and consecutive odd numbers differ by two.

EXAMPLE 1. *Write three consecutive numbers of which the middle one is 7. Write three consecutive numbers of which the middle one is* x.

Since consecutive numbers differ by unity, the number preceding 7 is $(7-1)$ *i.e.*, 6, and that following 7 is $(7+1)$ *i.e.*, 8.

Similarly, the number preceding x is $(x-1)$ and that following x is $(x+1)$.

EXAMPLE 2. *Write three consecutive even numbers of which the middle one is* 10. *Write three consecutive even numbers of which the middle one is* 2x.

Since consecutive even numbers differ by 2, the number preceding 10 is $(10-2)$ *i.e.*, 8, and that following 10 is $(10+2)$ *i.e.*, 12.

Similarly, the consecutive even number preceding $2x$ is $(2x-2)$ and that following $2x$ is $(2x+2)$.

EXAMPLE 3. *Write three consecutive odd numbers of which the middle one is 7. Write three consecutive odd numbers of which the middle one is* $(2x+1)$.

Since consecutive odd numbers differ by 2, the number preceding 7 is $(7-2)$ *i.e.*, 5, and that following 7 is $(7+2)$ *i.e.*, 9.

Similarly, the consecutive odd number preceding $(2x+1)$ is $(2x+1-2)$ *i.e.*, $(2x-1)$, and that following $(2x+1)$ is $(2x+1+2)$ *i.e.*, $(2x+3)$.

61. We know that 64 consists of two digits 4 and 6. 4 is the digit in the units place and 6 in the digit in the tens place. Thus 64 means $60+4$ or $6\times 10+4$. Hence, we see that in a number consisting of two digits, if the digit in the units place is 4 and the digit in the tens place is 6, the number can be written as $6\times 10+4$. Similarly, in a number of two digits, if the digit in the units place is x and the digit in the tens place is y, the number can be written as $(10\times y+x)$ or $(10y+x)$.

If the digits of the number 64 are interchanged, the new number thus formed will be 46 or $4\times 10+6$. Similarly, if the digits of the number represented by $(10y+x)$ be interchanged, the new number thus formed will be represented by $(10x+y)$.

As a number consisting of three digits such as 253, can be written as $(2\times 100+5\times 10+3)$, similarly a number of three digits, whose digit in the units place is x, digit in the tens place y and digit in the hundreds place z, can be written as $(100z+10y+x)$.

EXAMPLES XXXIII

1. Write down three consecutive numbers of which (i) the smallest is 12, (ii) the smallest is x, (iii) the greatest is 12, (iv) the greatest is x.

2. Write down three consecutive numbers of which the middle one is (i) 12, (ii) x.

3. Write down three consecutive even numbers of which (i) the smallest is 12, (ii) the smallest is $2x$, (iii) the greatest is 12, (iv) the greatest is $2x$, (v) the middle one is 12, (vi) the middle one is $2x$.

4. Write down three consecutive odd numbers of which (i) the smallest is 15, (ii) the smallest is $(2x+1)$, (iii) the greatest is 15, (iv) the greatest is $(2x+1)$, (v) the middle one is 15, (vi) the middle one is $(2x+1)$, (vii) the last is $(2x-1)$.

5. Write down four consecutive odd numbers of which the last is $(2x+1)$.

6. Write down four consecutive even numbers of which the first is $(2x-4)$.

7. What is the number whose (i) digit in the units place is 2 and digit in the tens place is 3 ? (ii) digit in the units place is x and digit in the tens place is y ? (iii) digit in the units place is y and digit in the tens place is x ?

8. What is the number whose digit in the units place is a and digit in the tens place is b? What number will be formed by interchanging the digits?

9. In a number the digit in the units place is twice the digit in the tens place. If the digit in the units place is x, what is the number? What number will be formed by interchanging the digits?

10. What is the number whose digit in the units place is x, digit in the tens place y and digit in the hundreds place z? What number will be formed by interchanging the digits in the units and hundreds places?

11. In a number the digit in the units place is twice the digit in the tens place and four times the digit in the hundreds place. If the digit in the units place is x, what is the number? What number will be formed by interchanging the digits in the units and hundreds places?

12. In a number the digit in the units place is y and the digit in the tens place is x. What will be the difference if the sum of the digits be subtracted from the number?

EXAMPLES XXXIV

Express the following sentences as equations:

1. Twice x equals 12.
2. Three times x equals 18.
3. One half of x equals 6.
4. One-fourth of x equals 8.
5. One-fifth of x equals 15.
6. The sum of x and 5 equals 15.
7. The sum of x and a equals b.
8. x equals ten times a.
9. Twice x is greater than 10 by 8.
10. Four times x is greater than a by 24.
11. One-fourth of x is greater than a by b.
12. Three times x equals the sum of $2x$ and 6.
13. Five times x equals the sum of half x and 18.
14. If 5 be subtracted from x, the difference equals 8.
15. If 4 be subtracted from three times x, the difference equals $2x$.
16. If twice x be subtracted from four times x, the difference equals 6.
17. If 20 be subtracted from $2x$, the difference equals the sum of x and 6.
18. x equals 5% of 500.
19. x equals 5% of y.
20. x equals z% of y.

If Mohan's age is $2x$ years, Sohan's age $(5x-10)$ years and Rohan's age $(4x+20)$ years ; express the following in algebraic equations :

21. Mohan's age is half of Sohan's age.

22. Mohan is 20 years younger than Sohan.

23. Mohan is 40 years younger than Rohan.

24. Rohan is 20 years older than Sohan.

25. Rohan is three times as old as Mohan.

26. Ten years before Rohan was five times as old as Mohan.

27. Ten years hence Mohan will be as old as Sohan was ten years before.

28. The sum of the ages of Mohan and Sohan equals Rohan's age.

29. The difference of the ages of Rohan and Mohan equals Sohan's age.

30. Five years before the sum of the ages of Mohan, Sohan and Rohan was 105 years.

If Sushila has $3x$ rupees, Prabha $(5x+15)$ rupees and Vimla $(16x-20)$ rupees ; express the following in algebraic equations :

31. Prabha has twice as many rupees as Sushila has.

32. Vimla has four times as many rupees as Sushila has.

33. Sushila and Prabha have together 45 rupees.

34. If Prabha gives 10 rupees to Sushila, Sushila will have 5 rupees more than Prabha.

35. If Vimla spends 30 rupees, she will have left with her as many rupees as Prabha has.

36. Sushila, Prabha and Vimla have together 105 rupees.

37. Vimla has 15 rupees more than what both Sushila and Prabha have.

CHAPTER XII

SUBSTITUTION IN FORMULÆ

62. We know that the area of a rectangle is equal to the product of its length and breadth. Thus, if A represents the area of the rectangle, l the length, and b the breadth, then their relation can be expressed as an equation

$$A = l \times b. \quad \text{.................................(1)}$$

Now (1) is a **formula** which enables us to find the area of any rectangle if we know its length and breadth, also the length if we know its area and breadth, and breadth if we know its area and length. Thus, we see that (1) is a relation between three letters, any one of which can be found if the other two are known.

In this chapter we will give some important formulæ, and other results expressed in symbols, and explain their uses to find the value of some of the symbols when the values of others are known. Students are already familiar with most of these in Arithmetic.

EXAMPLE 1. *In formula* (1), *find* A *when* l=8 *ft.*, *and* b=6 *ft.*, *i.e., find the area of a rectangle whose length is* 8 *ft. and breadth* 6 *ft.*

In (1) writing 8 for l and 6 for b,

$A = 8 \times 6$ sq. ft., $= 48$ sq. ft.

EXAMPLE 2. *In formula* (1), *find* b *when* A=48 *sq. ft.*, *and* l=8 *ft.*

In (1) writing 48 for A and 8 for l,

$$48 = 8b,$$

Dividing by 8, $b = 6$ feet.

EXAMPLE 3. *In formula* (1), *find* l *when* A=48 *sq. ft.*, *and* b=6 *ft.*

In (1), writing 48 for A and 6 for b,

$$48 = 6l.$$

Dividing by 6, $l = 8$ ft.

The above examples can be given in a tabular form thus :

If the area A of a rectangle, whose length is l and breadth b is given by the formula $A = l \times b$, fill up the blanks in the following :

A	l	b
	8 ft.	6 ft.
48 sq. ft.	8 ft.	
48 sq. ft.		6 ft.

63. The volume V of a cylinder, the radius of whose circular base is r and height h, is given by the formula

$$V = \tfrac{22}{7}\, r^2 h. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

EXAMPLE 1. *Find* V, *when* r=7 *inches and* h=10 *inches.*

$V = \frac{22}{7} \times 7^2 \times 10$ cu. inches

$= 1540$ cu. inches.

EXAMPLE 2. *Find* r, *when* V=396 *cubic centimetres and* h=14 *centimetres.*

In formula (2), writing 396 for V and 14 for h,

$396 = \frac{22}{7} r^2 \times 14,$

Dividing by 44,

$9 = r^2,$

$\therefore\ r = 3$ cm.

EXAMPLE 3. *Find* h, *if* V=231 *cubic yards, and* r=3½ *yards.*

In formula (2), writing 231 for V and $\frac{7}{2}$ for r,

$231 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h,$

$\therefore\ 231 = \frac{77}{2} h.$

Multiplying by 2,

$462 = 77\, h.$

Dividing by 77,

$h = 6$ yd.

EXAMPLES XXXV

1. From formula (1) of Article 62,

(i) find A, when $l=16$ yd., and $b=10$ yd. ;

(ii) find b, when $A=15$ sq. ft., and $l=5$ ft. ;

(iii) find l, when $A=105$ sq. ft., and $b=5$ ft. ;

(iv) fill in the blanks in the following :

A	l	b
	12 ft.	9 ft.
112 sq. yd.	16 yd.	
200 sq. m.		10 m.

2. If the area A of a triangle, whose base is b and height p is given by the formula $A=\frac{1}{2}bp$,

(i) find A, when $b=10$ ft., and $p=8$ ft. ;

(ii) find b, when $A=144$ sq. yd., and $p=18$ yd. ;

(iii) find p, when $A=35$ sq. m., and $b=10$ m. ;

(iv) fill in the blanks in the following :

A	p	b
	6 in.	5 in.
12 sq. ft.	6 ft.	
88 sq. yd.		11 yd.

3. If the circumference C of a circle, whose radius is r, is given by the formula $C=2\times\frac{22}{7}\times r$,

(i) find C, when $r=21$ ft. ;

(ii) find r, when $C=88$ mm.

4. If the area A of a circle, whose radius is r, is given by the formula

$$A=\frac{22}{7}r^2,$$

(i) find A, when $r=3\frac{1}{2}$ cm. ;

(ii) find r, when $A=154$ sq. yd.

5. If the distance S travelled by a train in time t moving with a velocity v is given by the formula

$$S=vt.$$

(i) find S, when $v=40$ ft. per sec., and $t=45$ sec. ;

(ii) find t, when $v=50$ ft. per sec., and $S=1320$ ft. ;

(iii) find v, when $t=48$ sec., and $S=2112$ ft. ;

(iv) fill in the blanks in the following :

S	v	t
	44 ft. per sec.	30 sec.
210 ft.		7 sec.
x ft.	y ft. per sec.	

6. From formula (2) of Article 63,

(i) find V, when $r=21$ in., and $h=16$ in. ;

(ii) find h, when $r=35$ m., and $V=770$ cu. m. ;

(iii) find r, when $h=7$ ft., and $V=22$ cu. ft.

7. If the area A of the four walls of a room, whose height is h, length l and breadth b, is given by the formula

$$A=2(l+b)\,h,$$

(i) find A, when $l=16$ ft., $b=12$ ft., and $h=11$ ft. ;

(ii) find h, when $l=20$ ft., $b=10$ ft., and $A=900$ sq. ft. ;

(iii) find l, when $b=6$ yd., $h=4$ yd., and $A=112$ sq. yd. ;

(iv) find b, when $l=10$ m., $h=8$ m., and $A=286$ sq. m.

8. If the volume V of a box, whose length is l, breadth b and depth d, is given by the formula

$$V=l\times b\times d,$$

(i) find V, when $l=3$ ft., $b=2$ ft., and $d=1$ ft. ;

(ii) find d, when $l=8$ yd., $b=1$ yd., and $V=4$ cu. yd. ;

(iii) find b, when $l=100$ mm., $d=75$ mm., and $V=600000$ cu. cm.;

(iv) find l, when $b=8$ in., $d=18$ in., and $V=6400$ cu. in.;

(v) fill in the blanks in the following:

V	l	b	d
	8 in.	5 in.	4 in.
100 cu. in.	10 in.	5 in.	
144 cu. cm.	12 cm.		2 cm.
1536000 cu. mm.		120 mm.	80 mm.
	$3x$ in.	$2x$ in.	x in.

9. If the hypotenuse h of a right-angled triangle, whose base is b and height is p is given by the formula

$$h=\sqrt{b^2+p^2},$$

(i) find h, when $b=16$ ft., and $p=12$ ft.;

(ii) find p, when $h=5$ ft., and $b=4$ ft.;

(iii) find b, when $h=13$ ft., and $p=5$ ft.;

(iv) fill in the blanks in the following:

h	b	p
	3 ft.	4 ft.
13 in.	5 in.	
15 yd.		9 yd.
	$8x$ cm.	$6x$ cm.

10. If the simple interest I, on principal P in time t at rate per cent r, is given by the formula

$$I = \frac{P \times r \times t}{100},$$

(i) find I, when P=Rs. 500, $t=4$ years, and $r=3$ per cent. ;
(ii) find r, when P=Rs. 400, $t=2$ yr., and I=Rs. 32 ;
(iii) find t, when P=Rs. 750, $r=4$ p. c., and I=Rs. 150 ;
(iv) find P, when $t=4$ yr., I=Rs. 84, and $r=3\frac{1}{2}$ p. c. ;
(v) fill in the blanks in the following :

I	P	r	t
	Rs. 1000	4%	3 yr.
Rs. 60	Rs. 500	3%	
Rs. 150		5%	4 yr.
	Rs. x	y%	n yr.

11. In the formula $v=u+ft.$,
(i) find v, when $u=10$, $f=2$ and $t=4$;
(ii) find t, when $u=80$, $f=16$ and $v=240$;
(iii) find f, when $u=120$, $t=5$ and $v=195$;
(iv) find u, when $f=-6$, $t=20$ and $v=0$.

12. In the formula $v^2=u^2+2fs$,
(i) find v, when $u=20$, $f=5$ and $s=80$;
(ii) find s, when $u=0$, $f=6$ and $v=12$;
(iii) find u, when $f=-10$, $s=15$ and $v=0$;
(iv) find f, when $u=80$, $s=18$ and $v=100$.

13. In the formula $s=ut-16t^2$,
(i) find s, when $u=80$ and $t=2$;
(ii) find u, when $t=\frac{3}{2}$ and $s=30$.

14. In the formula $s=\frac{n(a+l)}{2}$,

(i) find s, when $n=100$, $a=1$ and $l=100$;

(ii) find l, when $n=20$, $a=2$ and $s=420$;

(iii) find a, when $n=40$, $l=60$ and $s=1400$;

(iv) find n, when $a=-4$, $l=36$ and $s=160$.

15. In the formula $A=\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$, find A, when

(i) $a=13$, $b=12$ and $c=5$;

(ii) $a=13$, $b=14$ and $c=15$.

CHAPTER XIII

SIMPLE PROBLEMS

64. The method of solving problems in Algebra is the same as that in Arithmetic. We know that in every problem some quantities are known, with the help of which we have to find some other quantities whose relation with the known quantities is given. Hence, on reading a problem we should carefully consider the following :

(i) What quantities are to be found ?

(ii) What quantities are known ?

(iii) What are the relations between the known quantities and the quantities that are to be found ?

Thus, if we translate the problem word for word into algebraic language, we shall obtain two expressions which will be numerically equal, and, thus equating these, we shall obtain an equation, the solution of which gives the value of the quantity required.

For example, consider the following problem :

"What must be added to 7 to obtain 10 ?"

This can be written in algebraic language thus,

$$7 + ? = 10.$$

In the above statement writing the letter x for the sign ? we have

$$7 + x = 10.$$

Thus, we get an equation, the solution of which will give us the number required.

Subtracting 7 from both sides, we have

$$x = 3.$$

Hence, we see that the required number is 3, which when added to 7, will make it 10.

The quantity which it is required to find is called an **unknown quantity**. If only one unknown quantity is to be found it is generally represented by x ; if two, they are represented by x and y.

EXAMPLE 1. *Find a number such that, if* 15 *be subtracted from its four times, the difference will be equal to* 25.

This may be translated word for word into algebraic language thus :

Find a number (suppose that number is x) such that if 15 be subtracted from its four times (four times x is $4x$), the difference ($4x-15$) is equal to 25,

$$i.e.,\quad 4x-15=25,$$

Adding 15 to both sides,

$$4x=25+15,$$

$$\therefore\quad 4x=40.$$

Dividing by 4, $\quad x=10.$

Hence the required number is 10.

EXAMPLE 2. *In* 5 *years a man will be three times as old as he was* 15 *years ago. Find his present age.*

Suppose the present age of the man is x years. In 5 years (in 5 years his age will be $(x+5)$ years) he will be three times as old as he was 15 years ago (15 years ago his age was $(x-15)$ years),

$$i.e.,\quad (x+5)=3(x-15).$$

Simplifying, $\quad x+5=3x-45,$

$$\therefore\quad x-3x=-45-5,$$

$$\therefore\quad -2x=-50.$$

Dividing by -2, $\quad x=25.$

Hence his present age is 25 years.

Verification. In 5 years, the man will be $(25+5)$ or 30 years old ; 15 years ago he was $(25-15)$ or 10 years old ; but $30=3\times10$, hence the answer is correct.

EXAMPLE 3. *Rama and Shyama have equal sums of money. If at the end of a year, Rama gains* Rs. 300 *and Shyama loses* Rs. 100, *Rama will have twice as much as Shyama. How much had each at the beginning of the year ?*

Suppose both Rama and Shyama have x rupees at the beginning of the year. At the end of the year Rama gains Rs. 300 (*i.e.*, at the end of the year Rama will have $(x+300)$ rupees) and Shyama loses Rs. 100 (*i.e.*, at the end of the year Shyama will have $(x-100)$ rupees), then Rama's money which becomes $(x+300)$ rupees will be twice as much as Shyama's money which becomes $(x-100)$ rupees, *i.e.*, $(x+300)$ rupees $=2(x-100)$ rupees,

$$\therefore \quad (x+300)=2(x-100).$$

Simplifying, $$x+300=2x-200,$$

$$\therefore \quad x-2x=-200-300,$$

$$\therefore \quad -x=-500,$$

$$\therefore \quad x=500.$$

Hence both Rama and Shyama had Rs. 500 at the beginning of the year.

Verification. At the end of the year Rama will have $(500+300)$ or 800 rupees and Shyama will have $(500-100)$ or 400 rupees; but $800=2\times400$, hence the answer is correct.

NOTE. Students should remember, that the unknown quantity which is generally represented by x always stands for a number. Thus in example 2, x stands for the number of years and in example 3, x stands for the number of rupees. x does not represent years or rupees but represents the *number* of years and the *number* of rupees. It should also be remembered that in a problem, all concrete quantities of the same kind should be expressed in terms of the same unit. Thus, if a problem has sums of money in terms of rupees, annas and pies, they should all be expressed in terms of only one unit, either in terms of rupees or annas or pies.

EXAMPLE 4. *A boy, when asked about his age, said, "If 4 be subtracted from twice my age, then 4 be subtracted from twice the difference, and then again 4 be subtracted from twice the second difference, the last difference will be four times my age." Find his age.*

Suppose his age is x years. If 4 be subtracted from twice x (*i.e.*, from $2x$), the difference will be $(2x-4)$. If 4 be subtracted from twice this difference, *i.e.*, from $2(2x-4)$, the second difference will be $\{2(2x-4)-4\}$. Now if 4 be again subtracted from twice this difference *i.e.* $2\{2(2x-4)-4\}$, the difference will be $2\{2(2x-4)-4\}-4$. This is four times his age (*i.e.* is equal to $4x$),

$$\therefore \quad 2\{2(2x-4)-4\}-4=4x.$$

Removing brackets,

$$2\{4x-8-4\}-4=4x,$$

$$8x-16-8-4=4x,$$

$$\therefore \quad 8x-4x=16+8+4,$$

$$\therefore \quad 4x=28,$$

$$\therefore \quad x=7.$$

Hence his age is 7 years.

EXAMPLES XXXVI

1. Four times a certain number is 48, find the number.

2. One-third of a certain number is 17, find the number.

3. One-fifth of a certain number is 50, find the number.

4. One man has x rupees and another has $3x$ rupees. If both together have 160 rupees, how many rupees has each ?

5. Rama has $7x$ rupees and Shyama has $4x$ rupees. If Rama has 30 rupees more than Shyama, how many rupees has each ?

6. Mohini has some money. Her mother gives her four times as much as she has. If she then has 55 rupees, how many rupees had she at first ?

7. Mohammad has some money. He gives one-fourth of this to his friend, Ahmad. If he then has 60 rupees, how many rupees had he at first ?

8. If 7 be added to four times a number, the sum equals 71. Find the number.

9. If 13 be added to five times a number, the sum equals 68. Find the number.

10. If 9 be subtracted from eight times a number, the difference equals 63. Find the number.

11. If 2 be subtracted from three times a number, the difference equals 58. Find the number.

12. The sum of six times and seven times a certain number is 52. Find the number.

13. The difference of five times and twice a certain number is 45. Find the number.

14. What number added to its six times amounts to 49 ?

15. What number subtracted from its five times gives 100 ?

16. Twice a certain number added to its three times amounts to 105. Find the number.

17. Twice a certain number subtracted from its five times gives 39. Find the number.

18. Find a number which is less than its five times by 16.

19. Find a number which is greater than half itself by 26.

20. Find a number which is greater than one-tenth itself by 81.

21. If 9 be subtracted from a number, the difference is two-thirds that number. Find the number.

22. Find a number whose half exceeds its one-third by one.

23. Find a number whose one-third exceeds its one-seventh by 20.

24. If 3 be added to a number and the sum be multiplied by 7, the result is 35. Find the number.

25. If 5 be subtracted from a number and the difference be multiplied by 11, the result is 121. Find the number.

26. The sum of four times a number and 3 equals three times that number and 8. Find the number.

27. The sum of six times a number and 15 equals eleven times that number. Find the number.

28. Twice a certain number exceeds 20 by as much as its three times exceeds 65. Find the number.

29. The sum of twice a number and 8 equals half that number and 17. Find the number.

30. If 50 be added to a number and the sum be divided by 3, the quotient equals twice that number. Find the number.

31. Rama is 35 years old and Kailash 7 years old. In how many years will Rama be twice as old as Kailash ? Verify your answer.

32. Madan is 30 years older than Hari, who is 16 years old. How many years ago was Madan four times as old as Hari ? Verify your answer.

33. A man is 40 years old and his son 10 years old. In how many years will he be twice as old as his son ? Verify your answer.

34. In 40 years Rahim's age will be five times his present age. Find his present age and verify your answer.

35. In 15 years Prem will be three times as old as he was 5 years ago. Find his present age and verify your answer.

36. In 6 years a man will be twice as old as he was 12 years ago. Find his present age and verify your answer.

37. Mathura and Rama have equal sums of money. If Mathura gives Rs. 300 to Rama, then Rama will have five times as much as Mathura. How much had each at first ? Verify your answer.

38. Nagendra and Mahendra have an equal number of mangoes. Nagendra says to Mahendra, "If you give me 18 mangoes, I shall have twice as many mangoes as you will have left." How many mangoes had each at first ? Verify your answer.

39. Sohan and Mohan have equal sums of money. At the end of a year Sohan gains Rs. 400 and Mohan Rs. 2000. Then Mohan has twice as much as Sohan. How much had each at the beginning of the year ? Verify your answer.

40. A and B have equal sums of money. At the end of a year A gains Rs. 200 and B loses Rs. 100. Then A has three times as much as B. How much had each at the beginning of the year ? Verify your answer.

41. Sushila has Rs. 120 and Shakuntala Rs. 80. How many rupees should Sushila give to Shakuntala, so that both may have equal ums of money ?

42. A boy, when asked about his age, said, "If 5 be subtracted from twice my present age, then 4 be subtracted from twice the difference, and then 2 be added to three times the second difference, the result will be eight times my age." Find his age.

65. When a problem contains two unknown quantities, two different statements showing the relations between known and unknown quantities must be given.* In such cases we should denote one of the unknown quantities by x, and use one of the statements to express the other unknown quantity in terms of x. We should next use the other statement to translate the problem in algebraic language, the solution of which will give the values of the quantities required. This can be better understood from the following examples :

EXAMPLE 1. *A number exceeds another by* 4, *and their sum is* 36. *Find the numbers.*

This problem has two statements :

(i) One number exceeds another by 4.

(ii) The sum of the numbers is 36.

Suppose the smaller number is x, then by the first statement the other number will be $(x+4)$, and by the second statement the sum of x and $(x+4)$ equals 36.

$$x+(x+4)=36,$$
$$x+x+4=36,$$
$$\therefore\ 2x=32,$$
$$\therefore\ x=16.$$

Hence the smaller number is 16 and the greater number is $(16+4)$ *i.e.*, 20.

EXAMPLE 2. *Shafi has three times as much money as Rafi. If Shafi gives Rs.* 250 *to Rafi, Rafi will have twice as much as Shafi. How much had each at first ?*

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* If a problem contains three unknown quantities, three statements must be given, *i.e.*, there should be as many statements as there are unknown quantities in the problem.

This problem has two statements :

(i) Shafi has three times as much money as Rafi.

(ii) If Shafi gives Rs. 250 to Rafi, Rafi will have twice as much money as Shafi.

Suppose Rafi has x rupees, then by the first statement Shafi has $3x$ rupees, and by the second statement if Shafi gives 250 rupees to Rafi (Shafi will then have $(3x-250)$ rupees and Rafi $(x+250)$ rupees); Rafi's money *i.e.*, $(x+250)$ rupees will be twice Shafi's money *i.e.*, $(3x-250)$ rupees.

$$x+250=2(3x-250).$$

Removing brackets,

$$x+250=6x-500,$$
$$x-6x=-500-250,$$
$$-5x=-750.$$

Dividing by -5, $x=150$.

Hence Rafi has Rs. 150 and Shafi has Rs. 3×150 or Rs. 450.

NOTE. It is more convenient to denote the smaller of the quantities by x.

EXAMPLE 3. *Parbati is three times as old as Satyawati, 9 years ago the sum of their ages was 66 years. Find their present ages.*

This problem has two statements :

(i) Parbati is three times as old as Satyawati.

(ii) 9 years ago the sum of their ages was 66 years.

Suppose the present age of Satyawati is x years, then by the first statement Parbati's present age is $3x$ years. Now, since 9 years ago Parbati was $(3x-9)$ years old and Satyawati $(x-9)$ years old, the second statement gives

$$(3x-9)+(x-9)=66.$$

Removing brackets,

$$3x-9+x-9=66,$$
$$\therefore\quad 3x+x=66+9+9,$$
$$\therefore\quad 4x=84,$$
$$\therefore\quad x=21.$$

Hence the present age of Satyawati is 21 years, and that of Parbati 3×21 years or 63 years.

EXAMPLE 4. *Divide* 49 *into two parts in the ratio of* 4 : 3.

Since the two numbers are in the ratio of 4 : 3, if one of the numbers is $4x$, the other will be $3x$.

Now the sum of the numbers *i.e.*, $4x$ and $3x$ is 49.

$$\therefore \quad 4x+3x=49,$$
$$\therefore \quad 7x=49,$$
$$\therefore \quad x=7.$$

Hence the numbers are 4×7 and 3×7 *i.e.*, 28 and 21.

EXAMPLE 5. *Distribute Rs.* 233 *among Sohan, Mohan and Rohan, so that Sohan may get twice as much as Mohan, and Rohan may get Rs.* 5 *more than Mohan. How much does each get ?*

This problem has three statements :

(i) Sohan gets twice as much as Mohan.

(ii) Rohan gets Rs. 5 more than Mohan.

(iii) The sum of their money is Rs. 233.

Suppose Mohan gets x rupees, then by the first statement Sohan gets $2x$ rupees and by the second statement Rohan gets $(x+5)$ rupees. The third statement gives

$$2x+x+(x+5)=233,$$
$$2x+x+x+5=233,$$
$$\therefore \quad 4x=233-5,$$
$$\therefore \quad 4x=228,$$
$$\therefore \quad x=57.$$

Hence Mohan gets Rs. 57, Sohan gets Rs. 2×57 *i.e.*, Rs. 114 and Rohan gets Rs. $(57+5)$ *i.e.*, Rs. 62.

EXAMPLE 6. *A boy has a certain number of rupees, twice as many eight-anna pieces and three times as many four-anna pieces. The value of all the coins is Rs.* 55. *How many coins has he of each kind ?*

NOTE. Students should note that there is a great difference between the number of coins and the value of coins, thus 8 four-anna pieces are 8 in number but are equivalent to 2 rupees in value.

This problem has three statements :

(i) The number of eight-anna pieces is twice that of the rupees.

(ii) The number of four-anna pieces is three times that of the rupees.

(iii) The value of all the coins is 55 rupees.

Suppose the boy has x rupees, then by the first statement he has $2x$ eight-anna pieces and by the second statement he has $3x$ four-anna pieces.

Since 1 eight-anna piece is equivalent to $\frac{1}{2}$ rupee,

$\therefore$ $2x$ eight-anna pieces are ,, ,, $2x \times \frac{1}{2}$ or x rupees.

Similarly, since 1 four-anna piece is ,, ,, $\frac{1}{4}$ rupee,

$\therefore$ $3x$ four-anna pieces are ,, ,, $3x \times \frac{1}{4}$ or $\frac{3x}{4}$ rupees.

Now the third statement gives

$$x + x + \frac{3x}{4} = 55.$$

Multiplying by 4,

$$4x + 4x + 3x = 220,$$

$$\therefore \quad 11x = 220,$$

$$\therefore \quad x = 20.$$

Hence he has 20 rupees, 2×20 *i.e.*, 40 eight-anna pieces and 3×20 *i.e.*, 60 four-anna pieces.

EXAMPLES XXXVII

1. The sum of two numbers is 36, and one is twice the other. Find the numbers.

2. The sum of two numbers is 48, and one is three times the other. Find the numbers.

3. The sum of two numbers is 39, and one is half the other. Find the numbers.

4. The sum of two numbers is 44, and one is one-third the other. Find the numbers.

5. The sum of two numbers is 150, and one is $\frac{2}{3}$ of the other. Find the numbers.

6. The sum of two numbers is 200, and one is $\frac{3}{7}$ of the other. Find the numbers.

7. The sum of two numbers is 280, and their difference is 40. Find the numbers.

8. The sum of two numbers is 70, and their difference is 50. Find the numbers.

9. The difference of two numbers is 50, and the greater is twice the smaller. Find the numbers.

10. The difference of two numbers is 80, and the smaller is $\frac{2}{3}$ of the greater. Find the numbers.

11. The difference of two numbers is 36, and the smaller is $\frac{5}{11}$ of the greater. Find the numbers.

12. The difference of two numbers is 12, and if 1 be added to the greater, the sum will be twice the smaller. Find the numbers.

13. The difference of two numbers is 39, and twice the greater exceeds three times the smaller by 65. Find the numbers.

14. A number exceeds another by 5, and their sum is 35. Find the numbers.

15. The difference of two numbers is 51, and their sum is five times the smaller. Find the numbers.

16. Divide 140 into two parts such that their difference is 22.

17. Divide 50 into two parts such that their difference is 1.

18. Divide 100 into two parts such that twice the greater is equal to three times the smaller.

19. Divide 9 into two parts such that one-fourth of the one is equal to one-fifth of the other.

20. Divide 90 into two parts such that if three times one part be added to twice the other, the sum is equal to 230.

21. Divide 250 into two parts such that if twice one part be subtracted from five times the other, the difference is equal to 200.

22. Divide 36 into two parts such that 25% of the one is equal to 20% of the other.

23. Divide 55 into two parts in the ratio of 2 : 3.

24. Two numbers are in the ratio of 5 : 3, and their sum is 160. Find the numbers.

25. Two numbers are in the ratio of 7 : 11, and their difference is 32. Find the numbers.

26. The sum of two numbers is 189, and $\frac{3}{5}$ of the one exceeds $\frac{3}{4}$ of the other by 27. Find the numbers.

27. Mohammad has Rs. 31 more than Shaukat, and they have Rs. 157 together. How many rupees has each ?

28. Divide Rs. 56 between Prabha and Durga so that Prabha may have Rs. 16 more than Durga.

29. Divide Rs. 29 between Gopal and Krishna so that Gopal may have Rs. 7 less than Krishna.

30. Distribute 15 oranges between two boys so that one may get half the other.

31. Distribute 20 inkpots between two boys, so that one may get three times the other.

32. Divide a footscale into two parts such that the length of one part may be twice the other.

33. Two men together have Rs. 129. If one has twice as much as the other, how much does each have ?

34. Two men together have Rs. 98. If one has six times as much as the other, how much does each have ?

35. Satendra is 5 years older than Mathura ; 20 years ago Satendra was twice as old as Mathura ; find their present ages.

36. Rama is 20 years older than Shyama ; 5 years hence, Rama will be twice as old as Shyama ; find their present ages.

37. Ahmad is twice as old as Mohammad ; 8 years ago Ahmad was four times as old as Mohammad ; find their present ages.

38. A man is 32 years older than his son ; 10 years ago he was three times as old as his son ; find their present ages.

39. Hari says to Veda, "I am twice as old as you are, and 10 years ago I was four times as old as you were." Find their present ages.

40. Shankar says to Om, "I am 10 years older than you are, and in 5 years time I shall be twice as old as you will be." Find their present ages.

41. A man is 30 years older than his son ; 12 years hence he will be three times as old as his son ; find their present ages.

42. Munni is 10 years older than Chunni ; 5 years ago the sum of their ages was 30 years ; find their present ages.

43. A woman is 20 years older than her daughter ; the woman's age is as much greater than 50 years as the daughter's age is less than 50 years ; find their ages.

44. Five years ago a man's age was four times his son's age and fifteen years hence his age will be twice his son's age ; find their present ages.

45. A man's age is twice his son's age ; the man's age is as much greater than 62 years as his son's age is less than 43 years ; find their ages.

46. Find three numbers such that the second is twice the first, the third is three times the first, and the difference of the third and second is 15.

47. Find three numbers such that the second is twice the first, the third is greater than second by 2, and the sum of the first and third is 20.

48. Find three numbers such that the sum of the first two is 4, the third is five times the first, and the third is greater than the second by 2.

49. Mohan, Sohan and Rohan have Rs. 159 between them. Mohan has Rs. 10 more than Sohan and Rohan has Rs. 10 less than Sohan. How many rupees has each ?

50. Two boys have Rs. 22. 6a. between them. If the first had Rs. 2. 4a. more and the second 12a. less, the first would have three times as much as the second. How much has each ?

51. Divide Rs. 100 between Mahendra, Nagendra and Satendra, so that Nagendra may have Rs. 10 more than Mahendra and Satendra may have Rs. 8 more than Nagendra.

52. Divide Rs. 215 between Sushila, Lakshmi and Durga, so that Sushila may have twice as much as Lakshmi, and Lakshmi may have Rs. 15 less than Durga.

53. Chatterji has £ 3 more than Mukerji and £ 15 more than Banerji. If the three have £ 186 altogether, how much does each have ?

54. There are two rooms ; the number of chairs in the first is $\frac{3}{5}$ of that in the second ; if two chairs be transferred from the second room to the first, the number of chairs in both the rooms will be equal. How many chairs are there in each room ?

55. Three boys, Hira, Panna and Mathura climb up a tree to pluck guavas. Panna collects 6 more guavas than Hira, and Mathura collects twice as many as Panna and Hira together. If the three collect 54 guavas in all, how many does each collect ?

56. Three persons have between them Rs. 140. The second man has Rs. 5 more than the first, and the third man has three times as much as the first two together. How much does each have ?

57. The sum of the three sides of a triangle is 42 inches. If the first side is 1 inch smaller than the second and the second 1 inch smaller than the third, what is the length of each ?

58. The sum of the three sides of a triangle is 47 centimetres. If the first side is 3 cm. greater than the second and twice that of the third, what is the length of each ?

59. The sum of the three angles of a triangle is 180°. If the first angle is 22° greater than the second and 16° less than the third, what is the value of each ?

60. The sum of the three angles of a triangle is 180°. If the second angle is 20° greater than the first, and the third angle is greater than the sum of the first and second by 20°, what is the value of each ?

61. Three numbers are in the ratio of 1 : 2 : 3. If their sum is 30, find the numbers.

62. Three numbers are in the ratio of 3 : 5 : 8. If the sum of the greatest and the smallest exceeds twice the middle one by 2, find the numbers.

63. The population of a village is 2820. If the number of women exceeds that of men by 540 and the number of children exceeds that of women by 600, find the number of men, women and children in the village.

64. Uma is 9 years younger than Vimla and 6 years older than Prabha. If the sum of their ages is 51 years, what is the age of each ?

65. Rama is twice as old as Shyama and 5 years younger than Gopal. Five years ago the sum of the ages of Shyama and Gopal was 25 years. Find their ages.

66. A is 9 years younger than B and 14 years younger than C. The sum of half A's age, one-fifth of B's age and one-fourth of C's age is 30 years. Find their ages.

67. A man has four sons, and their ages differ from one another by 2 years. If the sum of their ages is 60 years, find the age of the youngest.

68. A boy has a certain number of rupees, twice as many eight-anna pieces and eight more two-anna pieces than eight-anna pieces. If the total value of all the coins is Rs. 55, find the number of coins of each kind.

69. A purse contains a certain number of rupees and the same number of eight-anna pieces. If their total value is Rs. 75, find the number of rupees.

70. A purse contains a certain number of sovereigns and three times as many shillings. If the total value is £ 20. 14s., find the number of coins of each kind.

71. A charity collection in a school amounts to 570 coins, consisting of one-anna pieces, two-anna pieces and four-anna pieces. There are twice as many two-anna pieces as one-anna pieces, the remaining coins being all four-anna pieces. If the total value is Rs. 76. 14a., find the number of four-anna pieces in the collection.

72. A purse contains rupees, eight-anna pieces, four-anna pieces, and two-anna pieces. The number of eight-anna pieces is 1 less than that of rupees, the number of four-anna pieces is three times that of rupees, and the number of two-anna pieces is 2 more than that of four-anna pieces. If the total value of the coins is Rs. 26, find the number of coins of each kind.

66. EXAMPLE 1. *The sum of three consecutive even numbers is 42 ; find the numbers.*

Let the first number be $2x$, then the next two consecutive even numbers will be $(2x+2)$ and $(2x+4)$. [see Article 60]

$$\therefore \quad 2x+(2x+2)+(2x+4)=42,$$
$$\therefore \quad 2x+2x+2+2x+4=42,$$
$$\therefore \quad 6x+6=42,$$
$$\therefore \quad 6x=36,$$
$$\therefore \quad x=6.$$

Hence the required numbers are 2×6, $(2\times 6+2)$ and $(2\times 6+4)$, *i.e.*, 12, 14 and 16.

EXAMPLE 2. *A number consists of two digits ; the digit in the units place is 2 and the sum of the digits is $\frac{1}{6}$ of the number. Find the number.*

Let the digit in the tens place be x.

Since the digit in the units place is 2, the number is $(10\times x+2)$. [see Article 61]

Now the sum of the digits $(x+2)$ is equal to $\frac{1}{8}$ of the number $(10x+2)$.

$$(x+2)=\tfrac{1}{8}(10x+2),$$

$$x+2=\frac{10x}{8}+\frac{2}{8},$$

$$x+2=\frac{5x}{4}+\frac{1}{4}.$$

Multiplying both the sides by 4,

$$4x+8=5x+1,$$

$$4x-5x=1-8,$$

$$\therefore \quad -x=-7,$$

$$\therefore \quad x=7.$$

The digit in the tens place is therefore 7, and hence the required number is 72.

EXAMPLE 3. *A number has two digits ; the digit in the units place is twice the digit in the tens place. If the digits be reversed the number is increased by* 18. *Find the number.*

Let the digit in the tens place be x, then the digit in the units place will be $2x$.

Therefore the number $=10\times x+2x=10x+2x=12x$.

If the digits be reversed, the number $=10\times 2x+x=20x+x=21x$.

$\therefore$ $21x$ exceeds $12x$ by 18.

$$i.e.,\ 21x-12x=18,$$

$$\therefore \quad 9x=18,$$

$$\therefore \quad x=2.$$

The digit in the tens place is therefore 2 and the digit in the units place is 2×2 *i.e.*, 4. Hence the required number is 24.

EXAMPLES XXXVIII

1. The sum of three consecutive numbers is 24 ; find the numbers.
2. The sum of four consecutive numbers is 50 ; find the numbers.
3. The sum of two consecutive even numbers is 26 ; find the numbers.
4. The sum of three consecutive even numbers is 30 ; find the numbers.
5. The sum of four consecutive even numbers is 84 ; find the numbers.

6. The sum of two consecutive odd numbers is 28 ; find the numbers.

7. The sum of three consecutive odd numbers is 63 ; find the numbers.

8. The sum of five consecutive odd numbers is 55 ; find the numbers.

9. Find three consecutive numbers such that if three times the smallest be added to twice the greatest, the sum equals 34.

10. Find three consecutive numbers such that three times the middle one exceeds the sum of the first and third by 26.

11. A number has two digits whose sum is 7 ; if 9 be added to the number, the digits are reversed ; find the number.

12. A number has two digits whose sum is 11 ; if 27 be added to the number, the digits are reversed ; find the number.

13. A number has two digits whose sum is 9 ; if 9 be subtracted from the number, the digits are reversed ; find the number.

14. A number lies between 10 and 100 ; the sum of its digits is 10 ; if 18 be subtracted from the number, the digits are reversed ; find the number.

(Since the number lies between 10 and 100, it will have two digits).

15. A number consists of two digits ; the digit in the units place is twice the digit in the tens place ; if the digits are reversed, the number will be increased by 27 ; find the number.

16. A number consists of two digits ; the digit in the tens place is three times the digit in the units place ; if the digits are reversed, the number will be diminished by 36 ; find the number.

17. A number consists of two digits ; the digit in the units place is twice the digit in the tens place ; if the digits are reversed, the new number will be less than twice the original number by 12 ; find the number.

18. A number consists of two digits ; the digit in the tens place exceeds the digit in the units place by 3, and the number is sixteen times the digit in the units place ; find the number.

19. A number consists of two digits ; the digit in the units place is 3, and the sum of the digits is $\frac{1}{7}$ of the number ; find the number.

20. A number consists of two digits ; the digit in the units place is twice the digit in the tens place, and the number exceeds twice the digit in the units place by 24 ; find the number.

21. A number consists of three digits ; the digit in the units place is three times the digit in the hundreds place ; the digit in the tens place is twice the digit in the hundreds place ; and the sum of the three digits is 12 ; find the number.

22. A number consists of three digits; the digit in the tens place is three times the digit in the units place, and the digit in the hundreds place exceeds the digit in the tens place by 2; if the digits are reversed, the number will be diminished by 396; find the number.

67. EXAMPLE 1. *Two men start at the same time from Dehra-Dun and Mussoorie, 15 miles apart, and walk towards each other at the rates of $3\frac{1}{2}$ and 4 miles an hour respectively. When and where will they meet?*

Let x represent the number of hours when they meet.

Since the first man in 1 hour walks 4 miles,
∴ ,, ,, ,, ,, x hours ,, $4x$,, ,
and the second ,, ,, 1 hour ,, $\frac{7}{2}$,, ,
∴ ,, ,, ,, ,, x hours ,, $\frac{7}{2}x$,, ,

Again, since both have walked for x hours before they meet, therefore in x hours the sum of the distances covered by them is $(4x+\frac{7}{2}x)$ miles, and this is equal to 15 miles,

$$\therefore \quad 4x+\tfrac{7}{2}x=15.$$

Multiplying by 2,

$$8x+7x=30,$$
$$\therefore \quad 15x=30,$$
$$\therefore \quad x=2.$$

Hence both meet after 2 hours, and the distance of their meeting place from Mussoorie is $4\times 2=8$ miles.

EXAMPLE 2. *A mail train leaves Aligarh for Allahabad, 320 miles distant, travelling at the rate of 46 miles an hour, and 2 hours later a passenger train leaves Allahabad for Aligarh. What is the speed of the passenger train if the two trains pass each other after 5 hours of the start of the mail train?*

Let the speed of the passenger train be x miles per hour.

The mail train in 1 hour goes 46 miles,
∴ ,, ,, ,, ,, 5 hours ,, $46\times 5=230$ miles.

Since the passenger train starts 2 hours later, therefore before passing the mail train, it has travelled for 3 hours.

The passenger train in 1 hour goes x miles,
∴ ,, ,, ,, ,, 3 hours ,, $3x$ miles.

Now the sum of the distances covered by both the trains before passing each other is $(230+3x)$ miles, and this is equal to 320 miles.

$$230+3x=320,$$
$$\therefore \quad 3x=320-230,$$
$$\therefore \quad 3x=90,$$
$$\therefore \quad x=30.$$

Hence, the speed of the passenger train is 30 miles per hour.

EXAMPLE 3. *Rama can do a piece of work in 4 days, Lakshman in 6 days and Krishna in 12 days ; in what time will they do it working together ?*

Let x represent the number of days in which they can do the work working together ?

Rama in	4 days	can	do	the whole work,	
∴ ,, ,,	1 day	,,	,,	$\frac{1}{4}$ of the work,	
,,	x days	,,	,,	$\frac{x}{4}$,, ,,	
Lakshman ,,	6 days	,,	,,	the whole work,	
∴ ,, ,,	1 day	,,	,,	$\frac{1}{6}$ of the work,	
∴ ,,	x days	,,	,,	$\frac{x}{6}$,, ,,	
And Krishna ,,	12 days	,,	,,	the whole work,	
∴ ,, ,,	1 day	,,	,,	$\frac{1}{12}$ of the work,	
,,	x days	,,	,,	$\frac{x}{12}$,, ,, ,,	

Since the whole work is 1,

$$\frac{x}{4}+\frac{x}{6}+\frac{x}{12}=1.$$

Multiplying by 12,

$$3x+2x+x=12,$$
$$\therefore \quad 6x=12,$$
$$\therefore \quad x=2.$$

Hence, the three can do the work in 2 days.

EXAMPLE 4. *I spent* Rs. 26 *in buying wheat at* 6 *seers per rupee and rice at* $3\frac{1}{2}$ *seers per rupee. I bought* 116 *seers altogether ; how many seers of each did I buy ?*

Let x represent the number of seers of wheat which I bought, then $(116-x)$ will be the number of seers of rice.

The price of 6 seers of wheat is Re. 1,

1 seer ,, ,, Re. $\frac{1}{6}$,

and ,, x seers ,, ,, Rs. $\frac{x}{6}$.

Again ,, $\frac{7}{2}$ seers of rice is Re. 1,

1 seer ,, ,, Re. $\frac{2}{7}$,

and ,, ,, $(116-x)$ seers ,, ,, Rs. $\frac{2}{7}\times(116-x)$.

Since both wheat and rice cost Rs. 26,

$$\frac{x}{6}+\frac{2}{7}(116-x)=26.$$

Multiplying both sides by 42,

$$7x+12(116-x)=42\times 26,$$
$$7x+1392-12x=1092,$$
$$\therefore\quad 7x-12x=1092-1392,$$
$$\therefore\quad -5x=-300,$$
$$\therefore\quad x=60.$$

Hence, I bought 60 seers of wheat and $(116-60)=56$ seers of rice.

EXAMPLE 5. *The length of a room is 2 yards greater than its breadth; if its length were increased by 3 yards and its breadth diminished by 2 yards, the area would remain the same. Find the dimensions of the room.*

Let x yards be the length of the room, then $(x-2)$ yards will be its breadth and $x\times(x-2)$ sq. yd. its area.

If the length be increased by 3 yards *i.e.*, the length be $(x+3)$ yards, and the breadth be diminished by 2 yards, *i.e.*, the breadth be $(x-2-2)$ *i.e.*, $(x-4)$ yards, its area would be $(x+3)\times(x-4)$ sq. yd.

Since the two areas are equal,

$$\therefore\quad x(x-2)=(x+3)(x-4).$$

Removing brackets,

$$x^2-2x=x^2-4x+3x-12,$$
$$x^2-2x-x^2+4x-3x=-12,$$
$$\therefore\quad -x=-12,$$
$$\therefore\quad x=12.$$

Hence, the length of the room is 12 yards and breadth $(12-2)=10$ yards.

EXAMPLE 6. *I invested a certain sum at 5% per annum and double that sum at 4% per annum, simple interest. If I get* Rs. 65 *as interest every year, how much did I invest at the two rates?*

Let x represent the number of rupees which I invested at 5% per annum, then $2x$ will represent the number of rupees invested at 4% per annum.

Since the interest on Rs. 100 for 1 year is Rs. 5,

,, Re. 1 ,, ,, Re. $\frac{5}{100}$ = Re. $\frac{1}{20}$,

,, Rs. x ,, ,, Rs. $\frac{x}{20}$.

Again, since ,, ,, Rs. 100 ,, ,, Rs. 4,

,, Re. 1 ,, ,, Re. $\frac{4}{100}$ = Re. $\frac{1}{25}$,

,, Rs. $2x$,, ,, Rs. $\frac{2x}{25}$.

Therefore the total interest for 1 year is Rs. $\left(\frac{x}{20}+\frac{2x}{25}\right)$ and this is equal to Rs. 65,

$$\frac{x}{20}+\frac{2x}{25}=65.$$

Multiplying both sides by 100,

$$5x+4x=6500,$$

$$\therefore\quad 13x=6500,$$

$$\therefore\quad x=500.$$

Hence, I invested Rs. 500 at 5% per annum and Rs. 1000 at 4% per annum.

EXAMPLES XXXIX

1. Two persons start at the same time from Cawnpore and Lucknow, 45 miles apart; and proceed towards each other at the rates of 10 and 5 miles an hour respectively; when and where will they meet?

2. Two men start at the same time from two places A and B, 36 miles apart and proceed towards each other; if the first walks at the rate of $3\frac{1}{2}$ miles an hour and meets the second in 6 hours, what is the speed of the second man?

3. A motor car leaves Cawnpore for Delhi, 365 miles distant, and runs at the rate of 38 miles an hour; at the same instant another leaves Delhi for Cawnpore and runs at the rate of 35 miles an hour; when and where will they pass each other?

4. Two trains starting at the same time, run towards each other from two stations 420 miles apart and meet in 7 hours ; if the speed of one is double the other, find their speeds.

5. A train leaves Aligarh for Bareilly, 100 miles distant, and travels at the rate of 28 miles an hour. One hour later another train leaves Bareilly for Aligarh and travels at the rate of 20 miles an hour. When and where will they pass each other ?

6. A train leaves a station and travels at the rate of 20 miles an hour ; 2 hours later another train leaves the same station, in the same direction, and travels at the rate of 30 miles an hour ; when will the second train overtake the first ?

7. A man starts from a place and walks at the rate of 3 miles an hour ; 2 hours later another man leaves the same place on horse-back and rides at the rate of 5 miles an hour ; when will the second man overtake the first ?

8. Two boys start at the same time on cycles from two places 35 miles apart and proceed towards each other meeting in 2 hours ; if their speeds are in the ratio of 4 : 3, find their rates per hour.

9. A man walks the first third of a journey at the rate of 4 miles an hour, the second third at the rate of 5 miles an hour, and the third third at the rate of 6 miles an hour, completing the whole journey in 6 hours and 10 minutes ; find the length of the journey.

10. A man travels one-fourth of a journey on foot at the rate of 4 miles an hour, one-third on a cycle at the rate of 12 miles an hour and the rest on horse-back at the rate of 10 miles an hour, completing the whole journey in 6 hours and 20 minutes ; find the length of the journey.

11. A train goes from A to B in 3 hours. Had its speed been 12 miles per hour faster, it would have completed the journey in 1 hour less. Find the distance between A and B.

12. A motor car covers a journey in 3 hours and 30 minutes. Had its speed been 8 miles an hour slower, it would have taken 40 minutes more to complete the journey. Find the length of the journey.

13. Mohan can do a piece of work in 5 days and Sohan can do it in 7 days ; how long will they take if both work together ?

14. A can do a piece of work in 6 days, B can do it in 9 days and C in 18 days ; how long will they take if all work together ?

15. Rama and Shyama can do a piece of work in 14 days, which Rama alone can do in 20 days ; in how many days can Shyama alone do it ?

16. A, B and C can do a piece of work in 2 days, which A alone can do in 4 days and B alone in 20 days ; In how many days can C alone do it ?

17. Mahendra and Brajendra can do a piece of work in 2 days ; Brajendra and Ramendra can do it in 3 days ; Ramendra and Mahendra can do it in 6 days ; in how many days can all do it working together ?

18. Two pipes can fill a cistern in 20 minutes and 40 minutes respectively ; a waste pipe can empty it in 30 minutes ; if the three pipes be opened at the same time, in what time will the cistern be filled ?

19. A man spent Rs. 22 in buying barley at 8 seers per rupee and gram at 7 seers per rupee. He bought 166 seers altogether ; how many seers of each grain did he buy ?

20. I bought 55 yards of silk and cotton cloths for Rs. 134. 1a. If the silk cloth cost Rs. 4 per yard and the cotton cloth 9a. per yard, how many yards of each did I buy ?

21. A man bought 45 lbs. of tea for Rs. 73. 12a. If a part were bought for Re. 1. 8a. per lb and the rest for Re. 1. 12a. per lb, how much was bought at the two prices ?

22. A teacher distributed Rs. 9. 11a. among 35 students giving 4a. each to some and 5a. each to the rest. How many students got 5a. each ?

23. If the length of a square room were increased by 2 yards and the breadth decreased by 2 yards, its area would be diminished by 4 sq. yds. ; find its area.

24. The length of a room is 5 feet greater than its breadth ; if both length and breadth were increased by 5 feet, the area would be increased by 250 sq. ft. ; find its area.

25. One room is 22 yards long and another 20 yards long ; the breadth of the first is 2 yards less than that of the second, and the sum of their areas is equal to 712 sq. yds. ; find the breadth of each.

26. The length of a field is 300 yards greater than its breadth ; had the length been 100 yards less and the breadth 50 yards more, the area would have been the same. Find the dimensions of the field.

27. The length of a room is 20 ft. greater than its breadth ; if the length were diminished by 5 ft. and the breadth increased by 10 ft., the area would increase by $33\frac{1}{3}$ sq. yds. Find the length of the room.

28. The simple interest at 5% per annum on a sum of money is equal to the simple interest at 4% per annum on another sum of money which is £ 100 greater than the first ; find the two sums.

29. A man invested a portion of Rs. 1000 at 5% per annum and the rest at 4%. If he gets Rs. 46 as interest every year, how much did he invest at 4% ?

30. I lent out a certain sum of money at 6% per annum simple interest. If the interest of 20 years exceeds the original sum by Rs. 40, find the sum.

31. A money-lender lent one-third of his wealth at 3% per annum, one-fourth at $3\frac{1}{2}$% per annum and the rest at 4% per annum, simple interest. If he receives Rs. 42. 8a. as interest every year, how much did he lend ?

32. I spent half of my money in purchasing rice, one-third of the rest in purchasing wheat and three-fourth of the rest in purchasing gram. If I had Rs. 2 left, how many rupees had I at first ?

33. A labourer is engaged for 30 days on the condition that he will be paid 7a. for every day he works and fined 3a. for every day he is absent ; if he gets Rs. 10 in all, for how many days was he absent ?

34. Murari has Rs. 15 more than Behari. If Murari gives one-third of his money to Behari and Behari gives one-fifth of what he will then have to Murari, then Murari will have Re. 1 more than Behari. How much had each at first ?

35. I purchased cotton and woollen cloths for Rs. 750, and sold the cotton cloth at a gain of 30% and the woollen cloth at a gain of 20%, and thus gained $26\frac{2}{3}\%$ on the whole. How much did I spend in purchasing each kind of cloth ?

36. I have some nuts ; if I give 10 nuts to each student of my class I have 20 nuts left over but if I give 11 to each, I have 13 short ; how many students are there in my class ?

CHAPTER XIV

SIMULTANEOUS EQUATIONS

68. When a single equation containing two unknown quantities is given, we can find an indefinite number of pairs of values of the unknown quantities which will satisfy that equation. Consider, for example, the equation

$$5x + y = 14. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This equation contains two unknown quantities x and y. If we regard y as an unknown quantity and x as a known quantity, we can express y in terms of x thus

$$y = 14 - 5x.$$

From this equation we see that for every value we choose to give to x we get a corresponding value of y. Thus, if

$$\begin{aligned} x &= 0, \quad y = 14\ ; \\ x &= 1, \quad y = 9\ ; \\ x &= 2, \quad y = 4\ ; \\ &\ldots\ldots\ldots\ldots \\ x &= -1, \quad y = 19\ ; \\ x &= -2, \quad y = 24\ ; \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

and so on.

Now, if for every value of x we take the corresponding value of y, we shall get an infinite number of values of x and y which will satisfy the equation, *i.e.*, we shall have an infinite number of solutions of the equation. Thus, for instance, if in equation (1), we write 0 for x and 14 for y, the left side $= 5 \times 0 + 14 = 14$, which is equal to the right side. Again, if in equation (1), we write 1 for x and 9 for y, the left side $= 5 \times 1 + 9 = 14$, which is equal to the right side. Similarly, by giving different values to x and y in equation (1) we can see that the equation is satisfied.

In the same way, if we are given another equation expressing a different relation between the two unknown quantities x and y, such as

$$15x - 2y = -3, \quad \text{..........................(2)}$$

we can, by regarding y as an unknown quantity, write it thus

$$-2y = -3 - 15x.$$

Multiplying by -1, $\quad 2y = 3 + 15x.$

Dividing by 2, $\quad y = \frac{3 + 15x}{2} \quad \text{........................(3)}$

Now, as before, by giving different values to x and thus obtaining corresponding values of y, if we substitute them in (2), we shall see that the equation is satisfied.

Thus, we see that, if we are given only one equation involving two unknown quantities, we can find *as many pairs of values of the unknown quantities as we please* which will satisfy that given equation. In such a case, the solution is said to be **indeterminate.** If, however, we are given two equations each involving two unknown quantities, such as

$$5x + y = 14, \quad \text{..............................(1)}$$

$$\text{and } 15x - 2y = -3, \quad \text{..........................(2)}$$

we cannot find indefinite pairs of values of the unknown quantities satisfying both of them simultaneously, but we can find only *one pair*.

Expressing y in terms of x, equations (1) and (2) can be written as

$$y = 14 - 5x,$$

$$\text{and } y = \frac{3 + 15x}{2}. \quad \text{........................[see (3)].}$$

If the equations are to be satisfied by the same values of x and y, the two values of y must be equal.

Hence, $\quad 14 - 5x = \frac{3 + 15x}{2}. \quad \text{..........................(4)}$

Multiplying by 2,

$$28 - 10x = 3 + 15x,$$
$$-10x - 15x = 3 - 28,$$
$$\therefore \quad -25x = -25,$$
$$\therefore \quad x = 1.$$

If we substitute this value of x in *either* of the given equations, say (1), we have

$$5 \times 1 + y = 14,$$
$$\therefore \quad y = 9.$$

Hence, $x = 1$, $y = 9$ is *the only solution* possible if both the equations are to be satisfied by the same pair of values of x and y.

Verification. Writing $x=1$ and $y=9$ in equation (1),

$$5 \times 1 + 9 = 14,$$
$$\therefore \quad 14 = 14.$$

Since both the sides are equal, the equation (1) is satisfied.

Again, writing $x=1$ and $y=9$ in equation (2),

$$15 \times 1 - 2 \times 9 = -3,$$
$$\therefore \quad 15 - 18 = -3,$$
$$\therefore \quad -3 = -3.$$

Since both the sides are equal, the equation (2) is also satisfied.

Thus we see that the solution of the two equations taken separately is *indeterminate*, but when taken *simultaneously*, is *determinate.*

When two or more equations are satisfied by the same values of the unknown quantities contained in them, they are called **simultaneous equations.**

NOTE. It should be noted that two equations can be solved simultaneously if they are *independent*. For example, the equations $5x + y = 14$, and $15x + 3y = 42$ are not independent equations, for the second equation is deduced from the first by multiplying both the sides of the equation by 3. These cannot be solved as each will give the same pairs of values of the unknown quantities.

69. We have seen in the previous article that in solving equations (1) and (2) in x and y, we got equation (4), which does not contain y. This process of combining the two equations so as to get rid of one of the unknown quantities is called **elimination**. In the above equations we first expressed y in terms of x and other quantities, and then equating the two values of y obtained from the two equations, we *eliminated* y.

70. There are various ways of eliminating one of the unknown quantities from a pair of equations. These will be explained below but it would be better to eliminate that unknown quantity which has the smaller coefficient in the two equations.

EXAMPLE 1. *Solve* $5x+y=14$, ...(1)

$15x-2y=-3$. ....(2)

Multiplying (1) by 3,

$15x+3y=42$. ....(3)

We now see that the coefficients of x in equations (2) and (3) are equal and *have the same sign*. Hence, subtracting (2) from (3),

$$5y=45,$$
$$\therefore\quad y=9.$$

Substituting this value of y in (2),

$$15x-2\times 9=-3,$$
$$\therefore\quad 15x=18-3,$$
$$\therefore\quad 15x=15,$$
$$\therefore\quad x=1.$$

Hence, the solution is $x=1$, $y=9$.

EXAMPLE 2. *Solve* $x+3y=3$,

$4x-5y=29$.

Multiplying the first equation by 5 and the second by 3,

$5x+15y=15$, ...(1)

$12x-15y=87$. ....(2)

We now see that the coefficients of y in the two equations are equal but *have opposite signs*. Hence, adding (1) and (2),

$$17x=102,$$
$$\therefore\quad x=6.$$

Substituting this value of x in the first equation,

$$6+3y=3,$$
$$\therefore \quad 3y=3-6,$$
$$\therefore \quad 3y=-3,$$
$$\therefore \quad y=-1.$$

Hence, the solution is $x=6$, $y=-1$.

From the above examples we see that there are two methods of eliminating one unknown quantity between two equations :

(i) *Multiply, if necessary, both the equations by such numbers as will make the coefficient of one of the unknown quantities to be eliminated equal, and then subtract the equations, if the signs of the coefficients are like, and add, if they are unlike.*

(ii) *Express in the two equations the values of one unknown quantity in terms of the other and then equate the two values of that unknown quantity thus expressed.*

By any of the above processes, a new equation is obtained involving only one unknown quantity by solving which the value of that unknown quantity can be found. When one of the unknown quantities has thus been found, substitute this value in *either* of the given equations to find the other unknown quantity.

The second method is called the method of elimination by **substitution.**

NOTE. The student is advised to apply any of the methods best suited to the particular equations.

EXAMPLES XL

Solve

1. $x+y=8$,
$x-y=3$.

2. $x+y=30$,
$x-y=6$.

3. $x+y=48$,
$-x+y=12$.

4. $x+y=31$,
$x-y=-1$.

5. $x+y=\frac{7}{2}$,
$x-y=\frac{1}{2}$.

6. $x+y=8$,
$x-y=8$.

7. $x+y=3$,
$x-y=-3$.

8. $x+y=a$,
$x-y=b$.

9. $x+y=47$,
$-x+y=13$.

10. $2x+3y=9$, $2x-3y=3$.

11. $4x+3y=10$, $4x-3y=-2$.

12. $x+2y=12$, $x-3y=2$.

13. $4x-y=10$, $2x-y=4$.

14. $x+2y=7$, $x-y=1$.

15. $2x+y=14$, $x+2y=13$.

16. $y-x=5$, $2y-3x=9$.

17. $x+y=0$, $2x-y=9$.

18. $6x+8y=35$, $-2x+y=3$.

19. $x-4y=0$, $x-8y=4$.

20. $3y+x=14$, $x-2y=-1$.

21. $x+y=3$, $4y-3x=-17$.

22. $2x+y=-12$, $x-7y=9$.

23. $x-3y=-2$, $4x-9y=7$.

24. $5x+3y=13$, $3x+5y=11$.

25. $5x-6y=17$, $x-y=10$.

26. $x-4y=0$, $3x-8y=-30$.

27. $7x=9y$, $x-y=2$.

28. $3x=4y$, $5x-2y=14$.

29. $x=y+2$, $2x-y=7$.

30. $5x=5-y$, $7x=y+13$.

31. $4x=1-2y$, $y=3-x$.

32. $3x=24-4y$, $6y=5x-2$.

33. $x+y=10$, $x=15$.

34. $2x+3y=4$, $x=-1$.

35. $3x-5y=2$, $y+2=1$.

36. $x+20=-35$, $4x-5y=-520$.

37. $x=a$, $y+2x=5a$.

38. $2x+y=3a$, $5x-3y=2a$,

39. $\cdot 5x+\cdot 25y=0$, $3\cdot 5x+\cdot 5y=9$.

40. $3x-2y=-1\cdot 5$, $5x+3y=35\cdot 5$.

41. $\cdot 4x+\cdot 2y=2$, $\cdot 3x-\cdot 2y=\cdot 01$.

42. $x+\cdot 2y=\cdot 6$, $3\cdot 4x-\cdot 02y=\cdot 01$.

71. Sometimes it will be found necessary to simplify the equations before proceeding to apply any of the methods of solution.

EXAMPLE 1. *Solve*

$$7x-2y=y+5=5x+3y-11.$$

Since three expressions are equated to one another, we may take either the first with the second or the second with the third or the third with the first.

Taking the first with the second,

$$7x-2y=y+5. \quad \text{..........(1)}$$

Taking the second with the third,

$$y+5=5x+3y-11. \quad \text{..........(2)}$$

Taking the third with the first,

$$5x+3y-11=7x-2y. \quad \text{..........(3)}$$

Thus we get three equations, and by solving any two of these we can find the values of the unknown quantities.

Taking (1) and (2),

$$7x-2y=y+5, \quad \ldots\ldots(1)$$
$$y+5=5x+3y-11. \quad \ldots\ldots(2)$$

From (1), $7x-2y-y=5,$

$$\therefore \quad 7x-3y=5. \quad \ldots\ldots(4)$$

From (2), $y-5x-3y=-11-5,$

$$\therefore -2y-5x=-16,$$
$$\therefore \quad 5x+2y=16. \quad \ldots\ldots(5)$$

Multiplying (4) by 2, and (5) by 3,

$$14x-6y=10, \quad \ldots\ldots(6)$$
$$15x+6y=48. \quad \ldots\ldots(7)$$

Adding (6) and (7),

$$29x=58,$$
$$\therefore \quad x=2.$$

Substituting this value of x in (4),

$$7\times2-3y=5,$$
$$\therefore \quad -3y=5-14,$$
$$\therefore \quad -3y=-9,$$
$$\therefore \quad y=3.$$

Hence, the solution is $x=2$, $y=3$.

EXAMPLE 2. *Solve*

$$5(x-2)+3(y-3)=27, \quad \ldots\ldots(1)$$
$$3x+5(y+1)=55. \quad \ldots\ldots(2)$$

Simplifying (1),

$$5x-10+3y-9=27,$$
$$\therefore \quad 5x+3y=27+10+9,$$
$$\therefore \quad 5x+3y=46. \quad \ldots\ldots(3)$$

Simplifying (2),

$$3x+5y+5=55,$$
$$\therefore \quad 3x+5y=50. \quad \ldots\ldots(4)$$

Multiplying (3) by 3, and (4) by 5,

$$15x+9y=138, \quad \ldots\ldots(5)$$
$$15x+25y=250. \quad \ldots\ldots(6)$$

Subtracting (6) from (5),

$$9y-25y=138-250,$$
$$\therefore\quad -16y=-112,$$
$$\therefore\quad y=7.$$

Substituting this value of y in (3),

$$5x+3\times 7=46,$$
$$\therefore\quad 5x=46-21,$$
$$\therefore\quad 5x=25,$$
$$\therefore\quad x=5.$$

Hence, the solution is $x=5$, $y=7$.

EXAMPLE 3. *Solve*

$$\frac{x}{3}-\frac{y}{4}=1, \qquad \text{(1)}$$
$$\frac{x}{8}+\frac{y}{3}=\frac{25}{12} \qquad \text{(2)}$$

Multiplying both sides of (1) by 12,

$$4x-3y=12. \qquad \text{(3)}$$

Multiplying both sides of (2) by 24,

$$3x+8y=50. \qquad \text{(4)}$$

Multiplying (3) by 3, and (4) by 4,

$$12x-9y=36, \qquad \text{(5)}$$
$$12x+32y=200. \qquad \text{(6)}$$

Subtracting (6) from (5),

$$-9y-32y=36-200,$$
$$\therefore\quad -41y=-164,$$
$$\therefore\quad y=4.$$

Substituting this value of y in (3),

$$4x-3\times 4=12,$$
$$\therefore\quad 4x=12+12,$$
$$\therefore\quad 4x=24,$$
$$\therefore\quad x=6.$$

Hence, the solution is $x=6$, $y=4$.

72. In some equations it is better to consider certain expressions involving x and y as unknown quantities, instead of considering x and y as unknown quantities; thus, for instance, in the following example $\frac{1}{x}$ and $\frac{1}{y}$ may be considered as unknown quantities.

EXAMPLE. *Solve.*

$$\frac{3}{x}+\frac{6}{y}=4, \quad \text{.......... (1)}$$

$$\frac{9}{x}-\frac{2}{y}=2. \quad \text{.......... (2)}$$

Multiplying (2) by 3,

$$\frac{27}{x}-\frac{6}{y}=6. \quad \text{.......... (3)}$$

Adding (1) and (3),

$$\frac{3}{x}+\frac{27}{x}=4+6,$$

$$\therefore \quad \frac{3}{x}+\frac{27}{x}=10.$$

Multiplying both sides by x,

$$3+27=10x,$$

$$\therefore \quad 10x=30,$$

$$\therefore \quad x=3.$$

Substituting this value of x in (1),

$$\frac{3}{3}+\frac{6}{y}=4,$$

$$\therefore \quad \frac{6}{y}=4-1$$

$$\therefore \quad \frac{6}{y}=3.$$

Multiplying both sides by y,

$$6=3y,$$

$$\therefore \quad y=2.$$

Hence, the solution is $x=3$, $y=2$.

EXAMPLES XLI

Solve

1. $x-y=2x-3y=6.$
2. $x+y-10=15-3x=0.$
3. $3x+3y=2x-56=y.$
4. $5x-4y+2=x-y+2=0.$
5. $5y=2y-x=20-5y.$
6. $3x-y=5x+2y-13=3.$
7. $x+y=1-6x-4y=11-4x-6y.$
8. $3x-4y-15=7y-6x=9y-4x+10.$
9. $2(y-3)+2(x-3)=0,$ $4(x-3)+3(y-1)=0.$
10. $7x+(5-y)=32,$ $(x+7)-12y=-12.$
11. $4(x+2)+3(y+3)=36,$ $7(x+3)-2(y+2)=14.$
12. $7(x-4)=y-9,$ $3(4y-3)=x+10.$
13. $2(x-2y)+5y-2x=3,$ $4(3y-x)=4-(y-x).$
14. $(7x+2)-6(y-2)=-2,$ $(7y+3)-6(x+2)=-3.$
15. $8(3x+1)+28(2x-y)-7(2y-x)=0,$ $20(2x-1)-15(4y-5x)-6(x+y)=0.$
16. $12(x-2)=20(10-x)+15(y-10),$ $16(y+2)=3(2x+y)+6(x+13).$
17. $\frac{x}{2}+\frac{y}{3}=6,$ $\frac{x}{2}-\frac{y}{3}=2,$
18. $\frac{x}{2}+\frac{y}{4}=6,$ $\frac{x}{5}-\frac{y}{2}=0.$
19. $\frac{x}{2}+\frac{y}{3}=3,$ $x+y=7.$
20. $\frac{x}{3}+\frac{y}{8}=12,$ $\frac{y}{4}-\frac{x}{3}=6.$
21. $\frac{x}{3}-\frac{y}{2}=4,$ $\frac{5x}{3}-\frac{y}{2}=12.$
22. $\frac{1}{x}+\frac{1}{y}=7,$ $\frac{1}{x}-\frac{1}{y}=1.$
23. $\frac{1}{x}-\frac{1}{y}=\frac{1}{2},$ $\frac{1}{x}+\frac{3}{y}=\frac{5}{2}.$
24. $\frac{6}{x}+\frac{3}{y}=15,$ $\frac{1}{x}+\frac{3}{y}=5.$
25. $\frac{3}{x}-\frac{2}{y}=\frac{1}{2},$ $\frac{3}{x}+\frac{4}{y}=2.$
26. $\frac{5}{x}+\frac{6}{y}=48,$ $\frac{2}{x}-\frac{3}{y}=3.$
27. $\frac{4}{x}+\frac{3}{y}=3,$ $\frac{1}{x}-\frac{1}{y}=\frac{1}{6}.$
28. $\frac{3}{x}+\frac{8}{y}=3;$ $\frac{15}{x}-\frac{4}{y}=4.$

29. $\frac{3}{x}+\frac{4}{y}=2,$
$\frac{9}{x}-\frac{4}{y}=2.$

30. $\frac{3}{x}-\frac{4}{y}=-4,$
$\frac{4}{x}-\frac{5}{y}=-3.$

31. $\frac{5}{x}-\frac{2}{y}=7,$
$\frac{1}{x}+\frac{1}{y}=0.$

32. $\frac{1}{x}+3y=13,$
$\frac{2}{x}-y=5.$

33. $\frac{5}{x}-2y=9,$
$\frac{3}{x}+4y=8.$

34. $2x+\frac{3}{y}=4,$
$5x-\frac{2}{y}=-6.$

35. $4x+\frac{1}{y}=-2,$
$5x-\frac{2}{y}=7\frac{1}{4}.$

36. $\frac{x+y}{3}+\frac{x-y}{2}=5,$
$\frac{x+y}{2}+\frac{x-y}{4}=\frac{13}{2}.$

37. $\frac{y-5}{2}-\frac{1-3x}{3}=\frac{x+2y}{6},$
$\frac{7(2x-1)}{2}=3y-10\frac{1}{2}.$

38. $\frac{2(x-2y)}{3}-\frac{5x+3y}{6}=5,$
$\frac{3x+7y}{2}-\frac{x+7y}{3}=2.$

39. $\frac{x-y}{3}=\frac{2x+1}{2}=\frac{y}{2}.$

40. $\frac{2x-y}{2}-\frac{x+3}{2}=-3,$
$\frac{7(2y-3x)}{2}-1=9x-y.$

CHAPTER XV

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

73. We have seen in Article 68 that, to find the values of two unknown quantities we should be given two equations involving those unknown quantities, for one equation involving two unknown quantities will give us an indefinite number of pairs of values for those unknowns which will satisfy the equation. Hence, in problems involving two unknown quantities, two independent equations containing those unknown quantities should be obtained from the conditions of the problem in order to find their values.

74. In Article 65, the problems involving two or more unknown quantities were so easy and the given relations so simple that it was easy to find only one equation giving one of the unknown quantities, and, having found one of the unknowns, the other was easily obtained. In complex problems it is advisable to represent each unknown quantity by a different letter and to express every statement in the form of an algebraic equation. By doing so, we can obtain as many equations as there are independent statements, and, by solving these, we can find the values of the unknown quantities.

EXAMPLE 1. *A number exceeds another by* 4, *and their sum is* 36. *Find the numbers.*

This problem has two statements :

(i) One number exceeds another by 4.

(ii) The sum of the numbers is 36.

Let x be one of the numbers and y the other, x being greater than y.

From the first statement, since x exceeds y by 4,

$$x-y=4. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From the second statement, the sum of x and y equals 36,

$$x+y=36. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Adding (1) and (2),

$$2x = 40,$$
$$x = 20.$$

Substituting this value of x in (1),

$$20 - y = 4,$$
$$\therefore \quad -y = 4 - 20,$$
$$\therefore \quad -y = -16,$$
$$\therefore \quad y = 16.$$

Hence, the two numbers are 20 and 16.

NOTE. This problem can also be solved by regarding one of the numbers as an unknown quantity and expressing the other in terms of this number. (See Article 65, Example 1.)

EXAMPLE 2. *Shafi has three times as much money as Rafi. If Shafi gives* Rs. 250 *to Rafi, Rafi will have twice as much as Shafi. How much had each at first ?*

This problem has two statements :

(i) Shafi has three times as much money as Rafi.

(ii) If Shafi gives Rs. 250 to Rafi, Rafi will have twice as much as Shafi.

Suppose Shafi has x rupees and Rafi y rupees, then by the first statement,

$$x = 3y. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If Shafi gives Rs. 250 to Rafi, Shafi will have $(x - 250)$ rupees and Rafi will have $(y + 250)$ rupees, then by the second statement,

$$(y + 250) = 2(x - 250).$$

Removing brackets,

$$y + 250 = 2x - 500,$$
$$\therefore \quad y - 2x = 750. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equation (1) can be written as

$$x - 3y = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Multiplying (3) by 2,

$$2x - 6y = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Adding (2) and (4),

$$-5y = -750,$$
$$\therefore \quad y = 150.$$

Substituting this value of y in (1),

$$x = 3 \times 150 = 450.$$

Hence, Shafi had Rs. 450 and Rafi had Rs. 150.

NOTE. This problem has also been solved by regarding one of the numbers as an unknown quantity and expressing the other number in terms of this number. (See Article 65, Example 2.)

EXAMPLE 3. *A man is five times as old as his son; 24 years hence the son's age will be equal to the father's present age; find their present ages.*

Let x represent father's age and y son's age in years, then by the first statement,

$$x=5y. \quad \ldots\ldots(1)$$

24 years hence the son's age will be $(y+24)$ years and this age is equal to the father's present age,

$$\therefore \quad x=y+24. \quad \ldots\ldots(2)$$

Subtracting (2) from (1),

$$0=5y-y-24,$$

$$-5y+y=-24,$$

$$\therefore \quad -4y=-24$$

$$\therefore \quad y=6.$$

Substituting this value of y in (1),

$$x=5\times 6=30.$$

Hence, the father's present age is 30 years and the son's present age is 6 years.

EXAMPLES XLII

1. Find two numbers whose sum is 40, and whose difference is 8.

2. Find two numbers whose sum is 63, and whose difference is 9.

3. Twice the sum of two numbers is 60, and their difference is 8; find the numbers.

4. Three times the sum of two numbers is 105, and twice their difference is 10; find the numbers.

5. The difference of two numbers is 10; if 8 be added to the greater number, the sum is equal to three times the smaller number; find the numbers.

6. The sum of two numbers is 80, and three times one number is equal to five times the other; find the numbers.

7. Find two numbers such that half their sum is 19 and one-seventh of their difference is 2.

8. Find two numbers such that the sum of three times the first and twice the second equals 22, and the sum of four times the first and three times the second equals 31.

9. Find two numbers such that if 3 be added to the first the sum will be equal to twice the second, and if 6 be subtracted from the second the difference will be one-fifth of the first.

10. The sum of the four sides of a rectangle is 100 ft., and the difference of the two adjacent sides is 10 ft. ; find the length and breadth of the rectangle.

11. At an election for the monitor of a class, 35 students were present. Mohan was elected monitor by a majority of 7 votes. If all the students exercised their rights of voting, how many students voted against, and how many for Mohan ?

12. In the Legislative Assembly at which 108 members were present, a resolution was carried by a majority of 19 votes. If 7 members remained neutral, find how many voted for and how many against the resolution.

13. Find two numbers such that twice the greater exceeds three times the smaller by 10, and one-fifth of the greater is equal to one-third of the smaller.

14. Find two numbers such that if three times their difference be added to their sum, the result is equal to 18, and if their difference be added to twice their sum, the result is equal to 26.

15. The sum of two numbers exceeds their difference by 10, and if 3 be added to one, the sum will be equal to six times the other ; find the numbers.

16. Divide Rs. 226 between Rama and Shyama, so that Rama may have Rs. 30 more than Shyama. Find the share of each.

17. Divide Rs. 115 between Gopal and Mohan, so that Gopal's share may be Rs. 5 less than twice Mohan's share. Find the share of each.

18. *A* has Rs. 7 more than *B*. If *A* gives *B* Rs. 4, *B* will have Re. 1 more than *A*. How much has each ? Find the share of each.

19. Osman has twice as much money as Hamid. If Osman gives Hamid Rs. 15, both will have equal sums of money. How much has each ?

20. If *A* were to give *B* Rs. 12, *A* would have half the sum which *B* then has ; but if *B* were to give *A* Rs. 13, *B* would have one-third of what *A* then has. How much had each at first ?

21. A labourer is engaged for 40 days on the condition that he will be paid 8a. for every day that he works, and fined $2\frac{1}{2}$a. for every day that he is absent. If after 40 days he receives Rs. 15. 6a. 6p., how many days did he work, and how many days was he absent from work ?

22. At an examination, *A* obtained twice as many marks as *B* ; if *A* had gained 20 marks less and *B* 20 marks more, both would have got equal marks. How many marks did each get ?

23. At an examination, Rajendra obtained 11 marks more than Brijendra ; if Brijendra had gained half as many marks again as he did, he would have beaten Rajendra by 11 marks. How many marks did each get ?

24. A purse contains Rs. 162. 8a. made up of rupees and eight-anna pieces ; the number of eight-anna pieces exceeds that of the rupees by 25 ; find the number of rupees in the purse.

25. Mohan is twice as old as Sohan ; 16 years ago Mohan was four times as old as Sohan ; how old are they ?

26. A man is 24 years older than his son ; 12 years ago he was five times as old as his son ; how old are they ?

27. A mother says to her daughter, "Five years ago I was five times as old as you were, and 10 years hence I shall be twice as old as you will be." Find their ages.

28. A is 13 years older than B ; 7 years ago A was twice as old as B ; find their present ages.

29. Rama is 4 years older than Krishna ; the sum of their ages 6 years ago was two-thirds of the sum of their ages at present. Find Rama's present age.

30. Mohammad is 2 years older than Ahmad ; 15 years hence the sum of their ages will be twice the sum of their ages at present. Find Mohammad's present age.

31. Five times the age of a man is twelve times the age of his son ; 5 years ago the ratio of their ages was 11 : 4 ; find their present ages.

32. A mother is 22 years older than her daughter ; 4 years hence the sum of their ages will be three times the difference of their present ages ; find their ages.

33. Two years hence an elder brother's age will be three times what his younger brother's age was 5 years ago ; at present $\frac{3}{7}$ of the elder brother's age is equal to $\frac{1}{2}$ of the younger brother's age. Find their present ages.

34. The sum of the present ages of father and son is 90 years. If both live until son is as old as father is at present, the sum of their ages will then be 154 years. Find their present ages.

75. EXAMPLE 1. *The sum of the two digits of a number is 7, and if 27 be added to the number, the digits are reversed. Find the number.*

Let x be the digit in the tens place, and y the digit in the units place ; then the number will be represented by $(10x+y)$. If the digits be reversed, *i.e.*, if x occupies the units place and y the tens place, the number will be represented by $(10y+x)$.

Since the sum of the digits is $x+y$,

$$\therefore \quad x+y=7. \quad \dots\dots\dots\dots\dots\dots(1)$$

If 27 be added to the number, *i.e*, to $(10x+y)$, the digits are reversed, *i.e.*, the number becomes $(10y+x)$.

$$\therefore \quad (10x+y)+27=(10y+x),$$
$$\therefore \quad 10x+y+27=10y+x,$$
$$10x-x+y-10y=-27,$$
$$9x-9y=-27,$$
$$\therefore \quad x-y=-3. \quad \text{.....................................(2)}$$

Adding (1) and (2),

$$2x=4,$$
$$\therefore \quad x=2.$$

Substituting this value of x in (1),

$$2+y=7,$$
$$\therefore \quad y=5.$$

Hence, the digit in the tens place is 2 and the digit in the units place is 5, *i.e.*, the number is 25.

EXAMPLE 2. *The cost of* 9 *seers of wheat and* 18 *seers of barley is* Rs. 3. 15a. ; *and the cost of* 21 *seers of wheat and* 9 *seers of barley is* Rs. 5. 1a. *Find the cost of one seer of wheat and one seer of barley.*

Let x annas be the cost of 1 sr. of wheat and y annas the cost of 1 sr. of barley ; then 9 sr. of wheat and 18 sr. of barley cost $(9x+18y)$ annas, and these are equal to Rs. 3. 15a. = 63a.

$$\therefore \quad 9x+18y=63,$$
$$\therefore \quad x+2y=7. \quad \text{.....................................(1)}$$

Similarly 21 sr. of wheat and 9 sr. of barley cost $(21x+9y)$ annas, and these are equal to Rs. 5. 1a. = 81a.

$$\therefore \quad 21x+9y=81,$$
$$\therefore \quad 7x+3y=27. \quad \text{.....................................(2)}$$

Multiplying (1) by 7,

$$7x+14y=49. \quad \text{.....................................(3)}$$

Subtracting (2) from (3),

$$11y=22,$$
$$\therefore \quad y=2.$$

Substituting this value of y in (1),

$$x+4=7,$$
$$\therefore \quad x=3.$$

Hence, 1 sr. of wheat costs 3a., and 1 sr. of barley costs 2a.

EXAMPLE 3. *I spent* Rs. 5 *in buying guavas at* 3 *for an anna and oranges at* 2 *for an anna, and then sold two-thirds of the guavas and three-fourths of the oranges for* Rs. 4, *and thus gained* 6 *annas ; how many of each did I buy ?*

Let x be the number of guavas and y the number of oranges.

Since 1 guava costs $\frac{1}{3}$ anna,

x guavas cost $\frac{x}{3}$ annas.

And since 1 orange costs $\frac{1}{2}$ anna,

$\therefore$ y oranges cost $\frac{y}{2}$ annas.

Therefore, the sum of $\frac{x}{3}$ annas and $\frac{y}{2}$ annas is equal to Rs. 5 or 80 annas,

$$i.e., \quad \frac{x}{3}+\frac{y}{2}=80.$$

Multiplying by 6,

$$2x+3y=480. \qquad \text{.......................................}(1)$$

I sold $\frac{2}{3}$ of x guavas and $\frac{3}{4}$ of y oranges for Rs. 4. Since these 4 rupees include a gain of 6 annas, therefore the cost price of $\frac{2}{3}x$ guavas and $\frac{3}{4}y$ oranges is Rs. 3. 10a., or 58a.

Since 1 guava costs $\frac{1}{3}$ anna,

$\therefore$ $\frac{2}{3}x$ guavas cost $\frac{1}{3}\times\frac{2}{3}x=\frac{2}{9}x$ annas

And since 1 orange costs $\frac{1}{2}$ anna,

$\therefore$ $\frac{3}{4}y$ oranges cost $\frac{1}{2}\times\frac{3}{4}y=\frac{3}{8}y$ annas,

$$\therefore \quad \frac{2x}{9}+\frac{3y}{8}=58.$$

Multiplying both sides by 72,

$$16x+27y=4176. \qquad \text{....................................}(2)$$

Multiplying (1) by 8,

$$16x+24y=3840. \qquad \text{....................................}(3)$$

Subtracting (3) from (2),

$$3y=336,$$
$$\therefore \quad y=112.$$

Substituting this value of y in (1),

$$2x+336=480,$$
$$\therefore \quad 2x=480-336,$$
$$\therefore \quad 2x=144,$$
$$\therefore \quad x=72.$$

Hence, the required number of guavas bought was 72 and that of oranges 112.

EXAMPLE 4. *A merchant bought* 120 *seers of ghee and* 160 *seers of oil for* Rs. 368. *He sold the ghee at a gain of* 20% *and the oil at a loss of* 25% *and thus gained* Rs. 16 *in all. At what rates did he buy the two commodities ?*

Let x rupees represent the price at which he bought 1 sr. of ghee and y rupees the price at which he bought 1 sr. of oil.

Since 1 sr. of ghee costs x rupees,

$\therefore$ 120 sr. ,, ,, ,, $120x$ rupees.

And since 1 sr. of oil costs y rupees,

$\therefore$ 160 sr. ,, ,, ,, $160y$ rupees,

$$\therefore \quad 120x+160y=368,$$

$$\therefore \quad 15x+20y=46. \qquad \text{......(1)}$$

Now on ghee worth 100 rupees he gains 20 rupees,

$\therefore$,, ,, ,, 1 rupee ,, ,, $\frac{20}{100}=\frac{1}{5}$ rupee,

$\therefore$,, ,, ,, $120x$ rupees ,, ,, $120x\times\frac{1}{5}=24x$ rupees.

Again on oil worth 100 rupees he loses 25 rupees,

$\therefore$,, ,, ,, 1 rupee ,, ,, $\frac{25}{100}=\frac{1}{4}$ rupee,

$\therefore$,, ,, ,, $160y$ rupees ,, ,, $160y\times\frac{1}{4}=40y$ rupees.

Therefore his net gain is $(24x-40y)$ rupees and this is equal to 16 rupees,

$$\therefore \quad 24x-40y=16,$$

$$\therefore \quad 12x-20y=8. \qquad \text{......(2)}$$

Adding (1) and (2),

$$27x=54,$$

$$\therefore \quad x=2.$$

Substituting this value of x in (1),

$$30+20y=46,$$

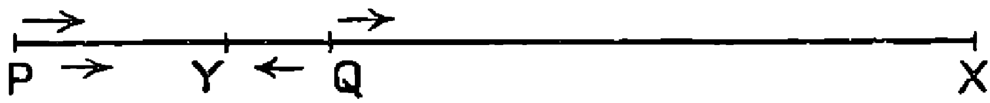
$$\therefore \quad 20y=16,$$

$$\therefore \quad y=\tfrac{4}{5}.$$

Hence, ghee was bought at Rs. 2 per seer, and oil at Re. $\frac{4}{5}$ or 12a. $9\frac{3}{5}$p. per seer.

EXAMPLE 5. *Two brothers start at the same time from two places* 20 *miles apart. If they walk in the same direction, the elder brother overtakes the younger in* 10 *hours but if they walk in opposite directions they meet in* 2 *hours. Find the rates at which they walk.*

Let P and Q be the two places 20 miles apart, and suppose the elder brother starts from P and the younger from Q.



Let x miles per hour be the speed of the elder brother and y miles per hour that of the younger.

When they walk in the same direction *i.e.*, to the right, the elder brother will overtake the younger at some place X to the right of Q.

Since the elder brother walks x miles and the younger y miles in 1 hour, therefore the elder gains $(x-y)$ miles on the younger in 1 hour, *i.e.*, he gains $10(x-y)$ miles in 10 hours. But he has to gain 20 miles altogether, which he does in 10 hours,

$$\therefore \quad 10(x-y)=20. \qquad \text{.......(1)}$$

When they walk in opposite directions, they will meet at some point Y between P and Q.

Since the elder brother walks x miles and the younger y miles in 1 hour, therefore they approach each other by $(x+y)$ miles in 1 hour, *i.e.*, they together cover $2(x+y)$ miles in 2 hours. But they have to cover 20 miles altogether, which they do in 2 hours,

$$\therefore \quad 2(x+y)=20. \qquad \text{.......(2)}$$

Dividing (1) by 10, and (2) by 2,

$$x-y=2, \qquad \text{.......(3)}$$

$$\text{and} \quad x+y=10. \qquad \text{.......(4)}$$

Adding (3) and (4)

$$2x=12,$$

$$\therefore \quad x=6.$$

Substituting this value of x in (3),

$$6-y=2,$$

$$\therefore \quad -y=-4,$$

$$\therefore \quad y=4.$$

Hence, the rate of walking of the elder brother is 6 miles per hour and of the younger brother is 4 miles per hour.

EXAMPLES XLIII

1. A number has two digits whose sum is 10, if the digits be reversed the number is diminished by 36. Find the number.

2. The sum of the digits of a number lying between 10 and 100 is 11, if the digits be reversed the number is increased by 9. Find the number.

3. A number has two digits whose sum is 7, if 9 be taken from the number, the digits are reversed. Find the number.

4. A number has two digits whose sum is 9, if the number be divided by 12, the quotient will be equal to the digit in the tens place. Find the number.

5. A number of two digits is eight times the sum of the digits, if 45 be taken from the number, the digits are reversed. Find the number.

6. A number of two digits is seven times the sum of the digits. If 9 be subtracted from the number obtained by reversing the digits, the sum of the digits of the difference thus obtained will be equal to the sum of the digits of the number obtained by adding 9 to the original number. Find the original number.

7. The price of 5 horses and 9 cows is Rs. 1020, and the price of 6 horses and 7 cows is Rs. 1110. Find the price of a horse and a cow.

8. The price of 9 goats and 7 sheep or the price of 6 goats and 13 sheep is Rs. 150. Find the price of a goat and a sheep.

9. I bought 15 maunds of grain consisting of wheat and barley for Rs. 100. If the cost of $5\frac{1}{3}$ seers of wheat or 8 seers of barley is Re. 1, how many maunds of wheat and how many maunds of barley did I buy?

10. The daily wages of 20 men and 25 boys amount to Rs. 25. 15a., and those of 25 men and 20 boys amount to Rs. 27. 8a.; find the daily wages of one man and one boy.

11. A man invested Rs. 10000 at simple interest, a part at 5% and the rest at 4% per annum, bringing a total yearly interest of Rs. 470. What was the amount of each investment?

12. A man invested Rs. 4000 at simple interest, partly at 5% and partly at 4% per annum, the 5% investment brings Rs. 2 more interest than the 4% investment annually. What was the amount of each investment?

13. The sums of Rs. 400 and Rs. 500 are invested at different rates and their annual interest is Rs. 31. If the rates of interest were exchanged, the annual interest would be Re. 1 more. Find the rates of interest.

14. The sums of Rs. 1000 and Rs. 800 are invested at different rates and the annual interest is Rs. 59. If the rates of interest were exchanged, the annual interest would be Re. 1 less. Find the rates of interest.

15. The sums of Rs. 650 and Rs. 400 are invested at different rates and the annual interest is Rs. 40. If the rates of interest were exchanged, the annual interest would be Rs. 38. 12a. Find the rates of interest.

16. A and B do a piece of work. The work is finished if A works for 12 days and B for 8 days, or if A works for 9 days and B for 12 days. In how many days can both of them separately and jointly finish the work ?

17. I spent Rs. 11 in buying nuts at 10 for an anna and almonds at 6 for an anna ; had I bought the same number of nuts as I bought of almonds and the same number of almonds as I bought of nuts, I would have spent Rs. 2 more. How many nuts and almonds did I buy ?

18. I have a sum of Rs. 36. 4a. in four-anna pieces and eight-anna pieces. If the number of four-anna and eight-anna pieces were interchanged I would lose Re. 1. 4a. How many eight-anna and four-anna pieces have I ?

19. A man spent Rs. 3520 in buying horses at Rs. 150 a horse and ponies at Rs. 40 a pony. He sold the horses at a gain of 20% and the ponies at a loss of 25% and thus gained Rs. 470 in all. How many horses and ponies did he buy ?

20. A man bought 17 almirahs and 34 tables for Rs. 680. He sold the almirahs at a loss of 10% and the tables at a gain of 20% and thus gained Rs. 13. 9a. $7\frac{1}{5}$p. in all. At what rates did he buy the two articles ?

21. A man spent £ 1. 17s. 6d. in buying 20 books and 25 pencils and realised £ 2. 3s. 9d. in selling the books at a gain of 15% and the pencils at a gain of 20%. At what rates did he buy the two articles ?

22. A and B start at the same time from two places 30 miles apart, A on cycle and B on foot. A overtakes B in $7\frac{1}{2}$ hours. If they had proceeded in opposite directions, they would have met in 3 hours. Find the rate of each.

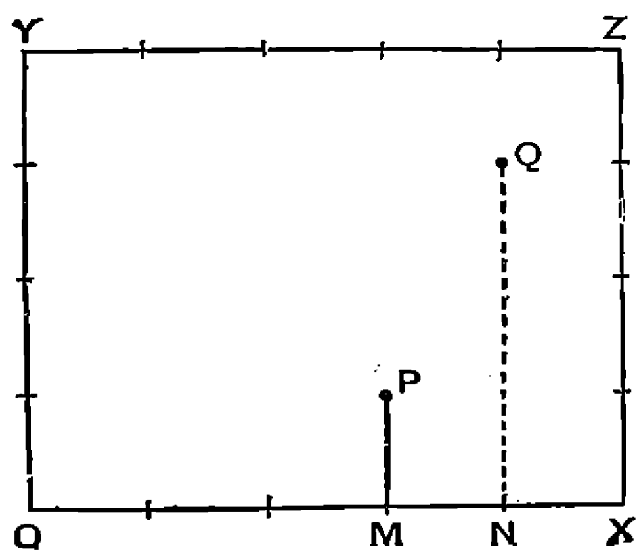
23. P and Q start at the same time from two places 21 miles apart. If they walk in the same direction P overtakes Q in 21 hours, but if they walk in opposite directions, they meet in 3 hours. Find the rate of each.

CHAPTER XVI

GRAPHS. CO-ORDINATES OF POINTS

Location of a Point

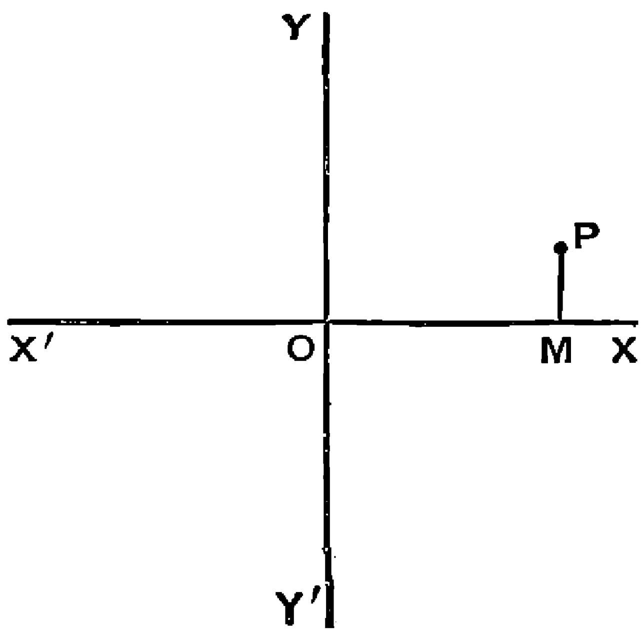
76. The figure OXZY represents a Black Board, 5 ft. long and 4 ft. broad, in which equal marked spaces represent 1 foot. In order to determine the position of any fixed point P on it, we have to find its distance from any two of its



adjacent edges. If we say that P is 3 feet to the right of OY, it does not locate the position of the point P, for there are an indefinite number of points at a distance of 3 feet to the right of OY. Similarly, if we say that P is 1 foot above OX, it also does not locate its position, for there are an indefinite number of points 1 foot from OX. But, if we say that P is 3 feet to the right of OY and 1 foot above OX, the position of P is completely determined, for P is the only point which is 3 feet to the right of OY and 1 foot above OX. Thus, we see that *in order to locate the position of a point on a plane we require two measurements.* We generally take two perpendicular straight lines OX and OY in the plane from which we take these measurements.

Consider the point P in the figure. In order to get to P, we start from O and first go 3 feet to the right along OX, that is, to the point M, and then go 1 foot straight up parallel to OY and arrive at P. If we measure first along OX and then along OY, the two measurements, that is, the perpendicular distances (3 ft., 1 ft.,) completely determine the position of the point P. This position can also be expressed by the numbers (3,1), if 1 foot be taken as a unit of measurement. Similarly, the position of the point Q can be expressed by (4,3), for the distance of Q from OY is 4 feet and that from OX is 3 feet.

77. If we take two fixed straight lines XOX′ and YOY′ cutting one another at right angles at O, the position of any point P with reference to these lines is known when we know the distances OM (along OX) and MP (parallel to OY).



The distance OM which is measured along OX is called the **abscissa** of the point P, and is denoted by x; and the distance MP which is measured parallel to OY is called the **ordinate** of the point P, and is denoted by y.

The abscissa and the ordinate taken together are called **co-ordinates** of the point P, and are denoted by (x, y).

The point O is called the **origin**, and the straight lines XOX′ and YOY′ are called the **axes of co-ordinates** or **axes of reference**, XOX′ being known as the **axis of x** and YOY′ as the **axis of y**.

The axes of co-ordinates divide the plane into four parts XOY, YOX′, X′OY′, Y′OX, known respectively as the first, second, third and fourth **quadrants**.

78. As we have seen in Article 14, if distances measured from a point in one direction are taken as *positive*, then those measured in the opposite direction must be taken as *negative*. Usually the distances measured along the axis of x to the right of the origin are positive, and those measured to the left of the origin are negative. Similarly, the distances measured along the axis of y above the origin are positive, and those measured below the origin are negative.

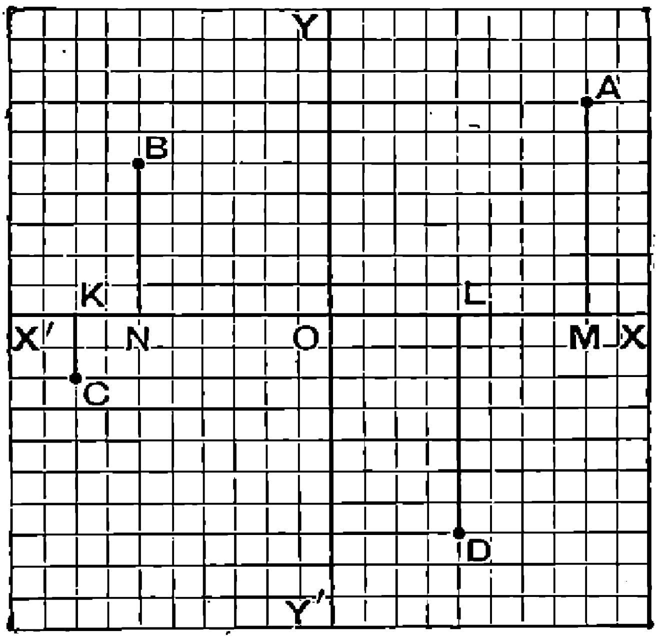
In order to get to any point in the first quadrant, we have to go first along the x-axis to the right of the origin and then straight up parallel to the y-axis, hence the signs of all the points in this quadrant are $(+,+)$. Similarly, all the points in the second quadrant, since they are measured first to the left of the origin and then straight up, have the signs $(-,+)$. All points in the third quadrant, since they are measured first to the left of the origin and then straight down, have the signs $(-,-)$. All points in the fourth quadrant, since they are measured first to the right of the origin and then straight down, have the signs $(+,-)$.

NOTE. The students should note that, in writing the co-ordinates of any point, the abscissa is always written first.

79. The process of locating a point, whose co-ordinates are given, is called **plotting the point.**

Points are generally plotted on squared paper which is ruled by horizontal and vertical lines drawn parallel to the axes of co-ordinates at equal intervals. The best paper is that which is ruled to show inches and tenths of an inch, or centimetres and millimetres.

80. If in the following figure, XOX′ and YOY′ are the axes of co-ordinates, O the origin, and A, B, C, D any four points, then



the abscissa of A is OM,
,, ,, ,, B ,, ON,
,, ,, ,, C ,, OK,
and ,, ,, ,, D ,, OL.

Similarly, the ordinate of A is MA,
,, ,, ,, B ,, NB,
,, ,, ,, C ,, KC,
and ,, ,, ,, D ,, LD.

Taking one-tenth of an inch as unit (*i.e.* the side of each small square as unit), and remembering that the right hand and upward directions are positive and the left hand and downward directions are negative, we see that

(i) $OM = 8$, $MA = 7$, therefore the co-ordinates of A are $(8, 7)$.

(ii) $ON = -6$, $NB = 5$, therefore the co-ordinates of B are $(-6, 5)$.

(iii) $OK = -8$, $KC = -2$, therefore the co-ordinates of C are $(-8, -2)$.

(iv) $OL = 4$, $LD = -7$, therefore the co-ordinates of D are $(4, -7)$.

It will also be seen from the figure that

(i) *The ordinates of all the points on the x-axis are* 0.

Thus the co-ordinates of the points M, N, K and L are (8, 0), (−6, 0), (−8, 0) and (4, 0).

(ii) *The abscissæ of all the points on the y-axis are* 0.

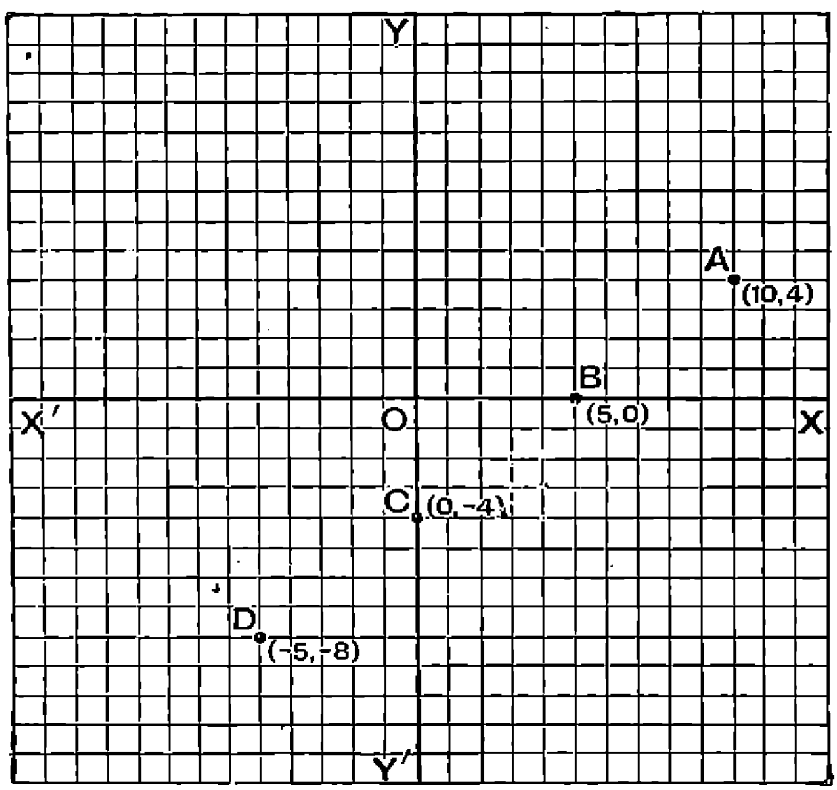
(iii) *The co-ordinates of the origin are* (0, 0).

If two-tenths of an inch *i.e.*, ·2″ be taken as unit, the co-ordinates of A, B, C and D will be (4, 3·5), (−3, 2·5), (−4, −1) and (2, −3·5).

NOTE. Before plotting points, the scale or the unit chosen for measurements should always be stated and the position of a point should be indicated either by a dot, a small cross-mark or a small circle.

EXAMPLE 1. *Plot the points* (10, 4), (5, 0), (0, −4) *and* (−5, −8).

Draw XOX′ and YOY′, axes of co-ordinates and take the side of a square as unit.



To plot the point (10, 4), first count 10 units along OX to the right, and then 4 units straight up parallel to OY. Put a dot there and call it A. This is the required point (10, 4).

To plot (5, 0), first count 5 units to the right, then no units either up or down. Put a dot and call it B.

To plot $(0, -4)$, first count no units either to the right or left, then 4 units down. Put a dot and call it C.

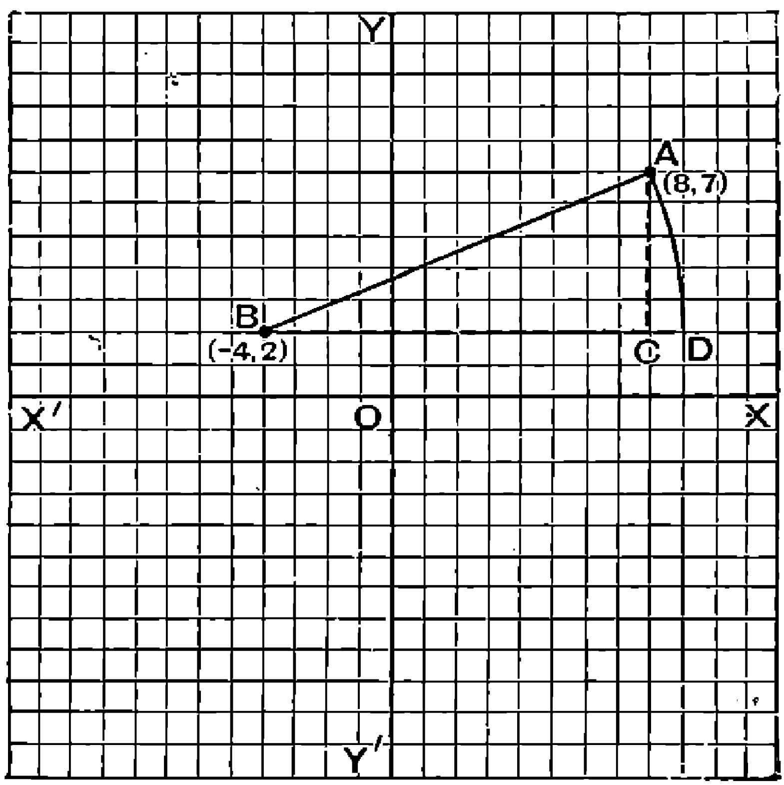
To plot $(-5, -8)$, first count 5 units to the left, and then 8 units down. Put a dot and call it D.

EXAMPLE 2. *Show that the points in Example* 1 *lie in a straight line.*

By joining the points A, B, C and D it will be seen that they all lie in a straight line.

EXAMPLE 3. *Find the distance between the points* (8,7) *and* $(-4,2)$.

Take XOX′ and YOY′ axes of co-ordinates. Plot the points as in the following figure and call them A and B.



First Method. Draw an arc of a circle with centre B and radius BA. Let it cut the line BC drawn through B parallel to X′OX at D. Then BA=BD.

By counting, BD=13 units, hence BA=13 units in length.

Second Method. Draw BC through B parallel to X′OX to meet AC drawn parallel to YO at C. Then ACB is a right-angled triangle in which BC=12 units, and AC=5 units.

$$AB^2 = BC^2 + CA^2$$
$$= 12^2 + 5^2$$
$$= 144 + 25$$
$$= 169.$$
$$\therefore \quad AB = 13.$$

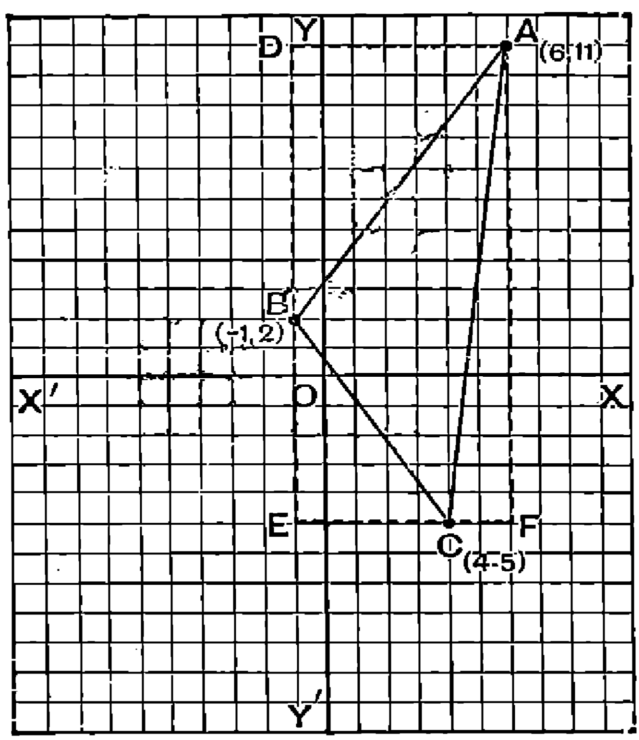
Hence, AB is 13 units in length.

Areas of Plane Figures on Squared Paper

81. Squared paper is sometimes a great help in finding the areas of plane figures, as will be clear from the following examples.

EXAMPLE 1. *Plot the points* (6, 11), (−1, 2) *and* (4, −5), *and find the area of the triangle formed by joining them.*

Plot the points as in the following figure, and form the triangle ABC by joining them.



First method. Count the number of squares within the triangle. Regard those which are equal to or greater than half a square as whole squares, and neglect those which are less than half a square.

Beginning with the right vertical row, the numbers in the different rows are 3, 10, 12, 10, 6, 4 and 2.

By addition, the total number of squares is 47.

Hence, the area of the triangle is 47 sq. units.

Second Method. Through A and B draw AF and DBE parallel to YOY′. Through A and C draw AD and FCE parallel to XOX′ meeting DBE at D and E and AF at F.

Now $\triangle$ ABC = rect. ADEF − $\triangle$ ADB − $\triangle$ BEC − $\triangle$ AFC

$= (7 \times 16 - \frac{1}{2} \times 7 \times 9 - \frac{1}{2} \times 7 \times 5 - \frac{1}{2} \times 2 \times 16)$ sq. units

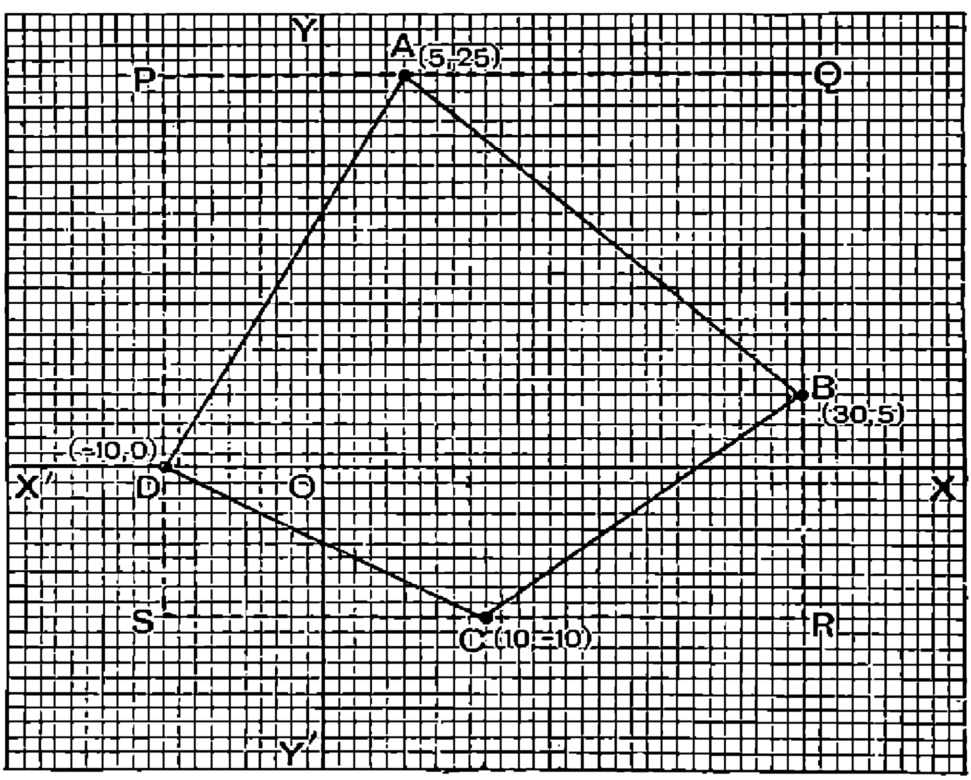
$= (112 - \frac{63}{2} - \frac{35}{2} - 16)$ sq. units

$= 47$ sq. units.

NOTE. The second method gives the correct result while the first an approximate result.

EXAMPLE 2. *Plot the points* (5, 25), (30, 5), (10, −10) *and* (−10, 0), *and find the area of the quadrilateral formed by joining them.*

Take the side of a small square as one unit. Plot the points as in the following figure and form the quadrilateral ABCD.



Through A and C draw PAQ and SCR parallel to X′OX and through B and D draw QBR and PDS parallel to YOY′ meeting PAQ at Q and P and SCR at R and S, and thus complete the rectangle PQRS.

Now the quadrilateral ABCD

$= \text{rect. } PQRS - \triangle APD - \triangle DSC - \triangle CRB - \triangle BQA$

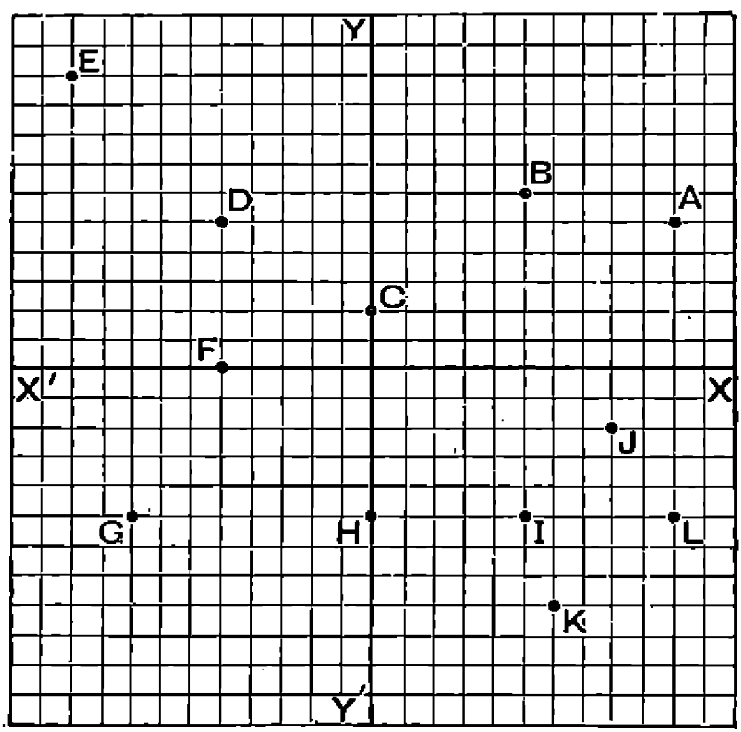
$= (40.35 - \frac{1}{2}.15.25 - \frac{1}{2}.10.20 - \frac{1}{2}.20.15 - \frac{1}{2}.20.25)$ sq. units

$= (1400 - \frac{375}{2} - 100 - 150 - 250)$ sq. units

$= 712\frac{1}{2}$ sq. units.

EXAMPLES XLIV

1. Write down the co-ordinates of the points A, B, C, D,...in the following figure taking as the unit length (i) one-tenth of an inch ; (ii) one-fifth of an inch ; (iii) one inch :



2. Taking one-tenth of an inch or one millimetre as unit, plot the following points :

(i) (10, 6), (−10, 6), (−10, −6), (10, −6).

(ii) (12, 0), (0, 12), (−12, 0), (0, −12), (0, 0).

(iii) (14, −16), (−10, 11), (9, −12), (−2, −2), (−3, 0), (20, −5), (−1, −3), (−7, −4), (8, −1).

3. Taking one inch or one centimetre as unit, plot the following points :

(i) (1, 1), (1, −1), (−1, 0), (0, −1), (2, 3), (−2, 3), (3, −1), (0, −3), (0, 0).

(ii) $(\cdot 5, \cdot 7)$, $(-\cdot 4, -\cdot 8)$, $(1\cdot 2, 0)$, $(0, -1\cdot 3)$, $(\cdot 9, \cdot 9)$, $(2\cdot 1, 1\cdot 2)$, $(-1\cdot 6, 1\cdot 2)$.

(iii) $(1, \frac{1}{2})$, $(-\frac{1}{2}, 1)$, $(1\frac{1}{2}, -\frac{1}{2})$, $(-1\frac{1}{2}, -1\frac{1}{2})$, $(1, \frac{1}{3})$, $(-\frac{1}{2}, -\frac{4}{5})$, $(1\frac{1}{2}, -\frac{3}{5})$, $(-1\frac{1}{5}, \frac{4}{5})$.

4. Plot the following sets of points and show that they all lie in a straight line :

(i) (5, 5), (5, 0), (5, 4), (5, −1), (5, 8), (5, −9).

(ii) (3, −7), (7, −7), (0, −7), (−2, −7), (−7, −7), (10, −7).

(iii) (0, 0), (1, 1), (−2, −2), (5, 5), (−8, −8).

(iv) (0, −3), (1, −1), (2, 1), (3, 3).

(v) (0, −2), (−2, 0), (−4, 2), (1, −3).

5. Plot the following pairs of points, and in each case find the distance between them :

(i) (7, 8), (−5, 3). (ii) (0, 0), (6, 8). (iii) (0, 0), (−5, 12).

(iv) (9, 8), (−10, 19). (v) (15, 0), (0, 8). (vi) (20, 25), (−15, −3).

(vii) (−3, −8), (15, −2).

6. Plot the points (2, 2), (−6, 2) and (−2, 11), and show that they are the vertices of an isosceles triangle. Find the length of each side.

7. Plot the following sets of points, and in each case find the area of the triangles formed by joining them :

(i) (2, 2), (2, 10), (6, 8). (ii) (−2, 4), (3, 4), (8, 0).

(iii) (8, 8), (5, 0), (−6, 0). (iv) (6, 8), (−12, −4), (4, −4).

8. Plot the following sets of points, and in each case find the area of the figures formed by joining them :

(i) (10, 5), (10, 0), (0, 0), (0, 5).

(ii) (7, 3), (−6, 3), (−6, 4), (7, −4).

(iii) (3, 0), (3, 4), (9, 0), (9, 10). (iv) (0, 0), (10, 0), (5, 6), (8, −4).

(v) (0, 0), (−2, 8ʾ, (−10, 8), (−16, 0).

(vi) (5, 4), (0, 8), (−5, 0), (0, −10).

9. Plot the following sets of points, and in each case find the area of the traingles formed by joining them :

(i) (5, 4), (−6, −4), (9, −1). (ii) (5, 7), (−8, 2), (3, −5).

(iii) (6, 4), (−7 −6), (−2, −15).

10. Plot the following sets of points, and in each case find the area of the figures formed by joining them :

(i) (17, 5), (6, 11), (−9, 6), (10, −3).

(ii) (6, 12), (10, 2), (8, −4), (−10, 0).

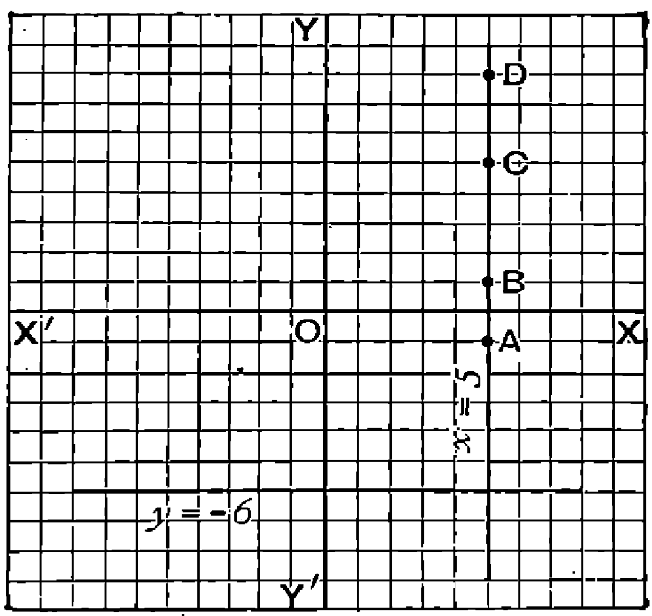
(iii) (0, 0), (−10, 4), (−14, −5), (−1, −3).

CHAPTER XVII

GRAPHS OF STRAIGHT LINES

82. A **graph** is a line, broken, straight, or curved, formed by joining a series of points.

83. If we plot a series of points such as $(5, -1)$, $(5, 1)$, $(5, 5)$, $(5, 8)$,... which have the same abscissa 5, we see that they all lie in a straight line, parallel to YOY′ and distant 5 units from it. A,B,C,D,... are such points in the figure.

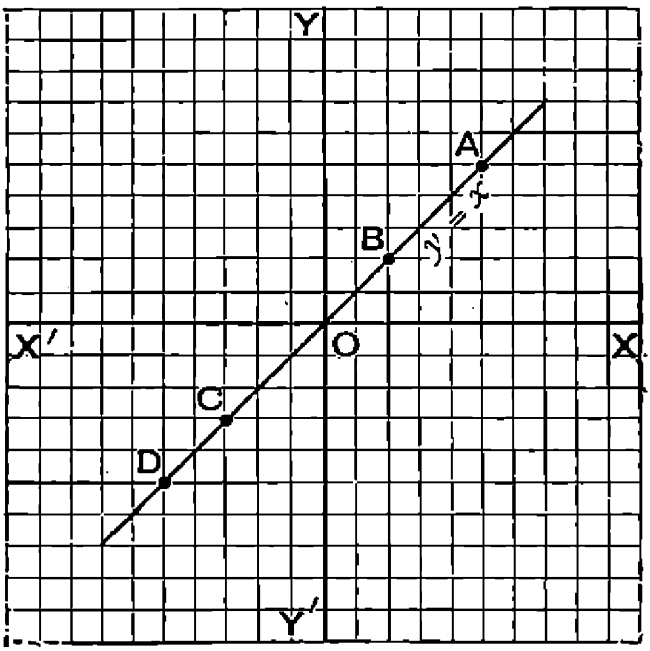


Since the abscissa is denoted by 'x', the straight line formed by joining them will be the graph of the equation $x = 5$. We also see that every point on this straight line will have 5 for its abscissa, whatever its ordinate may be, and this is true for no other point outside this straight line.

Similarly, if we plot another series of points whose ordinates are -6 and join them, we get a straight line, parallel to X′OX at a distance of 6 units below it, as shown in the figure. This will be the graph of $y = -6$.

NOTE. Since every point on the axis of x has its ordinate 0, the graph of $y=0$ is the axis of x; and every point on the axis of y has its abscissa 0, the graph of $x=0$ is the axis of y.

84. If we plot a series of points, each of which has its abscissa (represented by x) equal to its ordinate (represented by y), and join them, we get a straight line, the co-ordinates (x, y) of any point on which will satisfy the algebraic equation $x=y$. In the following figure, the straight line



obtained by joining the points A, B, C, D,...whose co-ordinates, (5, 5) (2, 2), (−3, −3), (−5, −5),...have abscissæ and ordinates equal, is the graph of $x=y$.

NOTE. From the figure it is clear that the origin whose co-ordinates are (0, 0) also lies on this straight line.

Graphs of Straight Lines

85. We have already seen in Article 68 that, if we are given a simple equation involving x and y, we can find an indefinite number of pairs of values of x and y which will satisfy that equation. Now, if we regard these values of x and y to be abscissæ and ordinates, plot the points and join them, we shall get a straight line which will be the graph of that equation.

EXAMPLE 1. *Draw the graph of* $y=2x$.

If we give a series of positive and negative values to x, then from the equation we can find the corresponding values of y. Thus

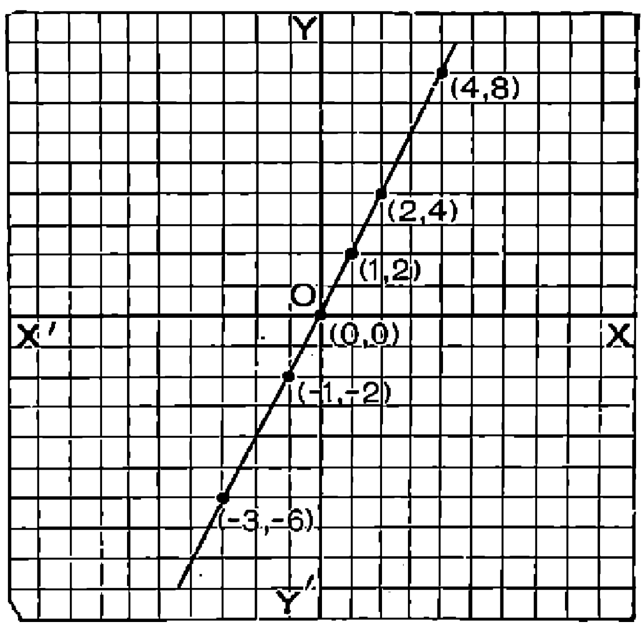
when

$x=$	1	2	4	0	-1	-3	
$y=$	2	4	8	0	-2	-6	

Now combining each value of x with the corresponding value of y, we get the following pairs of values of x and y.

$$(1, 2), (2, 4); (4, 8), (0, 0), (-1, -2), (-3, -6), \ldots$$

Taking one side of a square as unit, if we plot the points, as shown in the following figure, we see that all the points lie on a straight line which is the graph of the equation $y=2x$.



EXAMPLE 2. *Draw the graph of* $2x-2y-3=0$.

The equation can be written as

$$-2y=-2x+3,$$

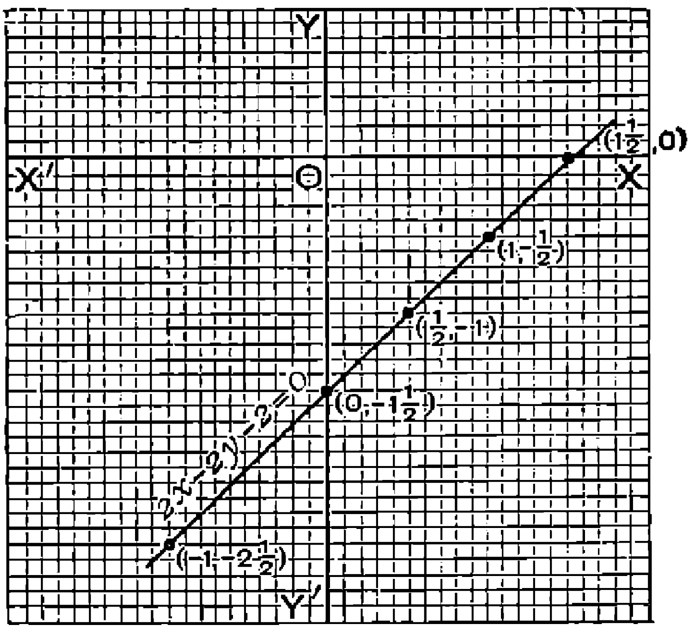
$$\therefore \quad 2y=2x-3,$$

$$\therefore \quad y=x-\tfrac{3}{2}.$$

We can now very easily find the value of y corresponding to any value of x, thus

when						
	$x =$	0	1	-1	$\frac{1}{2}$	$1\frac{1}{2}$	
	$y =$	$-1\frac{1}{2}$	$-\frac{1}{2}$	$-2\frac{1}{2}$	-1	0	

Taking *ten* divisions of the paper as one unit, we plot the points $(0, -1\frac{1}{2})$, $(1, -\frac{1}{2})$, $(-1, -2\frac{1}{2})$, $(\frac{1}{2}, -1)$, $(1\frac{1}{2}, 0)$,...as in the figure.



By joining these we see that the required graph is a straight line.

Graphs of Functions

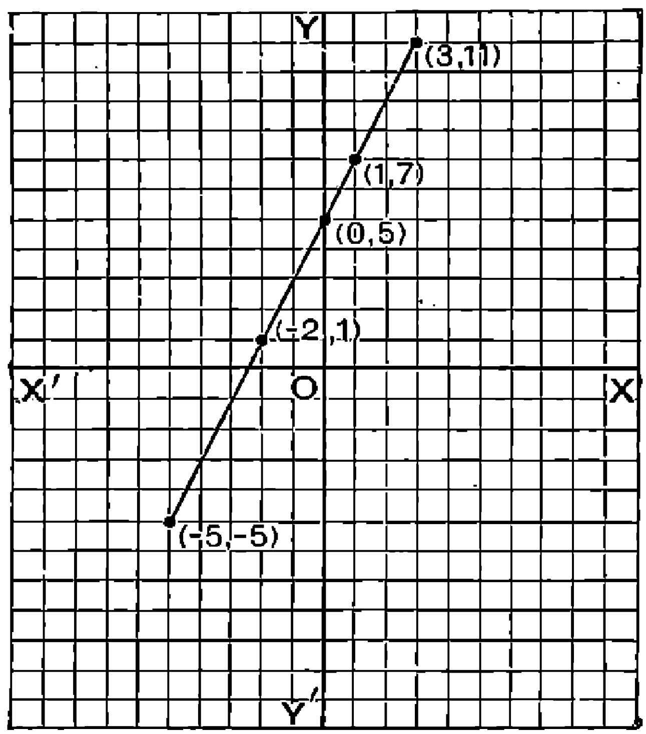
86. Any expression containing a variable x whose value depends upon the value of x is called a *function* of x. As the value of x varies, the value of the *function* also varies.

Thus, for instance, if in the function $2x+5$, we give to x the values 1,2,3,..., the function will assume the values 7,9,11,.... If we take every value of x with the corresponding value of the function, and consider them as abscissæ and ordinates respectively plot the points, the line obtained by joining these points will be the graph of the function. Thus, it is clear that the graph of the function $2x+5$ will be the same as that of the equation $y=2x+5$. Hence, to draw the graph of the function $2x+5$, we proceed thus :

Let y represent the value of the function. Now when

the variable	$x =$	0	1	3	-2	-5	
the function	$y =$	5	7	11	1	-5	

Taking one side of a square as unit, plot the points (0, 5), (1, 7), (3, 11), (− 2, 1), (− 5, − 5),... The graph of the



function is, therefore, the straight line passing through the points, as shown in the figure.

EXAMPLES XLV

1. Draw, on the same axes, the graphs of

(i) $x=2$. (ii) $x+2=0$. (iii) $x=5$.
(iv) $x+7=0$. (v) $y-10=0$. (vi) $y-11=0$.
(vii) $y=7$. (viii) $y+15=0$.

2. Draw, on the same axes, the graphs of

(i) $y=x$. (ii) $x=-y$. (iii) $y=4x$. (iv) $y+4x=0$.

3. Draw, on the same axes, the graphs of
(i) $y=2x$. (ii) $y=2x+5$. (iii) $y=2x-7$.

4. Draw, on the same axes, the graphs of
(i) $y+3x=0$. (ii) $y+3x-2=0$. (iii) $y+3x+5=0$.

Taking one-half of an inch or one centimetre as unit, draw the graphs of

5. $x+1=0$. **6.** $y=2x-1$. **7.** $x+y=2$.
8. $x-2y+1=0$. **9.** $x-2y-2=0$. **10.** $x+y=-2$.
11. $y=2x+3$. **12.** $y=-2x+1$. **13.** $x+y=\frac{1}{2}$.
14. $\frac{x}{2}+\frac{y}{3}=1$. **15.** $\frac{x}{2}-\frac{y}{3}=1$. **16.** $\frac{x}{3}-\frac{y}{4}=1$.

Draw the graphs of the following functions :

17. $2x+1$. **18.** $2x-1$. **19.** $3x+2$.
20. $3x-2$. **21.** $-3x+2$. **22.** $-\frac{x-1}{2}$.

Taking suitable units, draw the graphs of

23. $3y+5x=6$. **24.** $6x-3y+5=0$. **25.** $10x+5y=9$.
26. $\frac{x}{10}+\frac{y}{9}=1$. **27.** $y=\cdot 3x+\cdot 4$. **28.** $y=\cdot 2x-\cdot 3$.

Graphical Solution of Simultaneous Equations

87. We have already seen in Article 68 that, to find the values of two unknown quantities, x and y, we should be given two different equations involving these quantities, for from one equation we can find as many pairs of values of x and y as we please, which will satisfy the equations. This can also be seen graphically, for we know that the graph of a simple equation is a straight line which can be produced to any length on either side, and any pairs of values of x and y which satisfy the equation will give the co-ordinates of some point on this straight line. But, if we are given two equations, we shall have two straight lines, which, if not parallel, will intersect at *only one point*, whose co-ordinates will satisfy both the equations, since the point lies on both the straight lines.

Hence, to solve two simultaneous equations graphically, *draw the graphs of the two equations on the same scale and in the same diagram, and write down the co-ordinates of their point of intersection. The abscissa of the point will represent the value of x and the ordinate that of y.*

EXAMPLE 1. *Solve graphically the equations*

$$x+y=12,$$
$$x-y=4.$$

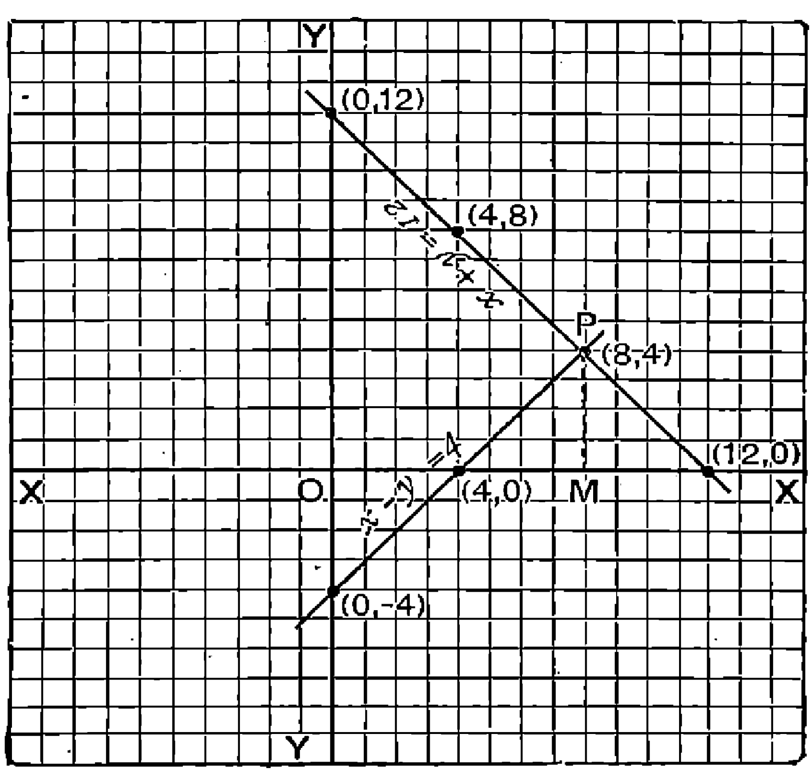
Verify your answer.

From the first equation,

when

$x=$	0	4	12
$y=$	12	8	0

Taking one-tenth of an inch as unit, plot the points (0, 12), (4, 8) and (12, 0), and draw a straight line passing through them.



Again, from the second equation,

when

$x =$	0	4	8
$y =$	-4	0	4

With the same scale plot the points (0, -4), (4, 0) and (8, 4), and draw a straight line passing through them.

The two straight lines intersect at the point P. From P draw PM perpendicular to the x-axis. Now x represents OM, the abscissa of P, and is equal to 8 ; and y represents MP, the ordinate of P, and is equal to 4.

Hence $x=8$, $y=4$ is the solution of the given equations.

Verification. Writing $x=8$ and $y=4$ in the first equation, the left side$=8+4=12$, which is equal to the right side, hence the point (8, 4) satisfies the equation.

Again, writing $x=8$ and $y=4$ in the second equation, the left side$=8-4=4$, which is equal to the right side, hence the point (8, 4) satisfies the equation.

Therefore the solution is correct.

NOTE. The student is advised to note the following :

(i) The same units should be used for both the graphs.

(ii) Two points are sufficient to determine a straight line, but to ensure accuracy it is advisable to plot one more point, *i.e.*, at least three.

(iii) The unit should not be too small, otherwise the correct answer will not be possible.

EXAMPLE 2. *Solve graphically*

$$4x-3y=6,$$
$$2x+y=8.$$

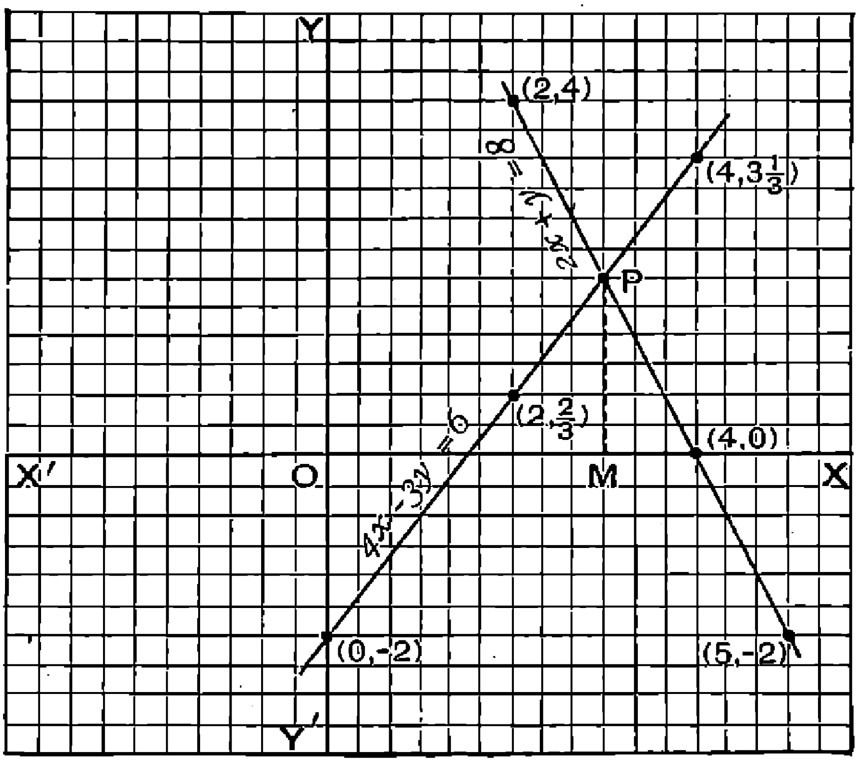
From the first equation,

when

$x =$	0	2	4
$y =$	-2	$\frac{2}{3}$	$3\frac{1}{3}$

NOTE. It is advisable to find whole number values, but when it is not possible units should be so chosen as to avoid fractions in plotting points.

Taking ·3″ as unit, plot the points $(0, -2)$, $(2, \frac{2}{3})$ and $(4, 3\frac{1}{3})$, and draw a straight line passing through them.



From the second equation,

when

$x =$	2	4	5
$y =$	4	0	-2

With the same scale plot the points $(2, 4)$, $(4, 0)$ and $(5, -2)$, and draw a straight line passing through them. The two straight lines intersect at the point P, whose co-ordinates are $(3, 2)$.

Hence $x=3$, $y=2$ is the solution of the given equations.

88. Measurement on Different Scales. In foregoing examples both abscissæ and ordinates have been measured on the same scale, but sometimes it will be found more convenient to use different scales for the two variables.

For example, in drawing the graph of

$$y = 10x + 6,$$

when we give to x the values 0, 1, 2, 3, 4,..., the corresponding values of y are 6, 16, 26, 36, 46,...

Here we see that as x increases, y increases more rapidly. In such a case it would be better to choose a smaller scale for the ordinate and a larger scale for the abscissa.

NOTE. The student is advised to draw the graph first by taking the same scale with a smaller unit and then with a larger unit for both the abscissæ and the ordinates, and then by taking different scales (for instance, one inch as unit for abscissa and one-tenth of an inch as unit for ordinate) and to compare the graphs.

Equation of Straight Line Graphs

89. We have seen in foregoing examples, that when equations or functions are of the first degree, *i.e.*, when they contain only the first power of the independent variable, their graphs are straight lines. For this reason they are sometimes called **linear equations** and **linear functions.**

It is also clear that any equation of the first degree in x and y can always be reduced to the form $y = mx$ or $y = mx + c$, where m and c are constant quantities.

EXAMPLE. *Find the equation to the straight line which passes through the points* $(0, -1\frac{1}{2})$, $(1, -\frac{1}{2})$, $(-1, -2\frac{1}{2})$, $(\frac{1}{2}, -1)$, $(1\frac{1}{2}, 0)$,...

In Article 85, Example 2, we have plotted these points and seen that they all lie on a straight line. Therefore the equation of the graph is of the first degree and is of the form $y = mx + c$.

We have, now, to find the values of m and c.

Since the point $(0, -1\frac{1}{2})$ lies on the graph, therefore $x = 0$ and $y = -1\frac{1}{2}$ must satisfy the equation $y = mx + c$.

$$\therefore \quad -\tfrac{3}{2} = m \times 0 + c,$$

$$\therefore \quad c = -\tfrac{3}{2}. \quad \text{.......................................(1)}$$

Again, since the point $(1, -\frac{1}{2})$ also lies on the graph, therefore $x = 1$ and $y = -\frac{1}{2}$ must also satisfy the equation $y = mx + c$.

$$\therefore \quad -\tfrac{1}{2} = m \times 1 + c,$$

$$\therefore \quad m + c = -\tfrac{1}{2}.$$

$$\text{from (1)}, \; m - \tfrac{3}{2} = -\tfrac{1}{2},$$

$$\therefore \quad m = 1.$$

The equation to the straight line is, therefore,

$$y=1\times x-\tfrac{3}{2},$$

or $$2x-2y-3=0.$$

NOTE. Since a straight line can always be drawn when any two points on it are known, hence two points are sufficient to determine the equation of a straight line.

EXAMPLES XLVI

Solve the following equations graphically and verify your results :

1. $x=5,\ x+2y=12.$

2. $3x-y=26,\ y+5=0.$

3. $4x-y=12,\ y=0.$

4. $x+y=8,\ x-y=2.$

5. $x+y+8=0,\ x-y=2.$

6. $4x-y-6=0,\ 3x-2y=7.$

7. $y=2x-3,\ 3x-2y=4.$

8. $\frac{1}{2}x+\frac{1}{3}y=2,\ y=x+1.$

9. $y+4=0,\ y=3x-2$, take ·3 inch as unit.

10. $x+y=10,\ x-y=1$, take ·5 inch or 1 cm. as unit.

11. $x+2y-12=0,\ x-3y-2=0$, take ·5 inch or 1 cm. as unit.

12. $3x-2y+1=0,\ 2x+y-11=0$, take ·2 inch as unit.

13. $3x+4y+1=0,\ x+2y=1$, take ·4 inch as unit.

14. $5x-3y+9=0,\ 3x+y-3=0$, take ·3 inch as unit.

15. $x+y=15,\ 3x-2y=-5$, take ·1 inch as unit.

16. Draw with the same axes the graphs of

$$y-2x=1,\ x+y=2,\ x-y=3\ ;$$

and hence find the solutions of the following pairs of equations :

$$\left.\begin{aligned}y-2x&=1\\x+y&=2\end{aligned}\right\},\qquad \left.\begin{aligned}x+y&=2\\x-y&=3\end{aligned}\right\},\qquad \left.\begin{aligned}x-y&=3\\y-2x&=1\end{aligned}\right\}.$$

17. Draw with the same axes the graphs of

$$2y=x+2,\ y=2x-3,\ 2y=3x-4\ ;$$

and hence find the solutions of the following pairs of equations ;

$$\left.\begin{aligned}2y&=x+2\\y&=2x-3\end{aligned}\right\},\qquad \left.\begin{aligned}y&=2x-3\\2y&=3x-4\end{aligned}\right\},\qquad \left.\begin{aligned}2y&=3x-4\\2y&=x+2\end{aligned}\right\}.$$

18. Shew that the straight lines represented by the equations

$$x+y=6,\ 3x-y=2,\ 2x+3y=16,$$

meet in a point, and find the co-ordinates of the point of intersection.

19. Shew that the straight lines represented by the equations

$$x+5y=0,\ x-y=6,\ 2x+5y=5,$$

meet in a point, and find the co-ordinates of the point of intersection.

Plot the following points and find the equation of the graph in each case :

20.

$x=$	0	2	4	6	-2	-4
$y=$	0	3	6	9	-3	-6

21.

$x=$	0	1	2	3	-1	-2
$y=$	5	8	11	14	2	-1

22.

$x=$	0	2	4	6	-2	-4
$y=$	3	$1\frac{1}{2}$	0	$-1\frac{1}{2}$	$4\frac{1}{2}$	6

23.

$x=$	0	1	2	3	-1	-2
$y=$	·6	1	1·4	1·8	·2	$-$·2

Find the equations of the straight lines joining the following pairs of points :

24. (0, 5), (1, 7). **25.** (0, 0), (1, 5).

26. $(2, \frac{1}{2})$, $(-2, -2\frac{1}{6})$. **27.** (2, ·4), (-1, -1·4).

28. Shew that the points (3, 6), (0, 3), (-3, 0) lie on a straight line ; find its equation.

29. If the equation $y=mx+c$ represents a straight line passing through the points (2, 9) and (-1, 3), find the values of m and c.

30. Find the values of m and c so that the straight line represented by $y=mx+c$ may pass through the intersection of the straight lines $2x-y-11=0$ and $x-2y-10=0$, and also through the point (-3, 4).

CHAPTER XVIII

HARDER CASES OF MULTIPLICATION AND DIVISION

90. Easy cases of Multiplication and Division of compound algebraic expressions have been dealt with in Chapters VI and VII. In this chapter we shall take up harder cases of Multiplication and Division. The principles already explained in those chapters will also be applicable to examples of this chapter.

EXAMPLE 1. *Multiply* x^2+5-2x *by* $3x+x^2-6$.

We first arrange both the expressions in the same order (*i.e.* in descending powers of x), and then proceed as in Article 34 thus,

$$\begin{array}{lllll}
x^2 & -2x & +5 & & \\
x^2 & +3x & -6 & & \\
\hline
x^4 & -2x^3 & +5x^2 & & \\
 & 3x^3 & -6x^2 & +15x & \\
 & & -6x^2 & +12x & -30 \\
\hline
x^4 & +x^3 & -7x^2 & +27x & -30
\end{array}$$

Here each term of the first expression is first multiplied by x^2, the first term of the second expression, and partial products written down; then each term of the first expression is multiplied by $+3x$, the second term of the second expression, and partial products written down so that terms of the same powers fall in the same vertical columns; and so on. The sum of these partial products $x^4+x^3-7x^2+27x-30$ is the required product.

NOTE. The arrangement according to ascending or descending powers of x is not necessary, but it is convenient for it makes the arrangement of terms of the same powers very easy.

EXAMPLE 2. *Multiply* x^3-2x+5 *by* x^2-3x+2.

Here the term in x^2 is missing from x^3-2x+5, hence to get terms of the same powers in vertical columns, we leave a gap in the place of x^2.

$$\begin{array}{llllll}
x^3 & & -2x & +5 & & \\
x^2-3x+2 & & & & & \\
\hline
x^5 & & -2x^3 & +5x^2 & & \\
 & -3x^4 & & +6x^2 & -15x & \\
 & & +2x^3 & & -4x & +10 \\
\hline
x^5 & -3x^4 & & +11x^2 & -19x & +10
\end{array}$$

Hence the product is $x^5-3x^4+11x^2-19x+10$.

EXAMPLE 3. *Find the product of* $a^2+4b^2+9c^2-2ab-6bc-3ac$ *and* $a+2b+3c$.

$$\begin{array}{llllllllll}
a^2+4b^2+9c^2-2ab-6bc-3ac & & & & & & & & & \\
a\ +2b\ +3c & & & & & & & & & \\
\hline
a^3 & +4ab^2 & +9ac^2 & -2a^2b & -6abc & -3a^2c & & & & \\
 & -4ab^2 & & +2a^2b & -6abc & & +8b^3 & +18bc^2 & -12b^2c & \\
 & & -9ac^2 & & -6abc & +3a^2c & & -18bc^2 & +12b^2c & +27c^3 \\
\hline
a^3 & & & & -18abc & & +8b^3 & & & +27c^3
\end{array}$$

Hence the product is $a^3+8b^3+27c^3-18abc$.

Here when multiplying by a, the terms are written in the order in which they are obtained, but when multiplying by $2b$ and $3c$, the terms are arranged so that like terms fall in the same vertical columns.

EXAMPLES XLVII

Multiply

1. $2x^2-3ax+a^2$ by $-x^2+3ax-2a^2$.
2. a^3-5a+6 by $a-2+3a^2$.
3. $l^2-2m^2-n^2$ by $-l^2-2m^2-n^2$.
4. $x-3+2x^2$ by $2-x^2-5x$.
5. $4+3a^2+a^3+3a$ by $1-2a+a^2$.
6. $k-l+m-n$ by $k-l-m+n$.
7. a^2+a-b^2+b by a^2+a+b^2-b.
8. $x^2+y^2+z^2-xy-yz-zx$ by $x+y+z$.
9. $a^3+b^3+3ab^2+3a^2b$ by $a^3-b^3+3ab^2-3a^2b$.
10. y^2-yz+z^2 by $xy-xz+yz$.
11. $ab+a+b+1$ by $ab-a-b+1$.

12. $a^2+b^2+c^2-2ab$ by $a^2+b^2-c^2+2ab$.

13. $a^2+ab+ac+b^2-bc+c^2$ by $a-b-c$.

14. $a^2+4b^2+c^2-ac-2ab-2bc$ by $a+2b+c$.

15. $a^2+4b^2+9c^2-2ab+3ac+6bc$ by $a+2b-3c$.

91. Consider the term $2x^5$. This can be written at full length as $2xxxxx$ and contains five factors represented by letters.

Each literal factor of a term is called a **dimension** of the term, and the number of these factors is called the **degree** of the term.

Thus the term $2x^5$ is said to be of the *fifth degree*. It is also said to be of *five dimensions* in x. Similarly $2a^5b^3c$ is of the nineth degree or of nine dimensions in a, b, c.

NOTE. A numerical quantity has no degree.

92. The **degree of an expression** is that of its term of the highest degree.

Thus in the expression $x^5+2x^3-7x^2+1$, the first, second, third and fourth terms are of the fifth, third, second and zero degrees respectively, hence the expression is of the fifth degree.

Similarly the expression $x^4y^2+x^3y+xy^2+y^3$ is of the fourth degree in x, third degree in y, and of the sixth degree in x and y.

93. Sometimes it is necessary to consider expressions involving letters, in which some stand for *fixed* values, while others have *different* values. We have already seen, letters which are supposed to be known are generally represented by the first letters of the alphabet a, b, c,... ; while letters which are supposed to vary in values are generally represented by the last letters of the alphabet x, y, z... In such a case, a, b, c... are called **constants**, and x, y, z,... are called **variables**.

In determining the degree of an expression containing constants as well as variables, constant letters are not counted ; thus ax^2+bx+c is an expression of the second degree in x.

Again, the expression ax^2+bx+c contains a first degree term, a second degree term, and a zero degree term ; such an expression is called a **general expression** of the second degree. Hence, to find a general expression, say of the fifth degree in x, we have to consider an expression in which all the terms beginning with the fifth degree to the first degree in x are present and there is also a term which does not contain x. For instance, if a, b, c, d, e, f are constants, the expression $ax^5+bx^4+cx^3+dx^2+ex+f$ is a general expression of the fifth degree, where $a, b, c,\ldots$ are the coefficients of $x^5, x^4, x^3,\ldots$

94. An expression is said to be **homogeneous** when all its terms are of the same degree. Thus

(i) $x+y$ is a homogeneous expression of the first degree in x and y.

(ii) x^2+xy+y^2 is a homogeneous expression of the second degree in x and y.

(iii) $a+2b+3c$ is a homogeneous expression of the first degree in a, b and c.

(iv) $a^3+b^3+c^3-3abc$ is a homogeneous expression of the third degree in a, b and c.

(v) $ax^3+bx^2y+cxy^2+dy^3$ is a homogeneous expression of the third degree in x and y, but is a homogeneous expression of the fourth degree in x, y, a, b, c and d.

Method of Detached Coefficients

95. When two compound expressions containing powers of one letter only are multiplied together, the labour of multiplication will be saved if we deal *with coefficients alone*. When this method is employed it is necessary that both the expressions should be arranged in *the same order*, either according to ascending powers or descending powers of the common letter, and zero coefficients should be supplied to represent terms corresponding to any missing power of that letter.

EXAMPLE 1. *Multiply* $4a-2a^2+5a^3-3$ *by* $6a-1+7a^2$.

Arranging both the expressions in descending powers of a and proceeding in the ordinary way as on the left side below :

$$\begin{array}{rrrrrr|rrrrrr}
5a^3 & -2a^2 & +4a & -3 & & & 5 & -2 & +4 & -3 & & \\
 & 7a^2 & +6a & -1 & & & & 7 & +6 & -1 & & \\
\hline
35x^5 & -14a^4 & +28a^3 & -21a^2 & & & 35 & -14 & +28 & -21 & & \\
 & 30a^4 & -12a^3 & +24a^2 & -18a & & & 30 & -12 & +24 & -18 & \\
 & & -5a^3 & +2a^2 & -4a & +3 & & & -5 & +2 & -4 & +3 \\
\hline
35a^5 & +16a^4 & +11a^3 & +5a^2 & -22a & +3 & 35 & +16 & +11 & +5 & -22 & +3
\end{array}$$

Hence, the product is $35a^5+16a^4+11a^3+5a^2-22a+3$.

The work has been shortened by *detaching the coefficients*, keeping each in its proper relative place, as on the right side above. In employing this method, which is known as the **Method of Detached Coefficients**, when supplying the letters in the last step, it should be remembered that the degree of the product is the sum of the degrees of the given expressions, that is in the above example the degree of the product is $3+2$ or 5. Thus the highest power of a in the product is a^5, and others follow in descending order.

EXAMPLE 2. *Multiply* a^3+2a^2-8 *by* a^2-4a-3.

Here the term a is missing from the first expression. Supplying this term with zero coefficient, we multiply a^3+2a^2+0a-8 by a^2-4a-3, and write down only the coefficients, thus

$$\begin{array}{rrrrrr}
1 & +2 & +0 & -8 & & \\
1 & -4 & -3 & & & \\
\hline
1 & +2 & +0 & -8 & & \\
 & -4 & -8 & -0 & +32 & \\
 & & -3 & -6 & -0 & +24 \\
\hline
1 & -2 & -11 & -14 & +32 & +24
\end{array}$$

Since one of the expressions is of the third degree and the other of the second degree, therefore the product will be of the fifth degree. Hence, the product is $a^5-2a^4-11a^3-14a^2+32a+24$.

96. The method of detached coefficients can also be applied to multiply expressions which are homogeneous in *two* letters.

EXAMPLE 1. *Multiply* $x^3-2xy^2+3y^3$ *by* $x^3+2x^2y-3y^3$.

Arranging the two expressions in descending powers of x, and writing a zero coefficient to represent the term containing x^2 missing from the first expression and a zero coefficient for x missing from the second expression, we proceed thus

$$\begin{array}{rrrrrrr}
1 & +0 & -2 & +3 & & & \\
1 & +2 & +0 & -3 & & & \\
\hline
1 & +0 & -2 & +3 & & & \\
 & 2 & +0 & -4 & +6 & & \\
 & & & -3 & -0 & +6 & -9 \\
\hline
1 & +2 & -2 & -4 & +6 & +6 & -9
\end{array}$$

Since each of the given expressions is homogeneous and of the third degree in x and y, the product is also homogeneous and of the sixth degree in x and y. The first term has the highest power of x, *i.e.*, x^6, and other terms follow in descending powers of x and ascending powers of y, the last term has the highest power of y, *i.e.*, y^6. Hence the product is $x^6+2x^5y-2x^4y^2-4x^3y^3+6x^2y^4+6xy^5-9y^6$.

NOTE. The third line of multiplication is not written as all the terms are zeroes.

EXAMPLE 2. *Find the three terms of the lowest degree in the product of* $4x^5+3x^4+x^2+2x+1$ *and* x^3+x-2.

As the terms of the *lowest* degree are required, we arrange the expressions in ascending powers of x, and omit all the terms of higher dimensions than x^3.

$$\begin{array}{rrrr|}
1 & +2 & +1 & +0 \\
-2 & +1 & +0 & +1 \\
\hline
-2 & -4 & -2 & -0 \\
 & 1 & +2 & +1 \\
 & & & 1 \\
\hline
-2 & -3 & +0 & +2
\end{array}$$

Hence the required result is $-2-3x+2x^3$.

EXAMPLES XLVIII

Using the method of Detached Coefficients, find the product of

1. x^3-2x^2+3x-4 and $4x^2+14x+9$.
2. $x^4-2x^3y+3x^2y^2-2xy^3+y^4$ and $x^2+2xy+y^2$.
3. $1-x+x^2-x^3$ and $1+x+x^2+x^3$.
4. $1+2x+2x^2+3x^3$ and $1-2x+2x^2-3x^3$.
5. $x^3+3x^2y+3xy^2+y^3$ and $x^3-3x^2y+3xy^2-y^3$.

6. $1+2x+3x^2+4x^3+5x^4$ and $1-2x+x^2$.

7. x^5-x^4+x-1 and x^3+x^2+x+1.

8. $x^4+3x^3+5x^2+3x+1$ and $x^4-3x^3+5x^2-3x+1$.

9. $x^6+2x^4+x^3+4x^2-2x+1$ and x^3-2x-1.

10. $x^8+x^7y-x^5y^3-x^4y^4-x^3y^5+xy^7+y^8$ and x^2-xy+y^2.

11. $x^8-2x^6y^2+3x^4y^4-2x^2y^6+y^8$ and $x^4+2x^2y^2+y^4$.

Multiply

12. $x^2-2x+\frac{1}{2}$ by $x^2+2x+\frac{1}{2}$.

13. $\frac{1}{3}x^2-\frac{1}{2}x+\frac{1}{4}$ by $\frac{1}{2}x^2+\frac{3}{2}x-\frac{3}{4}$.

14. $\frac{3}{2}x^3-5x^2y+\frac{1}{4}xy^2+9y^3$ by $\frac{1}{2}x^2-xy+3y^2$.

Find the continued product of

15. $x-a$, x^2+ax+a^2, x^3+a^3.

16. x^2-2y^2, $x^2-2xy+2y^2$, x^2+2y^2, $x^2+2xy+2y^2$.

Find the three terms of the lowest degree in the product of

17. x^3-3x^2-1 and x^2+2x-3.

18. x^3-3x^2+3x-1 and x^2-2x+1.

Find the three terms of the highest degree in the product of

19. $5x^3-4x^2+3x$ and x^2+4x+6.

20. x^3+3x^2-7x+6 and $3x^3-x^2+5x-4$.

97. In Chapter VII, we have already given simple cases of division of compound expressions by compound expressions. Here we will take up some harder cases of division.

EXAMPLE 1. *Divide* $x^4-35x+x^3-24x^2+57$ *by* x^2-3+2x.

Arranging both the dividend and the divisor in descending powers of x.

$$
\begin{array}{l}
x^2+2x-3)x^4+\ x^3-24x^2-35x+57(x^2-x-19 \\
\qquad\qquad\quad\ \underline{x^4+2x^3-3x^2\qquad\qquad} \\
\qquad\qquad\qquad\ -x^3-21x^2-35x \\
\qquad\qquad\qquad\ \underline{-x^3-\ 2x^2+\ 3x\qquad} \\
\qquad\qquad\qquad\qquad\ -19x^2-38x+57 \\
\qquad\qquad\qquad\qquad\ \underline{-19x^2-38x+57}
\end{array}
$$

Hence the quotient is x^2-x-19.

If we arrange both the dividend and the divisor in ascending powers of x, the working can be arranged thus

$$\begin{array}{r}
-3+2x+x^2)57-35x-24x^2+x^3+x^4(-19-x+x^2\\
\underline{57-38x-19x^2\qquad\qquad}\\
3x-5x^2+x^3\qquad\\
\underline{3x-2x^2-x^3\qquad}\\
-3x^2+2x^3+x^4\\
\underline{-3x^2+2x^3+x^4}
\end{array}$$

Hence the quotient is $-19-x+x^2$.

The work can be much shortened by using the *method of detached coefficients.*

Arranging according to descending powers of x, we have

$$\begin{array}{r}
1+2-3)\,1+1-24-35+57\,(1-1-19\\
\underline{1+2-3\qquad\qquad}\\
-1-21-35\qquad\\
\underline{-1-2+3\qquad}\\
-19-38+57\\
\underline{-19-38+57}
\end{array}$$

Now, since the dividend is of the fourth degree and the divisor of the second degree, therefore the quotient is of the second degree. Hence the quotient is x^2-x-19.

Again, arranging according to ascending powers of x, we have

$$\begin{array}{r}
-3+2+1)\,57-35-24+1+1\,(-19-1+1\\
\underline{57-38-19\qquad\qquad}\\
3-5+1\qquad\\
\underline{3-2-1\qquad}\\
-3+2+1\\
\underline{-3+2+1}
\end{array}$$

Hence the quotient is $-19-x+x^2$.

EXAMPLE 2. *Divide* $28x^4-xy^3+13x^2y^2+15y^4$ *by* $4x^2+3y^2+4xy$, *by the method of detached coefficients.*

Arranging both the dividend and the divisor in descending powers of x, and supplying the term involving x^3 which is missing in the dividend with zero coefficient, we proceed thus

$$\begin{array}{r}
4+4+3)\,28+0+13-1+15\,(7-7+5\\
\underline{28+28+21\qquad\qquad}\\
-28-8-1\qquad\\
\underline{-28-28-21\qquad}\\
20+20+15\\
\underline{20+20+15}
\end{array}$$

Hence the quotient is $7x^2-7xy+5y^2$.

EXAMPLE 3. *Divide* $\frac{1}{3}x^4 - \frac{11}{12}x^3 + \frac{41}{8}x^2 - \frac{23}{4}x + 6$ *by* $\frac{2}{3}x^2 - \frac{5}{6}x + 1$.

$$\frac{2}{3}x^2 - \frac{5}{6}x + 1\Big)\ \frac{1}{3}x^4 - \frac{11}{12}x^3 + \frac{41}{8}x^2 - \frac{23}{4}x + 6\ \Big(\frac{1}{2}x^2 - \frac{3}{4}x + 6$$

$$\frac{1}{3}x^4 - \frac{5}{12}x^3 + \frac{1}{2}x^2 \qquad \left[\ \frac{1}{3}x^4 \div \frac{2}{3}x^2 = \frac{x^4}{3} \times \frac{3}{2x^2} = \frac{x^2}{2}.\ \right]$$

$$-\frac{1}{2}x^3 + \frac{37}{8}x^2 - \frac{23}{4}x$$

$$-\frac{1}{2}x^3 + \frac{5}{8}x^2 - \frac{3}{4}x \qquad \left[-\frac{1}{2}x^3 \div \frac{2}{3}x^2 = -\frac{x^3}{2} \times \frac{3}{2x^2} = -\frac{3x}{4}.\right]$$

$$4x^2 - 5x + 6$$

$$4x^2 - 5x + 6 \qquad \left[\ 4x^2 \div \frac{2}{3}x^2 = 4x^2 \times \frac{3}{2x^2} = 6.\ \right]$$

Hence the quotient is $\frac{1}{2}x^2 - \frac{3}{4}x + 6$.

EXAMPLES XLIX

(Most of the following examples may be done by the method of Detached Coefficients)

Divide

1. $x^4 + 2x^2 + 9$ by $x^2 + 2x + 3$.
2. $x^4 - 10x^2 + 9$ by $x^2 - 2x - 3$.
3. $x^5 - 5x + 4$ by $x^2 - 2x + 1$.
4. $x^5 - 59x^3 - 31x^2 + 9$ by $x^2 + 7x - 3$.
5. $x^5 + 4x^4 + 48x - 32$ by $x^3 + 6x^2 + 8x - 8$.
6. $1 + 2x - x^2 + 5x^3 - x^5 - 2x^6$ by $1 - x + 2x^2$.
7. $x^6 - 2x^5 + 3x^4 - 5x^3 + 4x^2 + 9x - 6$ by $x^3 - 2x^2 + 3$.
8. $4 - 11x + 20x^2 - 30x^3 + 20x^4 - 11x^5 + 4x^6$ by $1 - 2x + 3x^2 - 4x^3$.
9. $x^5 - 5x^3y^2 - 2x^2y^3 - 8xy^4 + 4y^5$ by $x^2 + 2xy - y^2$.
10. $x^6 + 2a^3x^3 - 4a^4x^2 + a^6$ by $x^3 + 2a^2x + a^3$.
11. $x^6 - 3x^5y - 2x^4y^2 + 11x^3y^3 - 9x^2y^4 - 4xy^5 + 6y^6$ by $x^3 - 2xy^2 + 3y^3$.
12. $\frac{1}{3}x^3 - \frac{17}{36}x^2 + \frac{1}{3}x - \frac{1}{8}$ by $\frac{2}{3}x - \frac{1}{2}$.
13. $\frac{1}{2}x^3 + \frac{1}{2}x^2 - \frac{47}{8}x + \frac{1}{2}$ by $x^2 - 3x + \frac{1}{4}$.
14. $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$ by $\frac{1}{2}x^2 - x + 3$.
15. $1 + \frac{2}{3}y - \frac{1}{4}x^2 + \frac{1}{9}y^2$ by $1 + \frac{1}{2}x + \frac{1}{3}y$.
16. $x^4 - \frac{3}{2}ax^3 - \frac{5}{2}a^2x^2 + \frac{17}{4}a^3x - a^4$ by $x^2 - 2ax + \frac{1}{2}a^2$.

Important Cases in Division

98. We shall now take up some important cases in division.

EXAMPLE 1. *Divide* $a^3-27b^3+8c^3+18abc$ *by* $a-3b+2c$.

Arranging both the dividend and the divisor in descending powers of a, and placing the other letters in alphabetical order,

$$
\begin{array}{l}
a-3b+2c)a^3+18abc-27b^3+8c^3(a^2+3ab-2ac+9b^2+6bc+4c^2 \\
\qquad\qquad \underline{a^3-3a^2b+2a^2c} \\
\qquad\qquad\qquad 3a^2b-2a^2c+18abc \quad \text{[arranging in descending powers of } a\text{]} \\
\qquad\qquad\qquad \underline{3a^2b-9ab^2+\ 6abc} \\
\qquad\qquad\qquad\qquad -2a^2c+9ab^2+12abc \\
\qquad\qquad\qquad\qquad \underline{-2a^2c\qquad\quad +\ 6abc-4ac^2} \\
\qquad\qquad\qquad\qquad\qquad 9ab^2+\ 6abc+4ac^2-27b^3 \quad \text{[bringing down} \\
\qquad\qquad\qquad\qquad\qquad \underline{9ab^2\qquad\qquad\qquad -27b^3+18b^2c} \quad -27b^3\text{]} \\
\qquad\qquad\qquad\qquad\qquad\qquad 6abc+4ac^2-18b^2c \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{6abc\qquad\quad -18b^2c+12bc^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4ac^2-12bc^2+\ 8c^3 \quad \text{[bringing} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{4ac^2-12bc^2+\ 8c^3} \quad \text{down } 8c^3\text{]}
\end{array}
$$

Hence the quotient is $a^2+9b^2+4c^2+3ab-2ac+6bc$.

EXAMPLE 2. *Divide* x^5-y^5 *by* $x-y$.

$$
\begin{array}{l}
x-y)x^5-y^5(x^4+x^3y+x^2y^2+xy^3+y^4 \\
\qquad \underline{x^5-x^4y} \\
\qquad\quad x^4y \\
\qquad\quad \underline{x^4y-x^3y^2} \\
\qquad\qquad x^3y^2 \\
\qquad\qquad \underline{x^3y^2-x^2y^3} \\
\qquad\qquad\quad x^2y^3 \\
\qquad\qquad\quad \underline{x^2y^3-xy^4} \\
\qquad\qquad\qquad xy^4-y^5 \qquad \text{[bringing down } -y^5\text{]} \\
\qquad\qquad\qquad \underline{xy^4-y^5}
\end{array}
$$

Hence the quotient is $x^4+x^3y+x^2y^2+xy^3+y^4$.

99. In Article 41, Example 5, we have seen that in a division question when both dividend and divisor are arranged in descending powers of some letter, if the highest power of the *letter* in the remainder cannot be divided by the highest power of the same letter in the dividend, the division is *inexact* and cannot be carried any further. But if both dividend and divisor are arranged in ascending powers of the same letter, both the quotient and the remainder will differ greatly from those obtained in the previous case and the division can be *continued indefinitely.*

Now arranging both the dividend and the divisor of Example 5, Article 41, in ascending powers of x, we proceed as below :

$$\begin{array}{l}
1-2x+x^2)\,1+2x-3x^2+x^3+x^4\,(1+4x+4x^2+5x^3+7x^4\ldots \\
\qquad\qquad \underline{1-2x+x^2} \\
\qquad\qquad\quad 4x-4x^2+x^3 \\
\qquad\qquad\quad \underline{4x-8x^2+4x^3} \\
\qquad\qquad\qquad 4x^2-3x^3+x^4 \\
\qquad\qquad\qquad \underline{4x^2-8x^3+4x^4} \\
\qquad\qquad\qquad\qquad 5x^3-3x^4 \\
\qquad\qquad\qquad\qquad \underline{5x^3-10x^4+5x^5} \\
\qquad\qquad\qquad\qquad\qquad 7x^4-5x^5 \\
\qquad\qquad\qquad\qquad\qquad \underline{7x^4-14x^5+7x^6} \\
\qquad\qquad\qquad\qquad\qquad\qquad 9x^5-7x^6
\end{array}$$

Thus we see that the division can be continued indefinitely. If we stop after five terms in the quotient, the quotient is $1+4x+4x^2+5x^3+7x^4$, and the remainder is $9x^5-7x^6$.

EXAMPLE 1. *Divide* 1 *by* $1+x$ *to four terms in the quotient.*

$$\begin{array}{l}
1+x)\,1\quad(1-x+x^2-x^3 \\
\qquad \underline{1+x} \\
\qquad\ -x \\
\qquad\ \underline{-x-x^2} \\
\qquad\qquad x^2 \\
\qquad\qquad \underline{x^2+x^3} \\
\qquad\qquad\quad -x^3 \\
\qquad\qquad\quad \underline{-x^3-x^4} \\
\qquad\qquad\qquad x^4
\end{array}$$

Hence the quotient is $1-x+x^2-x^3$ and the remainder is x^4.

EXAMPLE 2. *Find the value of* a *so that* $x^4 - x^3 + x^2 - x + a$ *may be exactly divisible by* $x^2 + 2x + 3$.

Dividing the first expression by the second,

$$\begin{array}{l} x^2+2x+3)\,x^4 - x^3 + x^2 - x + a\,(x^2 - 3x + 4 \\ \qquad\qquad\quad x^4 + 2x^3 + 3x^2 \\ \qquad\qquad\quad\quad -3x^3 - 2x^2 - x \\ \qquad\qquad\quad\quad -3x^3 - 6x^2 - 9x \\ \qquad\qquad\qquad\qquad 4x^2 + 8x + a \\ \qquad\qquad\qquad\qquad 4x^2 + 8x + 12 \\ \qquad\qquad\qquad\qquad\qquad\quad a - 12 \end{array}$$

Now, in order that the division may be *exact*, the remainder must vanish, *i.e.*, $a-12$ must be equal to 0. Hence $a=12$.

EXAMPLE 3. *What must be added to* $x^4 - 3x^3 + 5x^2 - 2x - 5$ *so that the expression may be exactly divisible by* $x^2 - 2x + 1$?

Dividing the first expression by the second,

$$\begin{array}{l} x^2-2x+1)\,x^4 - 3x^3 + 5x^2 - 2x - 5\,(x^2 - x + 2 \\ \qquad\qquad\quad x^4 - 2x^3 + x^2 \\ \qquad\qquad\quad\quad - x^3 + 4x^2 - 2x \\ \qquad\qquad\quad\quad - x^3 + 2x^2 - x \\ \qquad\qquad\qquad\qquad 2x^2 - x - 5 \\ \qquad\qquad\qquad\qquad 2x^2 - 4x + 2 \\ \qquad\qquad\qquad\qquad\qquad 3x - 7 \end{array}$$

Now, in order that the division may be *exact*, the remainder must vanish, *i.e.*, the dividend should be diminished by $3x-7$, or increased by $-(3x-7)$ or $-3x+7$.

EXAMPLES L

Divide

1. $x^3 + y^3 - 1 + 3xy$ by $x + y - 1$.
2. $a^3 + 27b^3 - 8c^3 + 18abc$ by $a + 3b - 2c$.
3. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$.
4. $a^3 + 8b^3 + 27c^3 - 18abc$ by $a^2 + 4b^2 + 9c^2 - 6bc - 3ac - 2ab$.
5. $x(3x^2 + 5y^2) + 2y(3x^2 + 5y^2)$ by $x + 2y$.
6. $5x^3 - 7x^2y + 15xz^2 + 2xy^2 - 6yz^2$ by $x^2 - xy + 3z^2$.

7. x^5-y^5 by $x-y$.

8. x^5+y^5 by $x+y$.

9. x^6-y^6 by $x^3-2x^2y+2xy^2-y^3$.

10. x^8+64 by x^4-4x^2+8.

11. x^9-y^9 by x^2+xy+y^2.

12. $x^{12}-y^{12}$ by x^2-y^2.

13. $a^{12}+a^6-2$ by a^4+a^2+1.

14. a^9+x^9 by $a+x$.

15. 1 by $1-x$ to four terms.

16. 1 by $1-2x$ to six terms.

17. $1+x$ by $1-x$ to four terms.

18. 1 by $1-a+a^2$ to four terms.

19. $\frac{1}{2}-x$ by $\frac{1}{4}-x+x^2$ to five terms.

20. For what value of a is $6x^4+x^3-13x^2-ax-2$ exactly divisible by $2x^2+3x+1$?

21. Find the value of a so that $2x^4+2x^3-11x^2+13x+a$ may be exactly divisible by $2x^2-4x+3$.

22. For what value of p is $x^3+px+24$ exactly divisible by $x+4$?

23. Find, by division, what number must be added to $6x^4-x^3-6x^2+4$ in order that it may be exactly divisible by $3x^2+x-1$?

24. What number must be added to $x^5-2x^4-4x^3+19x^2-31x+7$ in order that it may be exactly divisible by x^3-7x+5 ?

25. What number must be added to $18x^6-9x^5-29x^2-115x-70$ in order that it may be exactly divisible by $3x^3-5x-7$?

26. Divide the continued product of $1+x+y$, $1+x-y$, $1-x+y$ and $x+y-1$ by $1+2xy-x^2-y^2$.

27. Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$ by $1+xy+yz+zx$.

MISCELLANEOUS EXAMPLES II

A

1. Add together $7ax$, x^2+4a^2, $x^2+ax-5a^2$ and x^2-a^2.

2. Express 345 algebraically when $a=3$, $b=4$ and $c=5$.

3. A man starts from O and first goes $a-b$ miles east and then $a+b$ miles west. How far is he from O ?

4. Find the value of $a^3+b^3+c^3-3abc$, when $a=2$, $b=3$ and $c=-5$.

5. Simplify $x-\{x+(x-\overline{x+1})\}$.

6. A room is 10 ft. broad. If its length be increased by 3 ft. and its breadth be decreased by 1 ft., its area would remain the same. Find its length.

B

1. Add together $a-\frac{3}{4}b, -2a-b, \frac{3}{2}a+\frac{3}{4}b$ and $\frac{1}{2}a+b$.

2. Divide $8x^4-8x^3-8x^2+11x-3$ by $2x^2-3x+1$.

3. Solve the equation

$$x-\frac{6x+1}{5}=\frac{x-13}{9}+\frac{2}{3}\left(6-\frac{3x}{2}\right).$$

4. Find the simple interest on Rs. x for y years at $2\frac{1}{2}\%$ per annum.

5. An article costs Rs. x and is sold at a gain of $y\%$. Find its selling price.

6. Show, on squared paper, that

(i) $8-6=2$. (ii) $8-6-4=-2$.

C

1. Subtract $3a-7a^3+5a^2+1$ from the sum of $2+8a^2-a^3$ and $2a^3-3a^2+a-2$.

2. Multiply $3x^2-a(a-x)$ by $x(x-a)-a(x+3a)$.

3. What is the difference between 32 and xy, when $x=3$ and $y=2$?

4. Find the value of $2\sqrt{ac}-3\sqrt{xy}+\sqrt{b^2c^4}$, when $a=-4$, $b=-3$, $c=-1$, $x=4$ and $y=1$.

5. The area of a rectangle is a^2-9 square feet, and its breadth is $a-3$ feet ; find its length.

6. A mother is 32 years old and her daughter 8 years old ; in how many years will the daughter's age be half of the mother's age ?

D

1. Find the cost of

(i) x articles, when one dozen articles cost y rupees.

(ii) x articles, when z dozen articles cost y rupees.

2. In an examination, Rama got $2x+y$ marks, Shyama $x-3y$ marks, and Gopal $x+2y$ marks. What is the total of the marks obtained by them ?

3. Find the value of $(x+y)^2+(y+z)^2+(z+x)^2$, when $x=-1$, $y=-2$ and $z=3$.

4. Divide $6x+12x^4+20-33x^2-5x^3$ by $4x^2+x-5$.

5. A room is x yards long, y feet broad, and z feet high. How many square yards of stones will be required to pave its floor ? How many square feet of paper will be required to cover its walls ?

6. Show, on squared paper, that

(i) $2x+6x-4x=4x$. (ii) $2x-6x-4x=-8x$.

(iii) $-2x-6x-4x=-12x$.

E

1. Find the sum of $2a-\{b-(2c-3d)\}$, $4a-(b-2c)+4d$ and $(a+2b)-(2c+3d)$.

2. Solve $(7x-5)-(7-5x)=(1-3x)-(5x+8)$.

3. Find the value of $a^2+b^2+c^2+2ab+2bc+2ca$, when $a=3$, $b=5$ and $c=-8$.

4. Divide $4a^4+6ax^3-x^4-9a^2x^2$ by $3ax-x^2+2a^2$.

5. Simplify $(a+b+c)-(c+b-a)+(a+c-b)-(a+b-c)$.

6. Distribute Rs. x among three persons in the ratio 2 : 3 : 5.

F

1. Find the sum of $(a+2)(a+4)$, $(a+1)(a-2)$, $(a+1)(a-3)$ and $(a-1)(a+3)$, and find the value of the sum when $a=-2$.

2. The area of a rectangle is x^2-y^2 square yards, and its length is $x+y$ yards. Find its breadth.

3. Simplify $2-[-2-\{-2-(-2)\}]$.

4. Solve $(x-3)(x+3)-40=(x+4)(x-7)$.

5. A's age is a years, B's b years and C's c years ; what will be the sum of their ages x years hence ?

6. Plot the following points and find the area of the triangle formed by joining them :

(10, 5), (−5, 15), (10, 22).

G

1. Kailash is x years old, and Prem is y years younger than Kailash,

(i) What is the sum of their ages ?

(ii) What will be the sum of their ages 15 years hence ?

(iii) What was the sum of their ages 15 years ago ?

(iv) What was the difference of their ages 15 years ago ?

2. Solve

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2},$$

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{3}.$$

3. Divide $4ax^2-5ax-16bx+20b$ by $ax-4b$.

4. Multiply the sum of $2x^2-5x+6$, x^2+4x+3 and $-3x^2+x+1$ by x^2+x+1.

5. Simplify $a-[a-\overline{b+a}-\{a-(a-\overline{b-a})\}]$.

6. Plot the following points and find the area of the figure formed by joining them :

$$(0,\ 8),\ (14,\ 8),\ (10,\ 0),\ (0,\ 0).$$

H

1. Subtract the sum of $2x^3-5x-7$ and $-3x^3+4x+4$ from $3x^3-3x^2+(x-6)-\{x^3-(x-6)\}$.

2. Divide $6x^5-x^4+4x^3-5x^2-x-15$ by $2x^2-x+3$.

3. Solve $\cdot375x-1\cdot875=\cdot12x+1\cdot185$.

4. Find the value of $\sqrt{a^2+b^2+c^2-2ab-2ac+2bc}$, when $a=-1$, $b=2$ and $c=3$.

5. A man first walks a hours at the rate of x miles an hour, and then b hours at the rate of y miles an hour ; how far does he walk ?

6. Draw the graphs of the following equations and find the area of the triangle enclosed by them :

$$x=5,\ y=5,\ x+y=18.$$

I

1. Express $2\frac{3}{4}$ algebraically, when $x=2$, $y=3$ and $z=4$.

2. Divide $6a^4+29a^3-28a^2+9a-4$ by a^2+5a-4.

3. Find the value of $\sqrt{x^2+2xz+z^2}\times\sqrt[3]{3y^2z-y^3}$, when $x=-4$, $y=-3$ and $z=2$.

4. Subtract $x-[x-\{x-(x-\overline{x-a})\}]$ from $a-[a-\{a-(a-\overline{a-x})\}]$.

5. A boy, after spending $\frac{1}{3}$ of his money in purchasing books, $\frac{3}{4}$ of the rest in purchasing sweets, $\frac{1}{3}$ of the rest in purchasing fruit, and $\frac{1}{2}$ of the rest at the pictures, has Re. 1 left with him. How much had he at first ?

6. Solve graphically the following equations :

$$6x+5y=60,\ 6x=5y.$$

J

1. Subtract the sum of $2x^2-3(x+2)$ and $2x-3(3-x^2)$ from the sum of $5x^2-(x-2)$ and $x^2-2(x+1)$.

2. Divide x^5+y^5 by $x+y$.

3. Solve $\frac{x-5}{5}+\frac{x-3}{3}=\frac{5x-3}{12}$.

4. Tabulate the values of the expression $2x^2-5x-3$, when $x=0, 1, 3, -3, -1$.

5. A can do a piece of work in 9 days, B can do it in 18 days, and C can do $\frac{3}{4}$ of what A does in one day in the same time; in how many days will all three finish the work together?

6. Find, by division, what number must be added to $x^5-2x^4-4x^3+19x^2-31x+9$ in order that the expression may be exactly divisible by x^3-7x+5?

TEST PAPERS

[*Questions in these papers have been selected from the High School Scholarship Examination papers of the United Provinces of Agra and Oudh*].

I

1. Simplify $3(a^2-b^2)-[2a^2-2\{b^2+ab+b(b-a+b)\}]$.

2. Find the value of $(a-c)(a+c)-(a+c)^2$, when $3a+2c=45$ and $3c+2a=15$.

3. If a is greater than $2b$, find an expression which exceeds $(a-b)$ by as much as $(a—b)$ exceeds b.

4. For what value of a is $6x^4-2x^3+4x^2+ax+a$ perfectly divisible by x^2-x+1?

5. Solve $\cdot75x-\cdot375+1=x-\cdot25+\cdot125x$.

6. Divide 28 into four parts, such that if 2 be added to the first, 4 be subtracted from the second, the third be multiplied by 3 and the fourth be divided by 2, the results are equal.

II

1. Plot the points (0, 0), (0, −12) and (−6, −7), and find the area of the triangle formed by joining them. Also show that the area of the triangle formed by joining the points (0, 0), (0, −12) and (−6, 0) is equal to the area of the above triangle.

2. Find the value of

$$5(a-b)-2\{3a—(a+b)\}+7\{(a-2b)-(5a-2b)\}, \text{ when } a=-\tfrac{1}{2}b.$$

3. Multiply $\cdot3+x^2$ by $\cdot2-x$ and find the value of the product when $x=\cdot1$.

4. Divide $(a+b)^3-8c^3$ by $a+b-2c$.

5. Solve $5x-\{8x-3(16-6x-\overline{4-5x})\}=x-(x-51)$.

6. Find two consecutive even numbers such that one-sixth of the smaller number exceeds one-seventh of the greater by 3.

III

1. Find the value of $c^4-(b-x)c^3+(b-x)x^3c-x^4$, when $b=-c=\frac{1}{2}$ and $x=1$.

2. Express 527 algebraically when $a=5$, $b=2$ and $c=7$.

3. Simplify $(x-1)(x+3)+5(x-4)^2-2(x-3)(x+2)-2x(2x-19)$.

4. Solve $(x-3)^2-(x+9)(x-1)=5(2-x)-13x$.

5. Solve the equations

$$1\cdot2x+\cdot6y=\cdot6,$$
$$\cdot3x-\cdot2y=\cdot01.$$

6. If the remainders obtained by dividing a^3+2a^2+3a+x and a^3+a^2+9 by $a+2$ separately are equal; find the value of x.

IV

1. Subtract $3x^3-7x+1$ from $2x^3-5x-3$, subtract the difference from 0, and then add the second difference to $2x^2-2x^3+2$.

2. Find the value of $a^3+b^3+c^3-3abc$, when $a=\cdot02$, $b=\cdot08$ and $c=-\cdot1$.

3. If $x^4+x^3-13x^2-31x+b$ is exactly divisible by x^2+3x+2, find the value of b.

4. Solve $\cdot2(x+4)-\cdot25(x-3)=\cdot2(x-1)+1$.

5. Solve $$\frac{x+3}{5}=\frac{8-y}{4}=\frac{3(x+y)}{8}.$$

6. A is 5 years older than B; 10 years ago $\frac{5}{6}$ of A's age exceeded $\frac{3}{5}$ of B's age by 15 years; find their ages.

V

1. Subtract $3a^2-b^2+c^2$ from 0.

2. Find the value of $9x^2+bc-20y$, when $x=0$, $y=4$, $b=3$ and $c=8$.

3. Simplify $(-a)^2\times(-a)^2\div a^5$.

4. Solve $(x-3)^2=x^2+4x+29$.

5. Solve the equations

$$4x+3=3y+2,$$
$$5x+4y=22.$$

6. Divide 57 into two parts such that the sum of one-third of the first part and one-seventh of the second equals 11.

VI

1. Solve $3x(x+1)-2x(x-1)=x^2+10$.

2. Solve $5x-3y=18y-10x=10$.

3. For what value of x is $(x+3)(x+4)$ equal to $(x+5)(x+7)$?

4. Find two numbers such that three times their sum is equal to 51 and their difference is equal to 7.

5. Solve the following equations graphically :

$$5x-3y=11,$$
$$2y-2x+4=0.$$

6. In an examination A got 11 marks less than B ; had A got half as many marks more, he would have beaten B by 17 marks. Find the number of marks obtained by each.

VII

1. Subtract $8x(x-1)$ from $16[1-x\{1-2x(1-x)\}+x^4]$, and divide the difference by $(1-2x)^2$.

2. Solve the following equation and verify your answer :

$$3x+1-\frac{5x-7}{17}=\frac{x-1}{2}+\frac{7x}{3}.$$

3. Solve the following equations graphically :

$$x+y=0,$$
$$4x+y=9.$$

4. What value of x will make the product of $(4x+3)$ and $(3x-4)$ exceed the product of $(6x+1)$ and $(2x-3)$ by 9 ?

5. A boy's present age is x years. (i) What will be his age 8 years hence ? (ii) What was his age 7 years ago ? If his age 8 years hence equals four times what his age was 7 years ago, find his present age.

6. A bill of Rs. 3. 3a. is paid in four-anna pieces and pice ; the sum of all the coins is 28, find the number of coins of each kind.

VIII

1. Find the value of $a(a^2+bc)+b(b^2+ac)-c(c^2-ab)$, when $a=\cdot 7$, $b=\cdot 08$ and $c=\cdot 78$.

2. Simplify $24\{x-\frac{1}{2}(x-2)\}\{x-\frac{2}{3}(x-1\frac{1}{2})\}\{x-\frac{3}{4}(x-1\frac{1}{3})\}$ and subtract the result from $(x+2)(x+3)(x+4)$.

3. Solve $\dfrac{3-x}{2}-\dfrac{1}{3}\left(\dfrac{3-2x}{4}\right)=\dfrac{2x+3}{7}+\left(\dfrac{1}{4}-\dfrac{1+3x}{2}\right).$

4. Solve the equations

$$x-\frac{y-2}{7}=5,\quad 4y-\frac{x+10}{3}=3.$$

5. Solve the following equations graphically :

$$2x+y=0,$$
$$y=\tfrac{1}{3}(x+5).$$

6. A man bought mangoes at the rate of 7 for 6*d.*; had he bought them at the rate of 13 for 1*s.*, he would have spent 6*d.* more; how many mangoes did he buy?

IX

1. There are $3x+4y-z$ passengers in a train, of whom $x-y+z$ are in the first class and $2x+3y-z$ in the third class. How many passengers are there in the second class? Find the number of passengers in each class when $x=45$, $y=36$ and $z=5$.

2. Divide $10(x^2-2ax)-3(ax-4a^2)$ by $2x-3a$.

3. Solve the following equations graphically:

$$y=3,\ \frac{x}{8}+\frac{y}{6}=1.$$

4. Solve $\frac{7x}{2}-\frac{x-8}{3}-\frac{4}{5}(4x+2)=\frac{5x-3}{7}$.

5. Solve the equations

$$\frac{x}{6}+\frac{y}{16}=6,\ \frac{y}{12}-\frac{x}{9}=2.$$

6. A railway train the speed of which is 40 miles an hour takes 2 hours less to complete a journey than another the speed of which is 24 miles an hour; find the length of the journey.

X

1. For what value of x is $\frac{x-3}{4}-\frac{2(x-7)}{3}+\frac{x+9}{12}$ equal to 0?

2. Solve the equations

$$(a+b)x-(a-b)y=4ab,$$
$$(a+b)y-(a-b)x=ab.$$

3. A man bought $4a$ sheep at the rate of $5p$ shillings a sheep and $5b$ sheep at the rate of $4q$ shillings a sheep; how many pounds did he spend in all?

4. Subtract the sum of the squares of $(2x+3y)$ and $(2x-3y)$ from $(3x-4y)(3x+4y)$ and find the value of the difference when $x=6y$.

5. A man bought oranges at the rate of 8*d.* per dozen. Of these 54 were spoiled and he sold the rest at the rate of 7 oranges for 5*d.* and thus lost 2*s.* 6*d.* How many oranges did he buy?

6. Solve the following equations algebraically and verify your answer graphically:

$$\frac{x}{10}+\frac{y}{12}=1,$$
$$5y=6x.$$

PART II

CHAPTER XIX

FACTORS

100. We have seen in Article 31 that the product of x and $y+z$ is $xy+xz$. Hence x and $y+z$ are the **factors** of the expression $xy+xz$.

Similarly, we have seen in Example 1 of Article 33, that the product of $a+b$ and $c+d$ is $ac+bc+ad+bd$, hence $a+b$ and $c+d$ are the factors of the expression $ac+bc+ad+bd$.

From the first example it is clear that in order to find the product of the expression $y+z$ and x, we *multiply* each term of the expression $y+z$ by x. Conversely, in order to find the factors of the product $xy+xz$, we have to find a quantity which *divides* each term of the expression $xy+xz$ exactly. In this case x divides both the terms of the expression, *i.e.*, *x is common to both the terms of the expression*, hence x is one of the factors. Now, dividing $xy+xz$ by x, we see that the quotient is $y+z$, which is therefore the other factor of the expression.

When all the terms of an expression are exactly divisible by a common factor, the expression is written down in the form of the product of factors by dividing each term of the expression by the common factor, enclosing the quotient within brackets, and placing the common factor outside the brackets.

Thus, since $xy+xz=x(y+z)$,

$\therefore$ x and $y+z$ are the factors of $xy+xz$.

Similarly $2x-2y=2(x-y)$,

$\therefore$ 2 and $x-y$ are the factors of $2x-2y$.

And $ab+ac+ad=a(b+c+d)$,

$\therefore$ a and $b+c+d$ are the factors of $ab+ac+ad$.

101. When an algebraic expression is expressed as a product of factors, the process of determining these factors is called **resolution into factors** or **factorising**.

From the examples of the previous article we see that resolution into factors is the converse problem of multiplication. In multiplication, the factors are given and we have to find the product, while in resolving into factors the product is given and we have to find its factors.

EXAMPLE 1. *Resolve* am+an *into factors.*

Here a is common to both terms,

$$\therefore \quad am+an=a(m+n).$$

Hence, the factors of $am+an$ are a and $m+n$.

EXAMPLE 2. *Resolve* 6x − 9y *into factors.*

Here 3 is common to both terms,

$$\therefore \quad 6x-9y=3(2x-3y).$$

Hence, the factors are 3 and $2x-3y$.

EXAMPLE 3. *Resolve* a^3-a^2 *into factors.*

Here a^2 is common to both terms,

$$\therefore \quad a^3-a^2=a^2(a-1).$$

EXAMPLE 4. *Resolve* $a^2bc+2ab^2c+3abc^2$ *into factors.*

Here each term of the expression is divisible by abc,

$$\therefore \quad a^2bc+2ab^2c+3abc^2=abc(a+2b+3c).$$

EXAMPLE 5. *Resolve* $7abc^2-14bc^3+21bc^4x$ *into factors.*

Here each term of the expression is divisible by $7bc^2$,

$$\therefore \quad 7abc^2-14bc^3+21bc^4x=7bc^2(a-2c+3c^2x).$$

EXAMPLE 6. *Resolve* $-x^4y^2+x^2y^3z-x^2y^2z^2$ *into factors.*

Here each term of the expression is divisible by $-x^2y^2$,

$$\therefore \quad -x^4y^2+x^2y^3z-x^2y^2z^2=-x^2y^2(x^2-yz+z^2).$$

NOTE. The above results should be verified by removing brackets.

EXAMPLES LI

Resolve into factors :

1. $ab+ac$. **2.** $ax-ay$. **3.** xy^2-xz. **4.** $3x+3y$.

5. $3x-6y$. **6.** $ax+2x$. **7.** $3x-9x^2$. **8.** x^2+x.

9. x^3-x^2. **10.** p^4-5p^3. **11.** a^2-ab. **12.** $5a^3-25a^2b$.
13. $39-91x$. **14.** $l^4-6l^2m^4$. **15.** xy^2z-3y^2. **16.** $-3a^2+2a$.
17. $abc+bcd$. **18.** $lmn-mpq$.
19. $lm^2n^3-l^3mn^2$. **20.** $4ax^2-6a^2x^4$.
21. $-2x^3y+4xy^3$. **22.** $ab+ac-ad$.
23. $xy-xz+xw$. **24.** $-al+am-an$.
25. $l^2a^2-xlab+lxyab$. **26.** $lmn^2+lm^2n-l^2mn$.
27. $7l^2mn^3-21l^3m^2n+35lm^3n^2$.
28. $-xy^2z+xyz^2-xyza+x^2yz$.

Factors found by Grouping Terms

102. In Example 1 of Article 33 in order to find the product of $a+b$ and $c+d$, we supposed $a+b$ to be equivalent to x, then

$$\begin{aligned}(a+b)\times(c+d) &= x(c+d)\\ &= xc+xd\\ &= (a+b)c+(a+b)d\\ &= ac+bc+ad+bd.\end{aligned}$$

Hence the product was $ac+bc+ad+bd$.

Now to find the factors of the expression $ac+bc+ad+bd$, we see that the expression as it stands has no common factor to all the terms, but if we notice carefully we see that the first two terms have a as a common factor and the last two terms have d as a common factor. Enclosing the first two terms in one bracket and last two terms in another, we have

$$\begin{aligned}&ac+bc+ad+bd\\ &\quad= (ac+bc)+(ad+bd)\\ &\quad= c(a+b)+d(a+b)\\ &\quad= cx+dx &&\text{[writing } x \text{ for } (a+b)]\\ &\quad= x(c+d)\\ &\quad= (a+b)(c+d). &&\text{[replacing } (a+b) \text{ by } x]\end{aligned}$$

Hence the factors are $a+b$ and $c+d$.

NOTE. It is not necessary to write x for $(a+b)$, for if we carefully examine the expression $c(a+b)+d(a+b)$, we see that it has two terms $c(a+b)$ and $d(a+b)$, of which $(a+b)$ is a common factor. Now dividing the expression by $(a+b)$, the quotient is $(c+d)$, which is the other factor of the expression.

103. If an expression is given in such a form that there is no common factor between its first two terms or last two terms, it is necessary to re-group the terms so as to reveal the presence of a common factor between the first two terms and another common factor between the last two.

For example, in the expression $ac+bd+bc+ad$, there is no common factor between the first two terms or the last two terms, but if it is re-grouped in the form $ac+bc+bd+ad$ or $ac+ad+bc+bd$, the presence of common factors is at once noticed.

NOTE. In the expression $ac+bd+bc+ad$, factors can also be found by grouping together the first and third terms, and also the second and the fourth; or the first and the fourth, and the second and the third, thus

$$\begin{aligned} &ac+bd+bc+ad \\ &\qquad=(ac+bc)+(bd+ad) \\ &\qquad=c(a+b)+d(a+b) \\ &\qquad=(a+b)(c+d). \end{aligned}$$

And also

$$\begin{aligned} &ac+bd+bc+ad \\ &\qquad=(ac+ad)+(bd+bc) \\ &\qquad=a(c+d)+b(c+d) \\ &\qquad=(c+d)(a+b). \end{aligned}$$

EXAMPLE 1. *Resolve* $ax-ay-bx+by$ *into factors.*

$$\begin{aligned} &ax-ay-bx+by \\ &\qquad=(ax-ay)-(bx-by) \\ &\qquad=a(x-y)-b(x-y) \\ &\qquad=(x-y)(a-b). \end{aligned}$$

EXAMPLE 2. *Resolve* x^3-x^2+x-1 *into factors.*

$$\begin{aligned} &x^3-x^2+x-1 \\ &\qquad=(x^3-x^2)+(x-1) \\ &\qquad=x^2(x-1)+(x-1) \\ &\qquad=(x-1)(x^2+1). \end{aligned}$$

EXAMPLE 3. *Resolve* ab^2-1-b^2+a *into factors.*

$$\begin{aligned} ab^2-1-b^2+a &= ab^2-b^2+a-1 \\ &= (ab^2-b^2)+(a-1) \\ &= b^2(a-1)+(a-1) \\ &= (a-1)(b^2+1). \end{aligned}$$

EXAMPLE 4. *Resolve* $x^2+xy+xz+xa+ya+za$ *into factors.*

$$\begin{aligned} x^2+xy+xz+xa+ya+za &= (x^2+xy+xz)+(xa+ya+za) \\ &= x(x+y+z)+a(x+y+z) \\ &= (x+y+z)(x+a). \end{aligned}$$

EXAMPLES LII

Resolve into factors :

1. $mx+nx+my+ny$.
2. $al-bl-am+bm$.
3. $cx+dx-cy-dy$.
4. $3x-3y-ax+ay$.
5. $a(l+m)+b(l+m)$.
6. $x^2(l-m)+(l-m)$.
7. $a(x+y)-b(x+y)$.
8. $2x(a^2+b^2)-3y(a^2+b^2)$.
9. $ab(c+d)+x(c+d)$.
10. $x^2+3x-4x-12$.
11. $x^2-2x+xa-2a$.
12. x^3+x^2+x+1.
13. y^3+y-y^2-1.
14. $ab+a+b+1$.
15. $3x^3+3x-7x^2-7$.
16. $6x^3-7x^2+6x-7$.
17. $3x^3-x^2+3x-1$.
18. x^5+x+x^4+1.
19. $x^2-yz-y+x^2z$.
20. $a^2-b^2+a-b^2a$.
21. $3l^3+12l-l^2-4$.
22. $x^3-xyz+x^2y-y^2z$.
23. $cx+3y-cy-3x$.
24. $a^3-5a^2-5a+25$.
25. a^2bc-a^2b-c+1.
26. $2a^2+7bc-7ab-2ac$.
27. $ax+bx+by+cy+cx+ay$.
28. $ax^2+ax+a+bx^2+bx+b$.
29. $lx-mx+my+ny-nx-ly$.
30. $ax^2+ay^2+3x^2+3y^2+axy+3xy$.

Trinomials

104. An algebraic expression consisting of three terms is called a **trinomial.**

We have seen in Article 43, that

$$(a+b)^2=a^2+b^2+2ab,$$

i.e., the square of the sum of two quantities is equal to the sum of the squares of the two quantities plus twice their product. Hence, if we can express a trinomial in the form of the sum of the squares of two quantities plus twice the product of the two quantities, the expression is equal to *the square of the sum of those two quantities.* Thus,

$$x^2+6x+9=(x)^2+2\times(3)\times(x)+(3)^2$$
$$=(x+3)^2.$$

Similarly

$$4x^2+20xy+25y^2=(2x)^2+2\times(2x)\times(5y)+(5y)^2$$
$$=(2x+5y)^2.$$

And

$$1+14a+49a^2=(1)^2+2\times(1)\times(7a)+(7a)^2$$
$$=(1+7a)^2.$$

105. We have seen in Article 45, that

$$(a-b)^2=a^2+b^2-2ab,$$

i.e., the square of the difference of two quantities is equal to the sum of the squares of the two quantities minus twice their product. Hence, if we can express a trinomial in the form of the sum of the squares of two quantities minus twice the product of the two quantities, the expression is equal to *the square of the difference of those two quantities.* Thus

$$x^2-6x+9=(x^2)-2\times(3)\times(x)+(3)^2$$
$$=(x-3)^2.$$

Similarly

$$4x^2-20xy+25y^2=(2x)^2-2\times(2x)\times(5y)+(5y)^2$$
$$=(2x-5y)^2.$$

And

$$1-14a+49a^2=(1)^2-2\times(1)\times(7a)+(7a)^2$$
$$=(1-7a)^2.$$

EXAMPLE. *Find the square root of* $(x+a)^2+2(x+a)(x+b)+(x+b)^2$.

Putting $(x+a)=p$ and $(x+b)=q$,

$$(x+a)^2+2(x+a)(x+b)+(x+b)^2$$
$$=p^2+2pq+q^2$$
$$=(p+q)^2$$
$$=\{(x+a)+(x+b)\}^2$$ [replacing p and q by $(x+a)$ and $(x+b)$ respectively].
$$=(x+a+x+b)^2$$
$$=(2x+a+b)^2.$$

Hence the square root $=2x+a+b$.

EXAMPLES LIII

What are the square roots of the following expressions? (*Oral*)

1. $x^2+2xy+y^2$. **2.** $x^2-2xy+y^2$. **3.** x^2+2x+1.
4. x^2+4x+4. **5.** $x^2-4xy+4y^2$. **6.** $a^2+10a+25$.
7. $a^2-12a+36$. **8.** $4k^2-4k+1$. **9.** $9p^4+6p^2+1$.
10. $4x^4+4x^2y^2+y^4$. **11.** $9l^4+6l^2p+p^2$.
12. $4x^2+12xy+9y^2$. **13.** $16x^4-8x^2y^2+y^4$.
14. $25y^2+20yz+4z^2$. **15.** $49p^4-70p^2q^3+25q^6$.
16. $64a^2b^2-16ab+1$. **17.** $1+22lm+121l^2m^2$.
18. $x^2+x+\frac{1}{4}$. **19.** $a^6-\frac{2}{3}a^3+\frac{1}{9}$.
20. $49a^4b^4+28a^2b^2c^2+4c^4$. **21.** $(p+q)^2+2(p+q)+1$.
22. $9+6(m-n)+(m-n)^2$. **23.** $(x+y)^2-4(x+y)z+4z^2$.
24. $(a-b)^2-10(a-b)c+25c^2$.
25. $(x+y)^2+2(x+y)(a+b)+(a+b)^2$.
26. $(l-m)^2-4(l-m)(p-q)+4(p-q)^2$.
27. $4(x-y)^2-12(x-y)(y-z)+9(y-z)^2$.

106. Trinomials of the second degree in which the coefficients of the highest power is unity. By multiplication we have

$$(x+2)(x+3)=x^2+2x+3x+6=x^2+5x+6,$$
and $$(x-2)(x-3)=x^2-2x-3x+6=x^2-5x+6.$$

From the above results we see that in the products, 5 (*irrespective of the sign, positive or negative*), which is the coefficient of x is equal to the sum of 2 and 3, and 6 which is the third term is equal to the product of 2 and 3. Hence to factorise x^2+5x+6 or x^2-5x+6, we have to find two numbers whose *sum* is 5 and *product* 6. These numbers are 2 and 3. Now writing $(2+3)$ for 5, we have

$$x^2+5x+6=x^2+(2+3)x+6$$
$$=x^2+2x+3x+6$$
$$=(x^2+2x)+(3x+6)$$ [enclosing the first two terms and the last two terms within brackets]
$$=x(x+2)+3(x+2)$$
$$=(x+2)(x+3).$$

Similarly

$$x^2-5x+6=x^2-(2+3)x+6$$
$$=x^2-2x-3x+6$$
$$=(x^2-2x)-(3x-6)$$
$$=x(x-2)-3(x-2)$$
$$=(x-2)(x-3).$$

Hence, we see that in order to factorise a trinomial expression of the second degree, in which the coefficient of the highest power of a certain letter is *unity* and the third term which is constant or of the lowest power of that letter is *positive*, we *find two numbers whose sum (irrespective of the sign) is equal to the coefficient of the second term and whose product is equal to the third term or the coefficient of the third term.*

EXAMPLE 1. *Resolve* $x^2+9x+14$ *into factors.*

Here we find two numbers whose sum is 9 and product 14. These numbers are 2 and 7, therefore writing $(2+7)$ for 9, we have

$$x^2+9x+14=x^2+(2+7)x+14$$
$$=x^2+2x+7x+14$$
$$=(x^2+2x)+(7x+14)$$
$$=x(x+2)+7(x+2)$$
$$=(x+2)(x+7).$$

EXAMPLE 2. *Resolve* $x^2-9xy+8y^2$ *into factors.*

Here we find two numbers whose sum is 9 and product 8. These numbers are 1 and 8.

$$\begin{aligned}\therefore\quad x^2-9xy+8y^2&=x^2-(1+8)xy+8y^2\\&=x^2-xy-8xy+8y^2\\&=(x^2-xy)-(8xy-8y^2)\\&=x(x-y)-8y(x-y)\\&=(x-y)(x-8y).\end{aligned}$$

EXAMPLE 3. *Resolve* x^4-11x^2+30 *into factors.*

Writing y for x^2, the expression is equivalent to $y^2-11y+30$, which is a trinomial of the second degree. Now we find two numbers whose sum is 11 and product 30. These numbers are 5 and 6.

$$\begin{aligned}\therefore\quad y^2-11y+30&=y^2-(5+6)y+30\\&=y^2-5y-6y+30\\&=(y^2-5y)-(6y-30)\\&=y(y-5)-6(y-5)\\&=(y-5)(y-6)\\&=(x^2-5)(x^2-6),\text{ writing } x^2 \text{ for } y.\end{aligned}$$

NOTE. It is not necessary to write y for x^2; x^4 may be regarded as the square of x^2.

EXAMPLE 4. *Resolve* $x^2+\frac{5}{6}x+\frac{1}{6}$ *into factors.*

Since $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ and $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$,

$$\begin{aligned}\therefore\quad x^2+\tfrac{5}{6}x+\tfrac{1}{6}&=x^2+(\tfrac{1}{2}+\tfrac{1}{3})x+\tfrac{1}{6}\\&=x^2+\tfrac{1}{2}x+\tfrac{1}{3}x+\tfrac{1}{6}\\&=(x^2+\tfrac{1}{2}x)+(\tfrac{1}{3}x+\tfrac{1}{6})\\&=x(x+\tfrac{1}{2})+\tfrac{1}{3}(x+\tfrac{1}{2})\\&=(x+\tfrac{1}{2})(x+\tfrac{1}{3}).\end{aligned}$$

EXAMPLE 5. *Resolve* $x^2-(a+b)x+ab$ *into factors.*

Here we find two quantities whose sum is $a+b$ and product ab. These are a and b.

$$\begin{aligned}\therefore\quad x^2-(a+b)x+ab&=x^2-ax-bx+ab\\&=(x^2-ax)-(bx-ab)\\&=x(x-a)-b(x-a)\\&=(x-a)(x-b).\end{aligned}$$

NOTE. It is clear from the above examples that if the coefficient of the second term of a trinomial is positive, the second terms of the factors are also positive, and if the coefficient of the second term is negative, the second terms of the factors are also negative.

EXAMPLES LIV

Resolve into factors :

1. x^2+6x+8.	2. $x^2+7x+10$.	3. $x^2-7x+12$.
4. x^2+7x+6.	5. x^2-8x+7.	6. a^2-5a+4.
7. $a^2+9a+14$.	8. $y^2-9y+20$.	9. $k^2-10k+24$.
10. $x^2-10x+21$.	11. $x^2-10x+9$.	12. $x^2-10x+25$.
13. $n^2+20n+96$.	14. $a^2+7ax+12x^2$.	15. $x^2-8xy+7y^2$.
16. $x^2-22xy+120y^2$.	17. $a^2+24ab+143b^2$.	18. $x^2y^2+9xy+18$.
19. $c^2d^2-14cd+40$.	20. $p^2q^2-18pq+17$.	21. $21+10x+x^2$.
22. $17-18x+x^2$.	23. $96-20n+n^2$.	24. $49-14a+a^2$.
25. a^4+18a^2+65.	26. x^4+12x^2+35.	27. a^4+12a^2+11.
28. a^4+25a^2+136.	29. c^6-10c^3+25.	30. $x^2+x+\frac{1}{4}$.
31. $p^2-p+\frac{1}{4}$.	32. $y^2-\frac{2}{3}y+\frac{1}{9}$.	33. $y^2+\frac{3}{4}y+\frac{1}{8}$.

34. $x^2+\frac{7}{12}x+\frac{1}{12}$.
35. $x^2+(a+b)x+ab$.
36. $x^2-(l+m)x+lm$.
37. $y^2-(2a+b)y+2ab$.
38. $y^2+(2a+3b)y+6ab$.
39. $x^2-(p+5q)x+5pq$.
40. $x^2+(m^2+n^2)x+m^2n^2$.

107. By multiplication we have

$$(x-2)(x+5)=x^2+5x-2x-10=x^2+3x-10,$$

and $(x+2)(x-5)=x^2-5x+2x-10=x^2-3x-10.$

From the above results we see that in the products, 3 (*irrespective of the sign, positive or negative*), which is the coefficient of x is equal to the difference of 5 and 2, and 10 which is the third term is equal to the product of 5 and 2. Hence, to factorise $x^2+3x-10$ or $x^2-3x-10$, we have to find two numbers whose *difference* is 3 and *product* 10. These numbers are 5 and 2. Now writing $(5-2)$ for 3, we have

$$x^2+3x-10=x^2+(5-2)x-10$$
$$=x^2+5x-2x-10$$
$$=(x^2+5x)-(2x+10)$$

[enclosing the first two terms and the last two terms within brackets]

$$=x(x+5)-2(x+5)$$
$$=(x+5)(x-2).$$

Similarly

$$\begin{aligned} x^2 - 3x - 10 &= x^2 - (5-2)x - 10 \\ &= x^2 - 5x + 2x - 10 \\ &= (x^2 - 5x) + (2x - 10) \\ &= x(x-5) + 2(x-5) \\ &= (x-5)(x+2). \end{aligned}$$

Hence, we see that in order to factorise a trinomial expression of the second degree, in which the coefficient of the highest power of a certain letter is *unity* and the third term which is constant or of the lowest power of that letter is *negative*, we find *two numbers whose difference (irrespective of the sign) is equal to the coefficient of the second term and whose product is equal to the third term or the coefficient of the third term.*

EXAMPLE 1. *Resolve* $x^2 + 5x - 14$ *into factors.*

Here we find two numbers whose difference is 5 and product 14. These numbers are 7 and 2, therefore writing $(7-2)$ for 5, we have

$$\begin{aligned} x^2 + 5x - 14 &= x^2 + (7-2)x - 14 \\ &= x^2 + 7x - 2x - 14 \\ &= (x^2 + 7x) - (2x + 14) \\ &= x(x+7) - 2(x+7) \\ &= (x+7)(x-2). \end{aligned}$$

EXAMPLE 2. *Resolve* $x^2 - 2xy - 35y^2$ *into factors.*

The two numbers, whose difference is 2 and product 35, are 7 and 5.

$$\begin{aligned} \therefore\quad x^2 - 2xy - 35y^2 &= x^2 - (7-5)xy - 35y^2 \\ &= x^2 - 7xy + 5xy - 35y^2 \\ &= (x^2 - 7xy) + (5xy - 35y^2) \\ &= x(x-7y) + 5y(x-7y) \\ &= (x-7y)(x+5y). \end{aligned}$$

EXAMPLE 3. *Resolve* $x^2 + \frac{1}{6}x - \frac{1}{6}$ *into factors.*

Since $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ and $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$,

$$\begin{aligned} \therefore\quad x^2 - \tfrac{1}{6}x - \tfrac{1}{6} &= x^2 + (\tfrac{1}{2} - \tfrac{1}{3})x - \tfrac{1}{6} \\ &= x^2 + \tfrac{1}{2}x - \tfrac{1}{3}x - \tfrac{1}{6} \\ &= (x^2 + \tfrac{1}{2}x) - (\tfrac{1}{3}x + \tfrac{1}{6}) \\ &= x(x+\tfrac{1}{2}) - \tfrac{1}{3}(x+\tfrac{1}{2}) \\ &= (x+\tfrac{1}{2})(x-\tfrac{1}{3}). \end{aligned}$$

EXAMPLE 4. *Resolve* $x^2-(a-b)x-ab$ *into factors.*

The two numbers, whose difference is $(a-b)$ and product ab, are a and b.

$$\therefore \quad x^2-(a-b)x-ab=x^2-ax+bx-ab$$
$$=(x^2-ax)+(bx-ab)$$
$$=x(x-a)+b(x-a)$$
$$=(x-a)(x+b).$$

EXAMPLES LV

Resolve into factors :

1. $x^2+x-12.$ **2.** $x^2-x-2.$ **3.** $x^2-2x-3.$

4. $x^2+x-6.$ **5.** $x^2-x-6.$ **6.** $y^2+2y-63.$

7. $y^2-y-72.$ **8.** $a^2+a-56.$ **9.** $a^2-4a-77.$

10. $x^2+10x-24.$ **11.** $b^2-20b-21.$ **12.** $c^2-7c-78.$

13. $x^2+2xa-15a^2.$ **14.** $x^2-3xa-28a^2.$ **15.** $x^2-xy-30y^2.$

16. $l^2+lm-110m^2.$ **17.** $x^2y^2+4xy-5.$

18. $c^2d^2-3cd-70.$ **19.** $a^2b^2-8ab-48.$

20. $a^2x^2+2ax-35.$ **21.** $a^2b^2-5abc-36c^2.$

22. $x^2-2xyz-63y^2z^2.$ **23.** $x^2-3xpq^2-10p^2q^4.$

24. $x^2-\frac{1}{6}x-\frac{1}{6}.$ **25.** $a^2+\frac{1}{4}a-\frac{1}{8}.$

26. $n^2-\frac{3}{10}n-\frac{1}{10}.$ **27.** $x^2+(a-b)x-ab.$

28. $x^2-(l-m)x-lm.$ **29.** $y^2-(a-2b)y-2ab.$

30. $y^2-(3a-2b)y-6ab.$ **31.** $x^2-(2p-5q)x-10pq.$

32. $x^4-2x^2-3.$ **33.** $x^4-4x^2-5.$

34. $a^4-a^2-6.$ **35.** $y^4-y^2-2.$

36. $x^4+3x^2-10.$ **37.** $x^4+2x^2y^2-8y^4.$

38. $l^4+2l^2m^2-63m^4.$ **39.** $p^4-5p^2q^2-66q^4.$

40. $a^6+6a^3-16.$

Difference of Two Squares

108. We have seen in Article 47, that

$$a^2-b^2=(a+b)(a-b),$$

i.e., the difference of the squares of two quantities is equal to the product of their sum and difference.

Hence, *an expression which is equal to the difference of the squares of two quantities can be resolved into two factors, one is equal to the sum and the other to the difference of those quantities.*

For example

(i) $x^2-9=x^2-3^2=(x+3)(x-3)$.

(ii) $36x^2-25y^2=(6x)^2-(5y)^2=(6x+5y)(6x-5y)$.

(iii) $\frac{25x^2}{49}-1=\left(\frac{5x}{7}\right)^2-(1)^2=\left(\frac{5x}{7}+1\right)\left(\frac{5x}{7}-1\right)$.

EXAMPLE 1. *Resolve* $(x+y)^2-(a-b)^2$ *into factors.*

Regarding $(x+y)$ as one quantity and $(a-b)$ as another, we have

$$(x+y)^2-(a-b)^2=\{(x+y)+(a-b)\}\{(x+y)-(a-b)\}$$
$$=(x+y+a-b)(x+y-a+b).$$

EXAMPLE 2. *Resolve* $4(a+b)^2-9(a-b)^2$ *into factors.*

$$4(a+b)^2-9(a-b)^2=\{2(a+b)\}^2-\{3(a-b)\}^2$$
$$=\{2(a+b)+3(a-b)\}\{2(a+b)-3(a-b)\}$$
$$=(2a+2b+3a-3b)(2a+2b-3a+3b).$$
$$=(5a-b)(-a+5b)$$
$$=(5a-b)(5b-a).$$

EXAMPLE 3. *Resolve* x^4-16y^4 *into factors.*

$$x^4-16y^4=(x^2)^2-(4y^2)^2$$
$$=(x^2+4y^2)(x^2-4y^2).$$

Since $\quad x^2-4y^2=(x+2y)(x-2y)$,

$$x^4-16y^4=(x^2+4y^2)(x+2y)(x-2y).$$

109. Sometimes by suitably grouping together terms, compound expressions can often be expressed as the difference of two squares, and then can be resolved into factors.

EXAMPLE 1. *Resolve* $a^2+2ab+b^2-x^2$ *into factors.*

The first three terms of the expression form a perfect square, hence writing them within one bracket, we have

$$a^2+2ab+b^2-x^2=(a^2+2ab+b^2)-x^2$$
$$=(a+b)^2-x^2$$

[Regarding $(a+b)$ as one quantity]

$$=\{(a+b)+x\}\{(a+b)-x\}$$
$$=(a+b+x)(a+b-x).$$

EXAMPLE 2. *Resolve* $12ab+2cd-c^2-d^2+4a^2+9b^2$ *into factors.*

$$\begin{aligned}12ab+2cd-c^2-d^2+4a^2+9b^2\\ &=4a^2+12ab+9b^2-c^2+2cd-d^2\\ &=(4a^2+12ab+9b^2)-(c^2-2cd+d^2)\\ &=(2a+3b)^2-(c-d)^2\\ &=\{(2a+3b)+(c-d)\}\{(2a+3b)-(c-d)\}\\ &=(2a+3b+c-d)(2a+3b-c+d).\end{aligned}$$

EXAMPLES LVI

(*Examples* 1 *to* 20 *may be attempted orally*)

Resolve into factors :

1. a^2-b^2. **2.** x^2-y^2. **3.** x^2-1.

4. x^2-4. **5.** a^2-16. **6.** $16-a^2$.

7. $9-y^2$. **8.** $36-n^2$. **9.** p^2-121.

10. x^2-4y^2. **11.** l^2-49m^2. **12.** $4l^2-1$.

13. $100x^2-1$. **14.** $4l^2-9m^2$. **15.** $25c^2-d^2$.

16. $144k^2-169l^2$. **17.** $64a^2b^2-1$. **18.** a^2b^2-64.

19. $81-25x^2y^2$. **20.** $225-4m^2n^2$. **21.** x^4-y^4.

22. a^4-1. **23.** $1-a^4$. **24.** l^4-4m^4.

25. p^4-16q^4. **26.** $16a^4-b^4$. **27.** $25a^4-9b^4$.

28. a^4-100x^4. **29.** $81a^2-b^4$. **30.** $64a^6-b^4$.

31. $10000x^4-1$. **32.** $625-y^4$. **33.** $81x^4-256y^4$.

34. $121l^2-p^2q^4$. **35.** $p^4q^4-4r^2s^2$. **36.** y^6-4x^{10}.

37. $a^4b^{10}-c^8$. **38.** x^8-y^{12}. **39.** $400-a^2b^4$.

40. $\frac{x^2}{y^2}-1$. **41.** $x^2-\frac{a^2}{b^2}$. **42.** $\frac{4x^2}{9}-25$.

43. $1-\frac{25x^2}{16}$. **44.** $\frac{16a^2}{25}-9$. **45.** $\frac{x^2}{y^2}-\frac{a^2}{b^2}$.

46. $\frac{4x^2}{9y^2}-\frac{25a^2}{16b^2}$. **47.** $\frac{4x^2}{25a^2}-\frac{49y^2}{16b^2}$. **48.** $(a+b)^2-c^2$.

49. $(a-b)^2-4x^2$. **50.** $x^2-(y+z)^2$. **51.** $x^2-(y-z)^2$.

52. $(a-b)^2-1$. **53.** $1-(l-m)^2$. **54.** $4m^2-(n+p)^2$.

55. $(x+y)^2-(a+b)^2$. **56.** $(x-y)^2-(a-b)^2$.

57. $(a+b)^2-(a-b)^2$. **58.** $(a-b)^2-(a+b)^2$.

59. $(2x+3y)^2-(2x-3y)^2$.

60. $(2a+3b)^2-(5a+4b)^2$.

61. $(x+1)^2-(x-1)^2$.

62. $(ab-1)^2-(xy+1)^2$.

63. $(x+y+z)^2-(a+b+c)^2$.

64. $(x+y+z)^2-(x+y-z)^2$.

65. $(x-y-z)^2-(x-y+z)^2$.

66. $(a+b-c)^2-(l-m+n)^2$.

67. $4(a-b)^2-9(c-d)^2$.

68. $49(a+b)^2-16(c+d)^2$.

69. $25(a+b-c)^2-1$.

70. $100(a+b-c)^2-(x-y+z)^2$.

71. $36(x+y+z)^2-121(x-y-z)^2$.

72. $(x+\frac{1}{2})^2-\frac{1}{9}$.

73. $(x+\frac{1}{3})^2-\frac{1}{4}$.

74. $(x-\frac{1}{3})^2-\frac{1}{9}$.

75. $(y+\frac{1}{2})^2-1$.

76. $(y+\frac{3}{4})^2-\frac{9}{16}$.

77. $(ab-\frac{2}{3})^2-\frac{4}{9}$.

78. $(k-\frac{2}{5}l)^2-\frac{16}{25}l^2$.

79. $\left(4p-\dfrac{3q}{2r}\right)^2-\dfrac{49}{16}$.

80. $x^2+2xy+y^2-a^2$.

81. $x^2-2xy+y^2-z^2$.

82. $a^2-2ab+b^2-c^2$.

83. $l^2+4lm+4m^2-n^2$.

84. $a^2-x^2-2xy-y^2$.

85. $x^2-a^2+2ab-b^2$.

86. $16a^2-x^2-4xy-4y^2$.

87. $9a^2-6ab+b^2-81c^2$.

88. $a^2+2ab+b^2-x^2-2xy-y^2$.

89. $l^2-2lm+m^2-p^2+2pq-q^2$.

90. $x^2-a^2-b^2+y^2-2xy+2ab$.

91. $a^2+b^2-c^2-d^2-2ab-2cd$.

92. $4x^4-25x^2+10x-1$.

110. Trinomials in which the coefficient of the highest power is not unity. The method of factorising trinomials, in which the coefficient of the highest power is not unity, is the same as that of factorising trinomials in which the coefficient of the highest power is unity, the only difference is (when the trinomial is arranged in descending powers of a certain letter) that the coefficient of the first term is multiplied by the third term or the coefficient of the third term (irrespective of the sign, positive or negative), and if the third term is *positive*, we find two factors of this product whose *sum* is equal to the coefficient of the second term, and if *negative*, we find two factors whose *difference* is equal to the coefficient of the second term, and then write the sum or difference of these two factors in place of the coefficients of the second term and then proceed as in Articles 106 and 107.

EXAMPLE 1. *Resolve* $6x^2+13x+5$ *into factors.*

Multiplying 6 the coefficient of x^2, by 5 the third term, we get the product 30. Since the third term is positive, we find two factors of 30 such that their sum is equal to 13 the coefficient of the second term. These factors are 10 and 3,

$$\begin{aligned}6x^2+13x+5&=6x^2+(10+3)x+5\\&=6x^2+10x+3x+5\\&=(6x^2+10x)+(3x+5)\\&=2x(3x+5)+(3x+5)\\&=(3x+5)(2x+1).\end{aligned}$$

EXAMPLE 2. *Resolve* $6x^2-7x-5$ *into factors.*

Multiplying 6 the coefficient of x^2, by 5 the third term, we get the product 30. Since the third term is negative, we find two factors of 30, such that their difference is equal to 7 the coefficient of the second term. These factors are 10 and 3,

$$\begin{aligned}\therefore\quad 6x^2-7x-5&=6x^2-(10-3)x-5\\&=6x^2-10x+3x-5\\&=(6x^2-10x)+(3x-5)\\&=2x(3x-5)+(3x-5)\\&=(3x-5)(2x+1).\end{aligned}$$

EXAMPLE 3. *Resolve* $5x^2-31xy+30y^2$ *into factors.*

Multiplying 5 the coefficient of the first term, by 30 the coefficient of the third term, we get the product 150. Since the two factors of 150, whose sum is equal to 31 the coefficient of the second term, are 25 and 6,

$$\begin{aligned}\therefore\quad 5x^2-31xy+30y^2&=5x^2-(25+6)xy+30y^2\\&=5x^2-25xy-6xy+30y^2\\&=(5x^2-25xy)-(6xy-30y^2)\\&=5x(x-5y)-6y(x-5y)\\&=(x-5y)(5x-6y).\end{aligned}$$

EXAMPLE 4. *Resolve* $10(x+y)^2+99(x+y)xy-10x^2y^2$ *into factors.*

Writing p for $(x+y)$ and q for xy, we have

$$\begin{aligned}&10(x+y)^2+99(x+y)xy-10x^2y^2\\&\quad=10p^2+99pq-10q^2\\&\quad=10p^2+(100-1)pq-10q^2\\&\quad=10p^2+100pq-pq-10q^2\\&\quad=(10p^2+100pq)-(pq+10q^2)\\&\quad=10p(p+10q)-q(p+10q)\\&\quad=(p+10q)(10p-q)\end{aligned}$$

[Replacing p by $(x+y)$ and q by xy]

$$\begin{aligned}&=\{(x+y)+10xy\}\{10(x+y)-xy\}\\&=(x+y+10xy)(10x+10y-xy).\end{aligned}$$

EXAMPLE 5. *Resolve* $acx^2+(bc-ad)x-bd$ *into factors.*

Multiplying ac the coefficient of x^2, by bd the third term, we get the product $abcd$. Since the two factors of $abcd$, whose difference is equal to $bc-ad$ the coefficient of the second term, are bc and ad,

$$\begin{aligned}\therefore\quad acx^2+(bc-ad)x-bd &= acx^2+bcx-adx-bd\\ &= (acx^2+bcx)-(adx+bd)\\ &= cx(ax+b)-d(ax+b)\\ &= (ax+b)(cx-d).\end{aligned}$$

EXAMPLES LVII

Resolve into factors :

1. $2x^2+5x+2$. 2. $2a^2+3a+1$. 3. $2a^2+a-1$.
4. $2a^2-a-1$. 5. $3y^2+5y+2$. 6. $2y^2-3y-2$.
7. $5y^2-12y+4$. 8. $3a^2-7a+2$. 9. $3x^2-4x-4$.
10. $2m^2+19m+9$. 11. $2m^2+9m+10$. 12. $3a^2-11a+6$.
13. $12x^2-x-20$. 14. $4x^2-13x-12$. 15. $6x^2-23x+7$.
16. $3k^2-13k+14$. 17. $4k^2-8k+3$. 18. $2b^2-5b+3$.
19. $2b^2+b-28$. 20. $3n^2+7n-6$. 21. $6n^2+35n+44$.
22. $14c^2+29c-15$. 23. $49c^2+21c+2$. 24. $5+11y+2y^2$.
25. $3-8x+4x^2$. 26. $6+17x+5x^2$. 27. $20-41x+20x^2$.
28. $15-11x+2x^2$. 29. $3-11y+6y^2$. 30. $12-25z+12z^2$.
31. $8+z-7z^2$. 32. $12x^2+8xy-15y^2$. 33. $2x^2-5xy-3y^2$.
34. $6x^2-7xy+2y^2$. 35. $2x^2+11xy-21y^2$. 36. $7a^2+6ab-16b^2$.
37. $10a^2-13ab-9b^2$. 38. $a^2b^2+4ab-60$.
39. $a^2b^2-ab-72$. 40. $32x^2y^2-84xy-135$.
41. $20x^2y^2-9xy-20$. 42. $15+xy-2x^2y^2$.
43. $2x^4+7x^2+3$. 44. $3x^4+7x^2+2$.
45. $3x^4-20x^2-7$. 46. $5x^4-41x^2+8$.
47. $6x^4-13x^2+5$. 48. $21x^4-101x^2+88$.
49. $7(a+b)^2+48(a+b)ab-7a^2b^2$.
50. $3(a+b)^2+5(a+b)ab-2a^2b^2$.
51. $6a^2+7a(b+c)-3(b+c)^2$.
52. $3(x+y)^2-7(x+y)(a+b)-6(a+b)^2$.
53. $2(x+y)^2+9(x+y)(x-y)-5(x-y)^2$.
54. $ax^2+(ac+b)x+bc$.
55. $ax^2+(ac-b)x-bc$. 56. $lnx^2+(l-mn)x-m$.
57. $pqx^2-(p+q)x+1$. 58. $abx^2+(3a+2b)x+6$.
59. $acx^2+(bc+ad)x+bd$. 60. $lnx^2-(lp+mn)xy+mpy^2$.

Sum and Difference of Two Cubes

111. By actual multiplication, we have

$$(a+b)(a^2-ab+b^2)=a^3+b^3,$$

and $$(a-b)(a^2+ab+b^2)=a^3-b^3.$$

Hence, we see that *an expression which is equal to*

(i) *the sum of the cubes of two quantities can be resolved into two factors, one of which is equal to the sum of the two quantities, and the other is equal to the sum of the squares of the two quantities minus their product* ;

(ii) *the difference of the cubes of two quantities can be resolved into two factors, one of which is equal to the difference of the two quantities, and the other is equal to the sum of the squares of the two quantities plus their product.*

For example,

$$\begin{aligned} x^3+8 &= x^3+2^3 \\ &= (x+2)(x^2-x\times 2+2^2) \\ &= (x+2)(x^2-2x+4)\ ; \end{aligned}$$

$$\begin{aligned} 27x^3y^3-64 &= (3xy)^3-(4)^3 \\ &= (3xy-4)\{(3xy)^2+(3xy)(4)+(4)^2\} \\ &= (3xy-4)(9x^2y^2+12xy+16)\ ; \end{aligned}$$

and $$\begin{aligned} \frac{x^3y^6}{27}-\frac{27}{z^9} &= \left(\frac{xy^2}{3}\right)^3-\left(\frac{3}{z^3}\right)^3 \\ &= \left(\frac{xy^2}{3}-\frac{3}{z^3}\right)\left\{\left(\frac{xy^2}{3}\right)^2+\left(\frac{xy^2}{3}\right)\left(\frac{3}{z^3}\right)+\left(\frac{3}{z^3}\right)^2\right\} \\ &= \left(\frac{xy^2}{3}-\frac{3}{z^3}\right)\left(\frac{x^2y^4}{9}+\frac{xy^2}{z^3}+\frac{9}{z^6}\right). \end{aligned}$$

EXAMPLE. *Resolve* $(a+b)^3+(x-y)^3$ *into factors.*

Writing p for $(a+b)$ and q for $(x-y)$, we have

$$\begin{aligned} &(a+b)^3+(x-y)^3 \\ &\quad =p^3+q^3 \\ &\quad =(p+q)(p^2-pq+q^2) \\ &\quad =\{(a+b)+(x-y)\}\{(a+b)^2-(a+b)(x-y)+(x-y)^2\} \\ &\quad =(a+b+x-y)(a^2+2ab+b^2-ax+ay+by-bx+x^2-2xy+y^2) \\ &\quad =(a+b+x-y)(a^2+b^2+x^2+y^2+2ab-2xy-ax+ay-bx+by). \end{aligned}$$

EXAMPLES LVIII

(*Examples* 1 *to* 31 *may be taken orally*)

Resolve into factors :

1. x^3+y^3.
2. x^3-y^3.
3. x^3+1.
4. x^3-1.
5. x^6-y^3.
6. $8a^3+1$.
7. a^3-8.
8. $27a^3-1$.
9. $8a^3+27$.
10. $27x^3-8$.
11. x^3-8y^6.
12. $64x^3+27a^3$.
13. $a^3-1000x^3$.
14. $125+p^3q^3$.
15. $8a^6+27x^3$.
16. $64x^3y^3-1$.
17. $343l^3m^6-1$.
18. $8+729x^3y^3$.
19. $1000a^3-27b^3c^3d^3$.
20. $x^3y^6z^9-27$.
21. $(2xy)^3-125$.
22. $(3pq)^3-343$.
23. $a^3+\frac{1}{b^3}$.
24. $a^3+\frac{1}{a^3}$.
25. $x^3-\frac{1}{x^3}$.
26. $x^3+\frac{8}{x^3}$.
27. $\frac{1}{x^3}-\frac{1}{y^3}$.
28. $\frac{x^3}{y^3}-\frac{a^3}{b^3}$.
29. $\frac{64}{a^3}-\frac{a^3}{64}$.
30. $\frac{a^3b^3}{c^3}-\frac{z^3}{x^3y^3}$.
31. $\frac{a^3b^6}{8}+\frac{64}{a^6b^3}$.
32. $(a+b)^3-c^3$.
33. $c^3-(a-b)^3$.
34. $l^3m^3+(l+m)^3$.
35. $(2x+3y)^3-8y^3$.
36. $8x^3-(2x-y)^3$.
37. $(x-y)^3-(x+y)^3$.
38. $(x+1)^3-(x-1)^3$.
39. $(p+2q)^3-(p-2q)^3$.
40. x^6+y^6.
41. x^6+1.
42. $1-x^6$.
43. a^6-b^6.
44. x^6-729.

Method of Completing the Square

112. We have already seen that

$$a^2+2ab+b^2=(a+b)^2,$$

and
$$a^2-2ab+b^2=(a-b)^2.$$

That is, *a trinomial is a perfect square of the sum or difference of two quantities, if two of its terms are perfect squares of those quantities with their signs positive and the third term is equal to twice their product with the sign positive or negative.*

If a trinomial is not a perfect square, it can be made into a perfect square by adding or subtracting certain terms to it. Thus, the expression a^2+2ab or a^2-2ab, in which the coefficient of a^2 is unity will become a perfect square if b^2 (*i.e.*, the square of half the coefficient of a irrespective of the sign, positive or negative) is added to it. Similarly, x^2+10x will become a perfect square if we add the square of half the coefficient of x, *i.e.*, $(\frac{10}{2})^2$ or 25, for then the expression

$$=x^2+10x+25=x^2+2\times5\times x+5^2=(x+5)^2.$$

In order to make $16x^2-24x$ a perfect square, in which the coefficient of x^2 is not unity, we write the expression thus :

$$16x^2-24x=(4x)^2-6(4x).$$

Now regarding $(4x)$ as one quantity the given expression can be made into a perfect square, if we add the square of half the coefficient of $(4x)$ *i.e.*, $(\frac{6}{2})^2$ or 9 to it, for then the expression

$$=(4x)^2-6\times(4x)+9=(4x)^2-2(4x)(3)+(3)^2=(4x-3)^2.$$

EXAMPLE. 1. *What should be added to* $9x^2-30xy$ *to make it a perfect square ?*

$$9x^2-30xy=(3x)^2-10y(3x).$$

The given expression can be made into a perfect square, if we add the square of half the coefficient of $(3x)$ *i.e.*, $\left(\frac{10y}{2}\right)^2$ or $25y^2$, for then the expression

$$\begin{aligned}&=(3x)^2-10y(3x)+25y^2\\&=(3x)^2-2(3x)(5y)+(5y)^2\\&=(3x-5y)^2.\end{aligned}$$

EXAMPLE 2. *What should be added to* x^4+y^4 *to make it a perfect square ?*

Since both the terms of the expression are perfect squares, the third term will be equal to twice the product of their square roots with the positive or negative sign. Hence, the expression will be a perfect square if $+2x^2y^2$ or $-2x^2y^2$ be added to it.

113. Sometimes when the factors of a trinomial cannot be easily found by inspection, we find its factors by expressing the trinomial as the difference of two squares by the *method of completing the square* explained in the previous article.

EXAMPLE 1. *Resolve* $x^2-7x+12$ *into factors.*

First complete the square of x^2-7x by adding $(\frac{7}{2})^2$ to it. Now in order that the expression may remain the same we also subtract $(\frac{7}{2})^2$ from it. Hence we have

$$\begin{aligned} x^2-7x+12 &= x^2-7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2+12 \\ &= x^2-2.\tfrac{7}{2}x+(\tfrac{7}{2})^2-\tfrac{1}{4} \\ &= (x-\tfrac{7}{2})^2-(\tfrac{1}{2})^2 \\ &= (x-\tfrac{7}{2}+\tfrac{1}{2})(x-\tfrac{7}{2}-\tfrac{1}{2}) \\ &= (x-3)(x-4). \end{aligned}$$

EXAMPLE 2. *Resolve* $9x^2+12x-5$ *into factors.*

$$\begin{aligned} 9x^2+12x-5 &= (3x)^2+4(3x)-5 \\ &= (3x)^2+2(3x)(2)+(2)^2-(2)^2-5 \\ &= (3x+2)^2-9 \\ &= (3x+2)^2-(3)^2 \\ &= (3x+2+3)(3x+2-3) \\ &= (3x+5)(3x-1). \end{aligned}$$

EXAMPLE 3. *Resolve* $143x^2-149x-184$ *into factors.*

Here it is not easy to find two numbers by inspection whose product is equal to 143×184 *i.e.*, 26312, and whose difference is equal to 149. Moreover the first term is not a perfect square. In such cases we proceed thus:

$$\begin{aligned} 143x^2-149x-184 &= 143\left(x^2-\frac{149}{143}x-\frac{184}{143}\right) \\ &= 143\left\{x^2-\frac{149}{143}x+\left(\frac{149}{286}\right)^2-\left(\frac{149}{286}\right)^2-\frac{184}{143}\right\} \\ &= 143\left\{\left(x-\frac{149}{286}\right)^2-\frac{127449}{81796}\right\} \\ &= 143\left\{\left(x-\frac{149}{286}\right)^2-\left(\frac{357}{286}\right)^2\right\} \\ &= 143\left(x-\frac{149}{286}+\frac{357}{286}\right)\left(x-\frac{149}{286}-\frac{357}{286}\right) \\ &= 143\left(x-\frac{208}{286}\right)\left(x-\frac{506}{286}\right) \\ &= 143\left(x+\frac{8}{11}\right)\left(x-\frac{23}{13}\right) \\ &= 11\left(x+\frac{8}{11}\right)13\left(x-\frac{23}{13}\right) \\ &= (11x+8)(13x-23). \end{aligned}$$

EXAMPLES LIX

(*Examples* 1—30 *may be taken orally*)

What should be added to the following expressions to make them into perfect squares ?

1. x^2+2x. **2.** x^2-2x. **3.** x^2+4x. **4.** x^2+9.
5. x^2-x+1. **6.** x^2+6x. **7.** x^2+6x+7. **8.** x^2+100.
9. x^2+14x. **10.** x^2-15x. **11.** x^2+2xy. **12.** a^2-4ab.
13. a^2+2ab. **14.** a^2+3ab. **15.** a^2-5ax. **16.** x^2+y^2.
17. a^2+4x^2. **18.** $4a^2+9b^2$. **19.** $49x^2+25y^2$.
20. $121x^2+36a^2$. **21.** $4x^2+4x$. **22.** $4x^2-8x$.
23. $25a^2-20a$. **24.** $9a^2+6ab$. **25.** $49x^2-28xy$.
26. $36y^2-y$. **27.** $9x^2y^2+3xy$. **28.** $\frac{1}{4}x^2-2x$.
29. $\frac{4}{9}p^2q^2-\frac{1}{3}pq$. **30.** $\frac{1}{4}x^2+\frac{1}{3}x+\frac{2}{3}$.

Resolve into factors, by the method of completing the square :

31. x^2+6x+8. **32.** $x^2-8x+15$. **33.** $x^2+4x-32$.
34. $x^2-6x-27$. **35.** x^2+5x+4. **36.** $x^2-13x+40$.
37. $x^2+7x-60$. **38.** x^2+x-20. **39.** $x^2-xy-132y^2$.
40. $x^2+3xy-154y^2$. **41.** $4x^2+8x+3$. **42.** $9x^2-3x-2$.
43. $25x^2+5x-6$. **44.** $16x^2+8xy-15y^2$.
45. $3x^2+8x-3$. **46.** $8x^2-65x+8$.
47. $6x^2+13xy+6y^2$. **48.** $35x^2+41xy-22y^2$.
49. $126x^2-89xy-26y^2$. **50.** $204x^2-89xy-143y^2$.

Harder Factors

114. EXAMPLE 1. *Resolve* $x^4+x^2y^2+y^4$ *into factors.*

Expressions of this type can be resolved into factors very easily by expressing them as the difference of two squares by the *method of completing the square*. Hence to make the expression a perfect square, we add x^2y^2, and in order that the expression may remain the same we also subtract x^2y^2, thus

$$\begin{aligned} x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy). \end{aligned}$$

EXAMPLE 2. *Resolve* $4x^4-16x^2+9$ *into factors.*

$$\begin{aligned}4x^4-16x^2+9&=4x^4-12x^2+9-4x^2\\&=(2x^2-3)^2-(2x)^2\\&=(2x^2+2x-3)(2x^2-2x-3).\end{aligned}$$

EXAMPLE 3. *Resolve* x^4+4y^4 *into factors.*

$$\begin{aligned}x^4+4y^4&=x^4+4x^2y^2+4y^4-4x^2y^2\\&=(x^2+2y^2)^2-(2xy)^2\\&=(x^2+2xy+2y^2)(x^2-2xy+2y^2).\end{aligned}$$

EXAMPLES LX

Resolve into factors :

1. $a^4+a^2b^2+b^4$. **2.** a^4+a^2+1. **3.** x^4+2x^2+9.

4. x^4+4x^2+16. **5.** $16x^4+4x^2+1$. **6.** a^4-7a^2+1.

7. $x^4+3x^2y^2+4y^4$. **8.** $9a^4+3a^2b^2+4b^4$.

9. $9a^4+14a^2b^2+25b^4$. **10.** $1-13y^2+4y^4$.

11. $81x^4+9x^2y^2+y^4$. **12.** $x^4-x^2y^2+16y^4$.

13. $x^4-11x^2y^2+y^4$. **14.** $9x^4-x^2y^2+16y^4$.

15. $49a^4-15a^2x^2+121x^4$. **16.** x^4+4.

17. $4x^4+y^4$. **18.** x^4+64. **19.** $1+64a^4$.

20. a^4b^4+4. **21.** $4k^4+81$. **22.** $4l^4+625$.

23. $x^8+x^4y^4+y^8$. **24.** x^8+x^4+1. **25.** $(x+y)^4+64(x-y)^4$.

115. By multiplication, we have

$$(b+c)^3=b^3+c^3+3b^2c+3bc^2=b^3+c^3+3bc(b+c),$$

$$b^3+c^3=(b+c)^3-3bc(b+c) \quad\text{.........................(1)}$$

and $(b-c)^3=b^3-c^3-3b^2c+3bc^2=b^3-c^3-3bc(b-c)$,

$$\therefore\quad b^3-c^3=(b-c)^3+3bc(b-c) \quad\text{.........................(2)}$$

Results (1) and (2) are very important. These can be expressed in words thus :

The sum of the cubes of two quantities is equal to the cube of the sum of the two quantities, minus three times the product of their sum and product.

The difference of the cubes of two quantities is equal to the cube of the difference of the two quantities, plus three times the product of their difference and product.

For example,

$$64x^3 + 1 = (4x)^3 + (1)^3$$
$$= (4x+1)^3 - 3 \times 4x \times 1 \times (4x+1)$$
$$= (4x+1)^3 - 12x(4x+1)\ ;$$

and $8y^3 - 27z^3 = (2y)^3 - (3z)^3$

$$= (2y-3z)^3 + 3 \times 2y \times 3z \times (2y-3z)$$
$$= (2y-3z)^3 + 18yz(2y-3z).$$

116. EXAMPLE 1. *Resolve* $a^3+b^3+c^3-3abc$ *into factors.*

$a^3+b^3+c^3-3abc$

$$=a^3+(b+c)^3-3bc(b+c)-3abc$$

[Writing $(b+c)^3-3bc(b+c)$ for b^3+c^3... See Article 115 (1)]

$$=\{a^3+(b+c)^3\}-\{3bc(b+c)+3abc\}$$

[Enclosing the first two terms which are perfect cubes within one bracket and the last two in another]

$$=\{a+(b+c)\}\{a^2-a(b+c)+(b+c)^2\}-3bc\{(b+c)+a\}$$

[Factorising the expression within the first bracket]

$$=(a+b+c)(a^2-ab-ac+b^2+2bc+c^2)-3bc(a+b+c)$$
$$=(a+b+c)\{(a^2-ab-ac+b^2+2bc+c^2)-3bc\}$$

[Taking out $(a+b+c)$ common]

$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ac).$$

EXAMPLE 2. *Resolve* $x^3+8y^3-27z^3+18xyz$ *into factors.*

$x^3+8y^3-27z^3+18xyz$

$$=x^3+(2y-3z)^3+18yz(2y-3z)+18xyz$$

[Writing $(2y-3z)^3+18yz(2y-3z)$ for $8y^3-27z^3$... See Article 115]

$$=\{x^3+(2y-3z)^3\}+\{18yz(2y-3z)+18xyz\}$$
$$=\{x+(2y-3z)\}\{x^2-x(2y-3z)+(2y-3z)^2\}+18yz\{(2y-3z)+x\}$$
$$=(x+2y-3z)(x^2-2xy+3xz+4y^2-12yz+9z^2)+18yz(x+2y-3z)$$
$$=(x+2y-3z)\{(x^2-2xy+3xz+4y^2-12yz+9z^2)+18yz\}$$
$$=(x+2y-3z)(x^2+4y^2+9z^2-2xy+3xz+6yz).$$

EXAMPLE 3. *If* $a+b+c=0$, *show that* $a^3+b^3+c^3=3abc$.

First Method.

Since $a+b+c=0$,

$\therefore \quad a+b=-c$ .... (1)

$\therefore \quad (a+b)^3=(-c)^3$,

$\therefore \quad a^3+b^3+3a^2b+3ab^2=-c^3$,

$\therefore \quad a^3+b^3+c^3+3ab(a+b)=0$,

$\therefore \quad a^3+b^3+c^3+3ab(-c)=0$ ....[From (1)]

$\therefore \quad a^3+b^3+c^3-3abc=0$

$\therefore \quad a^3+b^3+c^3=3abc$.

Second Method.

$a^3+b^3+c^3=a^3+(b+c)^3-3bc(b+c)$. [See Article 115 (1)]

Since $a+b+c=0$.

$\therefore \quad b+c=-a$.

$$\therefore \quad a^3+b^3+c^3=a^3+(-a)^3-3bc(-a)$$
$$=a^3-a^3+3abc$$
$$=3abc.$$

Third Method.

$a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$ [See Example 1]

$=0\times(a^2+b^2+c^2-ab-bc-ac)$

$=0$.

$a^3+b^3+c^3=3abc$.

EXAMPLES LXI

Resolve into factors :

1. $x^3+y^3+z^3-3xyz$.
2. $x^3+y^3-z^3+3xyz$.
3. $x^3-y^3-z^3-3xyz$.
4. $x^3+y^3+1-3xy$.
5. $1-a^3+b^3+3ab$.
6. $a^3+b^3-8c^3+6abc$.
7. $a^3-8b^3+27c^3+18abc$.
8. $a^3-b^3-27c^3-9abc$.
9. $8a^3-b^3+64c^3+24abc$.
10. $x^6+y^6-z^6+3x^2y^2z^2$.
11. $a^3+b^3-\frac{c^3}{8}+\frac{3}{2}abc$.
12. $a^3-\frac{b^3}{27}-c^3-abc$.

13. $a^3+27b^3-\frac{c^3}{27}+3abc.$ 14. $a^3-\frac{b^3}{8}+\frac{c^3}{125}+\frac{3}{10}abc.$

15. $(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a).$

16. $(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c).$

17. $(a-b)^3-(b-c)^3+(c-a)^3+3(a-b)(b-c)(c-a).$

Shew that, if

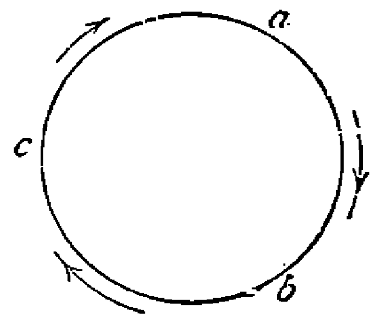
18. $a+b=c$, $a^3+b^3-c^3+3abc=0.$

19. $a+b=2c$, $a^3+b^3-8c^3+6abc=0.$

20. $a+2b+3c=0$, $a^3+8b^3+27c^3=18abc.$

Factors of Expressions in Cyclic Order

117. In the expression $ab+bc+ca$, letters a, b and c are arranged in a particular order, that is, the letters appear in the order in which they follow each other when placed round a circle and read in the directions of the arrows; thus b follows a, c follows b, and a follows c.



The expression $ab+bc+ca$, which deals with the letters a, b and c, is formed by taking the sum of the products of these letters taken two by two, beginning with a and following the above order. Such an arrangement is said to be in **cyclic order**.

Thus the sum of the letters taken two by two following the order of the arrowheads is $a+b$, $b+c$, $c+a$; and the difference $a-b$, $b-c$, $c-a$; while their sum in the opposite direction is $a+c$, $c+b$, $b+a$; and difference $a-c$, $c-b$, $b-a$. These are all in cyclic order.

NOTE. The expressions $(a-b)(b-c)(c-a)$ and $a^2(b+c)+b^2(c+a)+c^2(a+b)$ are also in cyclic order.

118. The method of factorising expressions in cyclic order can be easily understood by the following example.

EXAMPLE. *Resolve* $a^3(b-c)+b^3(c-a)+c^3(a-b)$ *into factors.*

Removing the brackets, the expression can be written as

$$a^3b-a^3c+b^3c-b^3a+c^3a-c^3b$$
$$=a^3b-a^3c-b^3a+c^3a+b^3c-c^3b$$

[Arranging in descending powers of a]

$$=(a^3b-a^3c)-(b^3a-c^3a)+(b^3c-c^3b)$$

[Enclosing the terms in pairs within brackets]

$$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$$
$$=a^3(b-c)-a(b-c)(b^2+bc+c^2)+bc(b-c)(b+c)$$
$$=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$$
$$=(b-c)(a^3-ab^2-abc-ac^2+b^2c+bc^2)$$
$$=(b-c)(b^2c-b^2a+bc^2-abc-ac^2+a^3)$$

[Arranging in descending powers of b]

$$=(b-c)\{(b^2c-b^2a)+(bc^2-abc)-(ac^2-a^3)\}$$

[Enclosing the terms of the second factor in pairs within brackets]

$$=(b-c)\{b^2(c-a)+bc(c-a)-a(c^2-a^2)\}$$
$$=(b-c)\{b^2(c-a)+bc(c-a)-a(c-a)(c+a)\}$$
$$=(b-c)(c-a)(b^2+bc-ac-a^2)$$
$$=(b-c)(c-a)(-ac+bc-a^2+b^2)$$

[Arranging in descending powers of c]

$$=(b-c)(c-a)\{-(ac-bc)-(a^2-b^2)\}$$

[Enclosing the terms of the third factor in pairs within brackets]

$$=(b-c)(c-a)\{-c(a-b)-(a-b)(a+b)\}$$
$$=(b-c)(c-a)(a-b)(-c-a-b)$$
$$=-(b-c)(c-a)(a-b)(a+b+c).$$

NOTE. Another method of factorising expressions in cyclic order is given in Article 126.

EXAMPLES LXII

Resolve into factors :

1. $a^2(b-c)+b^2(c-a)+c^2(a-b).$
2. $bc(b-c)+ca(c-a)+ab(a-b).$
3. $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2).$
4. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$
5. $x^3(y-z)+y^3(z-x)+z^3(x-y).$
6. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$
7. $a^4(b-c)+b^4(c-a)+c^4(a-b).$
8. $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2).$
9. $a^5(b-c)+b^5(c-a)+c^5(a-b).$
10. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2).$

Miscellaneous Factors

119. We shall now take some miscellaneous factors.

EXAMPLE 1. *Resolve* $x^5y^2-x^2y^5$ *into factors.*

$$x^5y^2-x^2y^5=x^2y^2(x^3-y^3)$$
$$=x^2y^2(x-y)(x^2+xy+y^2).$$

EXAMPLE 2. *Resolve* $(x-1)(x-2)(x-3)+(x-2)(x-3)(x-4)$ *into factors.*

Here $(x-2)$ and $(x-3)$ are common.

$$\therefore\quad (x-1)(x-2)(x-3)+(x-2)(x-3)(x-4)$$
$$=(x-2)(x-3)\{(x-1)+(x-4)\}$$
$$=(x-2)(x-3)(2x-5).$$

EXAMPLE 3. *Resolve* $(x-y)^2+a(x-y)-12a^2$ *into factors.*

Writing p for $(x-y)$, the expression

$$=p^2+ap-12a^2$$
$$=p^2+(4-3)ap-12a^2$$
$$=p^2+4ap-3ap-12a^2$$
$$=p(p+4a)-3a(p+4a)$$
$$=(p+4a)(p-3a)$$
$$=(x-y+4a)(x-y-3a). \qquad \text{[Replacing } p \text{ by } (x-y)]$$

EXAMPLE 4. *Resolve* $(x+1)(x+2)(x+3)(x+4)-48$ *into factors.*

Multiplying the first and the fourth, and the second and the third factors of the first part of the expression, we have

$$(x+1)(x+2)(x+3)(x+4)-48$$
$$=(x^2+5x+4)(x^2+5x+6)-48$$
$$=(p+4)(p+6)-48 \qquad \text{[Writing } p \text{ for } x^2+5x]$$
$$=p^2+10p+24-48$$
$$=p^2+10p-24$$
$$=p^2+12p-2p-24$$
$$=p(p+12)-2(p+12)$$
$$=(p+12)(p-2)$$
$$=(x^2+5x+12)(x^2+5x-2). \qquad \text{[Replacing } p \text{ by } x^2+5x]$$

EXAMPLES LXIII

Resolve into factors :

1. ax^3-a^4. **2.** $3ax^4-3a^4x$. **3.** $x^5+x^2y^3$.

4. $x^4-\dfrac{x}{27y^3}$. **5.** $(a+b)(a-b)-(a+b)^2$.

6. $a(b-1)-b+1$. **7.** $(x+y-3z)^2-x-y+3z$.

8. $a(a-1)-b(b-1)$. **9.** $(a^2-12)^2-a^2$.

10. $(x^2+3)^2-16x^2$. **11.** $(x+a)^4-(x-a)^4$.

12. $(x+1)(x+2)(x+3)+(x+1)(x+3)(x+6)$.

13. $x+y+8(x+y)^4$. **14.** $(x+1)(x+2)^3-(x+1)(x-1)^3$.

15. $x^2(y+z)+y^2(x-z)$. **16.** $a^2-2ab+b^2-a+b$.

17. $abx^2+ax+bx+1$. **18.** $x^2y^2-x^2-y^2+1$.

19. a^8-b^8. **20.** $a^{12}-b^{12}$. **21.** $a^{16}-b^{16}$.

22. $9a^4-82a^2b^2+9b^4$. **23.** $x^{16}+x^8+1$.

24. $a^2(a^4-1)+b^2(b^4-1)$. **25.** $(x^2+2x)^2-11(x^2+2x)+24$.

26. $(x^2-x)^2-26(x^2-x)+120$. **27.** $(x+y)^2-12a(x+y)+35a^2$.

28. $20a^2+9a(b+c)+(b+c)^2$. **29.** $6x^2+13x(a+b)+6(a+b)^2$.

30. $(a-2b)^2-2(a-2b)(2a-b)-8(2a-b)^2$.

31. $(x+y)^2-(a+b)(x+y)+ab$. **32.** $x^4+x^3+x^2y^2+y^3+y^4$.

33. $(x+2y)x^3+(2x-y)y^3$.

34. $(a+b)^2-(x+y)^2+(a+x)^2-(b+y)^2$.

35. $x^4-(z^2+4)x^2y^2+4y^4$. **36.** $1+2x+2yz+x^2-y^2-z^2$.

37. $a^2-b^2+c^2-d^2-2(ac+bd)$. **38.** $ab(x^2+y^2)-xy(a^2+b^2)$.

39. $(a^2-b^2)(x^2-y^2)-4abxy$. **40.** $(a+b+c)(ab+bc+ca)-abc$.

41. $(ab+cd)(c^2+d^2)+cd(a^2+b^2-c^2-d^2)$.

42. $(x+2)(x+3)(x+4)(x+5)-24$.

43. $x(x-1)(x-4)(x-5)-21$. **44.** $x(x-2)(2x+1)(2x-3)-35$.

CHAPTER XX

THE REMAINDER THEOREM

Functional Notation

120. We have already seen in Article 86, that any expression whose value depends upon the value of x is called a *function* of x. The symbol $f(x)$, for the sake of brevity, is sometimes used to denote function of x. This functional notation is generally used to save the repeated writing of the same function in some particular piece of work or example. If, in any piece of work, the symbol $f(x)$ denotes a certain function of x, the symbol $f(p)$ will denote the same function of p as $f(x)$ is of x, and similarly $f(2)$ will denote the same function of 2 as $f(x)$ is of x. Thus if

$f(x)$ represents $ax^2 + bx + c$,

$f(p)$ will represent $ap^2 + bp + c$,

and $f(2)$,, $a2^2 + b2 + c$.

That is, the values of $f(p)$ and $f(2)$ are obtained by replacing x by p and 2 in the function $ax^2 + bx + c$.

Similarly, if

$f(y)$ represents $2y^2 - 5y + 7$,

$f(a)$ will represent $2a^2 - 5a + 7$,

$f(3)$,, $2.3^2 - 5.3 + 7 = 10$,

and $f(0)$,, $2.0^2 - 5.0 + 7 = 7$.

NOTE. Sometimes $F(x)$ and $\phi(x)$ are also used to represent functions of x.

The Remainder Theorem

121. We have already seen in inexact division that if one function of x is divided by another function of x, there is a remainder which is of a lower degree in x than the divisor. Hence, if the divisor is of the first degree in x, the remainder will be *independent* of x.

Suppose we divide $ax^2 + bx + c$ by $x - p$.

$$\begin{array}{l} x - p)ax^2 + bx + c(ax + (ap + b) \\ \quad ax^2 - apx \\ \hline \qquad (ap + b)x + c \\ \qquad (ap + b)x - ap^2 - bp \\ \hline \qquad\qquad ap^2 + bp + c \end{array}$$

Thus we see that when the division is complete, the remainder is $ap^2 + bp + c$, which is the same function of p as the dividend is of x, *i.e.*, the remainder could be obtained by substituting p for x in the dividend.

Hence, we see that *if a function of x is divided by $x - p$, until the remainder does not contain x, the remainder is the same function of p as the dividend is of x.* In other words, if $f(x)$ is divided by $x - p$, the remainder is $f(p)$. This theorem is known as the **Remainder Theorem.**

If the remainder $ap^2 + bp + c$ vanishes, then it is clear that *$ax^2 + bx + c$ is exactly divisible by $x - p$* *i.e.*, $x - p$ is a factor of $ax^2 + bx + c$. Hence, we see that *if in the function $ax^2 + bx + c$, p is substituted for x and the remainder vanishes, then $x - p$ is a factor of $ax^2 + bx + c$,* *i.e.*, if $f(p) = 0$, $x - p$ is a factor of $f(x)$.

NOTE. If the divisor is $x+p$ *i.e.*, $x - (-p)$, the remainder is obtained by substituting $(-p)$ for x in the dividend.

EXAMPLE 1. *What is the remainder when* x^2+3x+5 *is divided by* $x-2$?

Writing 2 for x in the expression x^2+3x+5, the remainder

$$=2^2+3\times2+5=4+6+5=15.$$

This can be verified by ordinary division thus :

$$\begin{array}{l} x - 2)x^2+3x+5(x+5 \\ \quad x^2-2x \\ \hline \qquad 5x+5 \\ \qquad 5x-10 \\ \hline \qquad\quad 15 \end{array}$$

EXAMPLE 2. *Find the remainder when* x^3+2x^2+3x+4 *is divided by* $x+3$.

Since $x+3=x-(-3)$, therefore writing -3 for x in the expression, the remainder

$$=(-3)^3+2(-3)^2+3(-3)+4=-27+18-9+4=-14.$$

EXAMPLE 3. *Find the remainder when* $2x^3-3x^2-2x+1$ *is divided by* $2x-1$.

Since $2x-1=2(x-\frac{1}{2})$, therefore writing $\frac{1}{2}$ for x in the expression, the remainder

$$=2(\tfrac{1}{2})^3-3(\tfrac{1}{2})^2-2(\tfrac{1}{2})+1=\tfrac{1}{4}-\tfrac{3}{4}-1+1=-\tfrac{1}{2}.$$

EXAMPLE 4. *What is the remainder when* x^3+3x^2+3x+1 *is divided by* $x+1$?

Since $x+1=x-(-1)$, therefore writing -1 for x in the expression, the remainder

$$=(-1)^3+3(-1)^2+3(-1)+1=-1+3-3+1=0.$$

NOTE. Since the remainder is zero, hence $x+1$ is a factor of x^3+3x^2+3x+1.

EXAMPLE 5. *Shew that* $x-2$ *is a factor of* $3x^3-5x^2+4x-12$.

Writing 2 for x in the expression, the remainder

$$=3(2)^3-5(2)^2+4(2)-12=24-20+8-12=0.$$

Since the remainder is 0, hence $x-2$ is a factor of the given expression.

EXAMPLE 6. *For what value of* p *is* x^3-3x^2+5x-p *exactly divisible by* $x-3$?

Writing 3 for x in the expression, the remainder

$$=(3)^3-3(3)^2+5(3)-p=27-27+15-p=15-p.$$

If the expression is exactly divisible by $x-3$, the remainder $15-p$ must be equal to zero, that is

$$15-p=0,$$

$$\therefore\ p=15.$$

Hence the required value of p is 15.

EXAMPLE 7. *For what values of* a *and* b *is* $4x^3+ax^2-11x+b$ *exactly divisible by* $2x^2-5x+2$?

Since $2x^2-5x+2=(x-2)(2x-1)$, hence if $4x^3+ax^2-11x+b$ is exactly divisible by $2x^2-5x+2$, it must be divisible by its factors $x-2$ and $2x-1$.

If the expression is divisible by $x-2$, writing 2 for x in the expression, the remainder must be equal to zero, that is

$$4(2)^3+a(2)^2-11(2)+b=0,$$
$$\therefore\quad 32+4a-22+b=0,$$
$$\therefore\quad 10+4a+b=0,$$
$$\therefore\quad 4a+b=-10 \qquad \text{(i)}$$

Similarly, if the expression is divisible by $2x-1$, writing $\frac{1}{2}$ for x in the expression, the remainder must be equal to zero, that is

$$4(\tfrac{1}{2})^3+a(\tfrac{1}{2})^2-11(\tfrac{1}{2})+b=0,$$
$$\therefore\quad \tfrac{1}{2}+\tfrac{1}{4}a-\tfrac{11}{2}+b=0,$$
$$\therefore\quad 2+a-22+4b=0,$$
$$\therefore\quad a+4b=20 \qquad \text{(ii)}$$

Solving the equations (i) and (ii) for a and b, we get $a=-4$ and $b=6$.

EXAMPLES LXIV

Without actual division, find the remainder when

1. x^2+3x+5 is divided by $x-1$.
2. x^2+5x+7 is divided by $x-2$ and $x+2$.
3. x^3+2x^2+3x+4 is divided by $x-1$.
4. $2x^3-5x^2+x-7$ is divided by $x-1$, $x+1$, $2x-1$, $2x+1$ and $2x-3$.
5. x^6-1 is divided by $x-1$ and $x+2$.
6. x^4-x^3-x+1 is divided by $x-p$ and $x+p$.
7. $x^3-3ax^2+3a^2x-a^3$ is divided by $x-a$ and $x+a$.
8. $x^2-4ax+4a^2$ is divided by $x-2a$.
9. x^5-b^5 is divided by $x-b$.
10. x^6-y^6 is divided by $x-y$.
11. $x^3+x^2y+xy^2+y^3$ is divided by $x+y$.

Show that

12. $x-1$ and $x-3$ are the factors of x^3-2x^2-5x+6.
13. $x+1$ and $x+2$ are the factors of $x^3+7x^2+14x+8$.
14. $x+1$ and $x+3$ are the factors of $x^3+6x^2+11x+6$.
15. $x-1$, $x-2$ and $x-3$ are the factors of $x^3-6x^2+11x-6$.

16. $x+2$, $x+3$ and $x+4$ are the factors of $x^3+9x^2+26x+24$.

17. $2x-1$ is a factor of $8x^3+4x-3$.

18. $x-1$, $x+3$ and $2x+1$ are the factors of $2x^3+5x^2-4x-3$.

19. For what value of a is x^2+2x-a exactly divisible by $x-2$?

20. For what value of p is $x^3-2px^2+11x-2p$ exactly divisible by $x-3$?

21. For what value of k is $4x^4-kx^3+7x+2k$ exactly divisible by $x+2$?

22. For what value of l is lx^5+3x^3+16l exactly divisible by $x-2$?

23. For what value of a is $x^3-2(a-4)x^2+ax-1$ exactly divisible by $x-1$?

24. If $x^3-qx^2+2prx+pr^2$ is exactly divisible by $x+r$ and r is not equal to zero, prove that $p+q+r=0$.

25. For what values of a and b is $2x^3+ax^2-x-b$ exactly divisible by x^2+x-2 ?

26. Find l and m in order that $lx^3+31x^2-mx-10$ may be exactly divisible by $2x^2+9x-5$.

Use of the Remainder Theorem in Factorization

122. The Remainder Theorem is sometimes very useful in factorising certain expressions as will be evident from the following examples :

EXAMPLE 1. *Resolve* x^3+4x^2+x-6 *into factors.*

The last term is 6, whose factors can be the numbers 1, 2, 3 and 6. Hence, we can find the factors of the expression, if we write $+1$ or -1, $+2$ or -2, $+3$ or -3, $+6$ or -6, for x in the expression and it vanishes. For instance, writing 1 for x in the expression, the remainder$=1+4+1-6=0$, hence $x-1$ is a factor of the expression. Now writing the expression in a form that the factor $x-1$ may easily be detected, thus

$$\begin{aligned}
&x^3+4x^2+x-6\\
&=x^3-x^2+5x^2-5x+6x-6\\
&=(x^3-x^2)+(5x^2-5x)+(6x-6)\\
&=x^2(x-1)+5x(x-1)+6(x-1)\\
&=(x-1)(x^2+5x+6)\\
&=(x-1)(x^2+2x+3x+6)\\
&=(x-1)\{(x^2+2x)+(3x+6)\}\\
&=(x-1)\{x(x+2)+3(x+2)\}\\
&=(x-1)(x+2)(x+3).
\end{aligned}$$

EXAMPLE 2. *Resolve* $x^3+2x^2-9x-18$ *into factors.*

The factors of 18 can be 1, 2, 3, 6, 9 and 18. First writing 1 for x in the expression, we see that it does not vanish, hence $x-1$ is not a factor. Similarly $(x+1)$ and $(x-2)$ are not the factors. Now writing -2 for x, the remainder vanishes, hence $x+2$ is a factor.

$$\begin{aligned}\therefore\quad & x^3+2x^2-9x-18\\ &=(x^3+2x^2)-(9x+18)\\ &=x^2(x+2)-9(x+2)\\ &=(x+2)(x^2-9)\\ &=(x+2)(x+3)(x-3).\end{aligned}$$

EXAMPLE 3. *Resolve* $x^3+17x^2+95x+175$ *into factors.*

Since 1, 5, 7, 25, 35 and 175 can be the factors of 175, we find by trial that the expression does not vanish when $x=1, -1$ or 5, but vanishes when $x=-5$, hence $x+5$ is a factor.

$$\begin{aligned}\therefore\quad & x^3+17x^2+95x+175\\ &=x^3+5x^2+12x^2+60x+35x+175\\ &=x^2(x+5)+12x(x+5)+35(x+5)\\ &=(x+5)(x^2+12x+35)\\ &=(x+5)(x+5)(x+7)\\ &=(x+5)^2(x+7).\end{aligned}$$

EXAMPLE 4. *Resolve* $12x^3+4x^2-3x-1$ *into factors.*

Dividing all the terms by 12 the coefficient of x^3, the last term becomes $\frac{1}{12}$, whose factors can be 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$ or $\frac{1}{12}$. Now we find by trial that the expression does not vanish when $x=1$ or -1, but vanishes when $x=\frac{1}{2}$, hence $(2x-1)$ is a factor. (See Example 8, Article 121).

$$\begin{aligned}\therefore\quad & 12x^3+4x^2-3x-1\\ &=12x^3-6x^2+10x^2-5x+2x-1\\ &=6x^2(2x-1)+5x(2x-1)+(2x-1)\\ &=(2x-1)(6x^2+5x+1)\\ &=(2x-1)(2x+1)(3x+1).\end{aligned}$$

EXAMPLE 5. *Resolve* $x^3-6xy^2+9y^3$ *into factors.*

The expression vanishes when $x=-3y$, hence $x+3y$ is a factor.

$$\begin{aligned}\therefore\quad & x^3-6xy^2+9y^3\\ &=x^3+3x^2y-3x^2y-9xy^2+3xy^2+9y^3\\ &=(x^3+3x^2y)-(3x^2y+9xy^2)+(3xy^2+9y^3)\\ &=x^2(x+3y)-3xy(x+3y)+3y^2(x+3y)\\ &=(x+3y)(x^2-3xy+3y^2).\end{aligned}$$

EXAMPLES LXV

Resolve into factors :

1. $x^3-6x^2+11x-6$.

2. x^3-7x+6.

3. $x^3+6x^2+11x+6$.

4. x^3-7x-6.

5. $a^3-7a^2+14a-8$.

6. $a^3+8a^2+19a+12$.

7. $a^3+a^2-10a+8$.

8. $a^3-13a-12$.

9. $x^3+x^2-14x-24$.

10. $x^3-19x-30$.

11. $x^3-4x^2-27x+90$.

12. $x^3+6x^2-25x-150$.

13. $x^3+5x^2-25x-125$.

14. $x^3+x^2-16x-16$.

15. x^4-5x^2+4.

16. $x^4+7x^3+17x^2+17x+6$.

17. $2x^3-13x^2+22x-8$.

18. $12x^3-4x^2-3x+1$.

19. $20a^3+4a^2-5a-1$.

20. $4a^4+20a^3-5a^2-75a+36$.

21. $a^3+4a^2b+5ab^2+2b^3$.

22. $5a^3-3a^2b-28b^3$.

23. $x^3+3x^2y-xy^2-3y^3$.

24. $x^3+8x^2y+17xy^2+10y^3$.

123. An expression which remains unaltered when any two of its letters are interchanged is said to be **symmetrical.**

Thus, the expressions $a+b$, a^2+b^2, a^2+ab+b^2, a^2b+ab^2, etc., are symmetrical with respect to a and b, for the expressions remain the same if a is written for b or b for a.

Similarly, the expressions $a+b+c$, $ab+bc+ca$ and $a^3+b^3+c^3-3abc$ are symmetrical with respect to a, b and c ; for the expressions will remain the same, if any two of the letters are interchanged.

124. The expression $k(a+b+c)$ is the general form of a **homogeneous symmetrical expression of the first degree** in a, b, c, where k is any numerical coefficient. Whatever be the numerical value of k, the expression will remain the same, if any two of the letters are interchanged.

Similarly, the expression $k'(a^2+b^2+c^2)+l(ab+bc+ca)$ is the general form of a **homogeneous symmetrical expression of the second degree** in a, b, c, where k and l are any numerical coefficients, and the expression $k(a^3+b^3+c^3)+l(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+mabc$ is the general form of a **homogeneous symmetrical expression of the third degree** in a, b, c, where k, l and m are any numerical coefficients.

125. We see in the foregoing examples of this chapter that

(i) an expression of the second degree has two factors of the first degree ;

(ii) an expression of the third degree has three factors of the first degree, or one factor of the first degree and one of the second degree ;

(iii) an expression of the fourth degree has four factors of the first degree, or two factors of the second degree, or one factor of the second degree and two factors of the first degree, or one factor of the first degree and one factor of the third degree ; and so on.

That is, *the degree of an expression is equal to the sum of the degrees of its factors.*

126. The Remainder Theorem is also useful in factorising expressions in cyclic order.

EXAMPLE 1. *Resolve* $a^2(b-c)+b^2(c-a)+c^2(a-b)$ *into factors.*

If we put $b=c$, the expression

$$=a^2(c-c)+c^2(c-a)+c^2(a-c)$$
$$=a^2c-a^2c+c^3-c^2a+c^2a-c^3$$
$$=0.$$

$\therefore$ $b-c$ is a factor of the expression.

Similarly, by putting $c=a$ or $a=b$, we see that the expression again vanishes, therefore $c-a$ and $a-b$ are also factors of the expression.

Since the expression is of the third degree in a, b, c, it can have no other factors, except some numerical coefficient. Suppose the numerical coefficient is k,

$$\therefore \quad a^2(b-c)+b^2(c-a)+c^2(a-b)=k(a-b)(b-c)(c-a) \quad \ldots\ldots\ldots(1)$$

Since (1) is an identity, which is true for all values of a, b, c, therefore in order to find k, we give to a, b, c, some simple numerical values, say $a=0$, $b=1$ and $c=-1$.

Then (1) becomes

$$0\times\{1-(-1)\}+1^2\times\{(-1)-0\}+(-1)^2(0-1)=k(0-1)(1+1)(-1-0)$$

$$\therefore\quad -1-1=2k,$$

$$\therefore\quad k=-1.$$

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a).$$

EXAMPLE 2. *Resolve* $a^3(b-c)+b^3(c-a)+c^3(a-b)$ *into factors.*

If we put $a=b$, or $b=c$, or $c=a$, the expression vanishes, hence $a-b$, $b-c$ and $c-a$ are the factors of the expression.

Since the expression is of the fourth degree, and homogeneous and symmetrical in a, b, c; therefore there is another factor of the first degree which is also homogeneous and symmetrical in a, b, c. Hence, this factor must be of the form $k(a+b+c)$, where k is a numerical coefficient. (See Article 124).

$$\therefore\quad a^3(b-c)+b^3(c-a)+c^3(a-b)=k(a+b+c)(a-b)(b-c)(c-a).$$

Now to find k, if we put $a=1$, $b=-1$ and $c=2$ in the above, we have

$$-3-1+16=k(2)(2)(-3)(1)$$

$$\therefore\quad 12=-12k,$$

$$\therefore\quad k=-1.$$

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=-(a+b+c)(a-b)(b-c)(c-a).$$

EXAMPLE 3. *Resolve* $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$ *into factors.*

If we put $a=b$, or $b=c$, or $c=a$, the expression vanishes, hence $a-b$, $b-c$ and $c-a$ are the factors of the expression.

Since the expression is a homogeneous and symmetrical expression of the fifth degree, therefore there is another factor of the second degree which is also homogeneous and symmetrical in a, b, c. Hence this factor must be $\{k(a^2+b^2+c^2)+l(ab+bc+ca)\}$, where k and l are numerical coefficients. (See Article 124).

$$\therefore\quad b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$$
$$=(a-b)(b-c)(c-a)\{k(a^2+b^2+c^2)+l(ab+bc+ca)\}\ \ldots\ldots\ldots(1)$$

Putting $a=1$, $b=-1$ and $c=0$ in (1), we have

$$2=2(2k-l),$$

$$\therefore\quad 2k-l=1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (i)$$

Again, putting $a=2$, $b=0$ and $c=1$ in (1) we have

$$-4=2(5k+2l),$$

$$\therefore\quad 5k+2l=-2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (ii)$$

Solving the equations (i) and (ii) for k and l, we get $k=0$ and $l=-1$.

$$\therefore\quad b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)=-(a-b)(b-c)(c-a)(ab+bc+ca).$$

EXAMPLES LXVI

Resolve into factors by means of the Remainder Theorem :

1. $bc(b-c)+ca(c-a)+ab(a-b)$.
2. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.
3. $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$.
4. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.
5. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
6. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.
7. $(b-c)^5+(c-a)^5+(a-b)^5$.
8. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.

CHAPTER XXI

HIGHEST COMMON FACTOR

H.C.F. by Factors

127. When an expression divides two or more expressions exactly, it is called a **common factor** of those expressions.

Thus a is a common factor of $2a$ and $3a$;

y is a common factor of axy, byz and cwy ;

and 2, a, b^2, c^3 are common factors of $2a^2b^3c^4$, $6a^3b^2c^5$ and $8ab^3c^3$.

In the third example, the product of the common factors $2ab^2c^3$, which contains *all* the factors common to the given expressions is called their **Highest Common Factor**. Hence, *the Highest Common Factor of two or more simple expressions is the expression of the highest degree which exactly divides each of them.*

NOTE. For brevity, the Highest Common Factor is often written H.C.F.

Take, for example, the expressions $6x^5y^2z^5w^2$, $9x^3yz^7w$ and $15x^4y^3z^6$. The H.C.F. of 6, 9 and 15 is 3. Now the first expression is clearly divisible by x, x^2, x^3, x^4 or x^5, but by no higher power of x ; the second expression is divisible by x, x^2 or x^3, but by no higher power of x ; and the third expression is divisible by x, x^2, x^3 or x^4, but by no higher power of x. Hence, x^3 can divide all the three expressions and is therefore the highest power of x which will divide all of them exactly. Similarly y and no higher power of y above the first, and also z^5 and no higher power of z above z^5, can divide the expressions exactly. Since w is not found in the three expressions, therefore it is not a common factor. Hence, the H.C.F. of the expressions is $3x^3yz^5$, which is the product of 3, x^3, y and z^5.

From the above example it is clear that *the Highest Common Factor of two or more simple expressions is the product of the H.C.F. of the numerical coefficients, if any, and the lowest powers of the letters which occur in each of the expressions.*

EXAMPLE 1. *Find the H.C.F. of* $24a^2bc^3$ *and* $16a^3b^5c^2$.

The H.C.F. of 24 and 16 is 8.

The lowest power of a which occurs in both the expressions is a^2; and the lowest powers of b and c which occur in both the expressions are b and c^2.

Hence the H.C.F. is $8a^2bc^2$.

EXAMPLE 2. *Find the H.C.F. of* $8a^2bc^4$, $12bc^2d$ *and* $20b^2c^3e$.

The H.C.F. of 8, 12 and 20 is 4; and the lowest powers of b and c, which are common to all, are b and c^2.

Hence the H.C.F. is $4bc^2$.

NOTE. From the above examples it is evident that the Highest Common Factor of simple expressions can be written down by inspection.

EXAMPLES LXVII

(*Oral*)

Write down the H.C.F. of

1. ab, bc. **2.** a^2b, ab^2. **3.** $3a^2b^2$, a^2b.

4. $2a^3b^2$, $3a^2b^3$. **5.** ab^2c^3, a^2bc^2. **6.** $9ab^3$, $6a^2b^2$.

7. $10a^4b^2$, $25a^2b^4$. **8.** $6a^2xy$, $9ax^2z$. **9.** x^2, x^3, x^4.

10. ax, a^2x^2, a^3x^3. **11.** $7abc$, $14a^2bc^2$. **12.** $16a^4b^2c$, $20a^3b^3$.

13. $9a^2b^3x^4y^5$, $18a^3x^3$. **14.** $2ab^3$, a^2bc, abc^2.

15. $3x^2yz^3$, $10x^2y^2z^2$, $15xy^3z^2$. **16.** $ab^2c^3x^4$, $a^4b^2c^5x^3$, $a^2bc^2x^2$.

17. $5a^2bc^2$, $12a^3b^4c^5$, $9a^5b^3c^4$, $24a^3c^5$.

18. $60x^3y^4z^2$, $15x^2y^3z^3$, $36x^4y^2z^4$, $12x^5y^5z^5$.

128. The method of finding the H.C.F. of compound expressions, whose factors are known or can be easily found, is similar to that of finding the H.C.F. of simple expressions. This has been explained in the previous article. *The H.C.F. is the product of the lowest powers of all the factors which occur in each of the expressions.*

Take, for example, the expressions $(a+b)^3(a-2b)^3$ and $(a+b)^4(a-2b)^2$. It is clear that both the expressions are divisible by $(a+b)^3$ but by no higher power of $(a+b)$ and also by $(a-2b)^2$ but by no higher power of $(a-2b)$. Hence, the H.C.F. is $(a+b)^3(a-2b)^2$.

EXAMPLE 1. *Find the H.C.F. of* a^2-b^2, a^2b-ab^2 *and* $a^2-2ab+b^2$.

$$a^2-b^2=(a+b)(a-b),$$
$$a^2b-ab^2=ab(a-b),$$
$$a^2-2ab+b^2=(a-b)^2.$$
$$\text{H.C.F.}=(a-b).$$

EXAMPLE 2. *Find the H.C.F. of* $2x^2-2x-4$, $6x^2+6x-36$ *and* $12x^2-52x+56$.

$$2x^2-2x-4=2(x^2-x-2)=2(x-2)(x+1),$$
$$6x^2+6x-36=6(x^2+x-6)=6(x+3)(x-2),$$
$$12x^2-52x+56=4(3x^2-13x+14)=4(x-2)(3x-7).$$
$$\text{H.C.F.}=2(x-2).$$

EXAMPLE 3. *Find the H.C.F. of* x^3+4x^2+x-6 *and* x^3-3x^2-6x+8.

Finding the factors of the two expressions by means of the Remainder Theorem, we have

$$\begin{aligned}x^3+4x^2+x-6&=x^3-x^2+5x^2-5x+6x-6\\&=x^2(x-1)+5x(x-1)+6(x-1)\\&=(x-1)(x^2+5x+6)\\&=(x-1)(x+2)(x+3).\end{aligned}$$

$$\begin{aligned}x^3-3x^2-6x+8&=x^3-x^2-2x^2+2x-8x+8\\&=x^2(x-1)-2x(x-1)-8(x-1)\\&=(x-1)(x^2-2x-8)\\&=(x-1)(x+2)(x-4).\end{aligned}$$

$$\text{H.C.F.}=(x-1)(x+2).$$

EXAMPLES LXVIII

Find the H.C.F. of

1. $7x-14$, $4x-8$.
2. a^2+ab, $ab+b^2$.
3. x^2+3xy, $xy+3y^2$.
4. x^2+xy, x^2-y^2.
5. x^3-xy^2, xy^2+y^3.
6. a^2-b^2, a^3-b^3.
7. $a^2b(a+b)^2$, $a^4-a^2b^2$.
8. $xy^2(x+y)$, $x^2y(x+y)^2$.
9. $x^2(x-y)^3$, $x^3(x-y)^3$.
10. $2ab(a+b)^2$, $3(a-b)(a+b)^3$.
11. x^3-a^2x, $x^2-2ax+a^2$.
12. x^3+y^3, $(x+y)^3$.
13. $5a^2-10ab$, $3ac-6bc$.
14. a^2-1, a^3+1.
15. a^3+b^3, a^2-b^2.
16. $a^3b^3+ab^5$, $a^6-a^2b^4$.
17. $a^2b^3+2a^3b^2$, $a^2b^4-4a^4b^2$.
18. $12(a^2+b^2)^2$, $18(a^4-b^4)$.
19. $ab(b^3-a^3)$, a^2b-a^3.
20. $(a^2x-ax^2)^2$, $ax(a^2-x^2)^2$.
21. y^4-y, y^3+y^2+y.
22. $y^2+7y-60$, $y^2-7y+10$.
23. y^2+3y+2, y^2+7y+6.
24. x^2+3x-4, x^2-x-20.
25. x^3+y^3, $x^2y-xy^2+y^3$.
26. $2x^2+3x-2$, $6x^2-5x+1$.
27. $a^4x^2-4a^2x^4$, $a^2x^2(a^2-3ax+2x^2)$.
28. a^2+3a+2, a^2+6a+8.
29. $a^2+7a-18$, $a^2+10a+9$.
30. x^3+y^3, $x^4+x^2y^2+y^4$.
31. $a^3+3a^2b+2ab^2$, $a^4+6a^3b+8a^2b^2$.
32. x^3+7x^2+12x, x^3-2x^2-15x.
33. x^2-x, $(x-1)^2$, x^3-1.
34. x^2-1, x^2+x-2, x^3+3x^2-4x.
35. x^2-y^2, x^4-y^4, x^6-y^6.
36. x^2+x-6, x^2+2x-8, $2x^2-5x+2$.
37. x^2+2x-3, x^2-4x+3, $2x^2+x-3$.
38. a^2-1, a^3-1, a^2+a-2.
39. a^4x-ax^4, $a^4x^2-a^2x^4$, $a^2x(a^6-x^6)$,
40. $a^2+b^2+c^2+2ab+2bc+2ca$, $a^3+b^3+c^3-3abc$.
41. x^3-7x+6, x^3+3x^2-x-3, x^3+4x^2+x-6.
42. $x^3-6x^2+11x-6$, $x^3+2x^2-3x-36$, $x^3+5x^2-9x-45$.
43. x^3+3x^2+7x+5, x^3+x^2+3x-5, $x^3+4x^2+9x+10$.
44. $x^3+x^2y-xy^2-y^3$, $x^3-3xy^2+2y^3$, $x^3-2x^2y-xy^2+2y^3$.
45. $6x^3+5x^2-2x-1$, $6x^3-7x^2+1$, $12x^3-8x^2-x+1$.

H.C.F. by 'Division Method'

129. When the given expressions cannot be easily resolved into factors, their H.C.F. can be determined by the following method, which is similar to the Division Method used in Arithmetic to find the Greatest Common Measure of two numbers :

First arrange the two expressions in descending powers of some common letter, then divide the expression of the higher degree by the other until the remainder is of a lower degree than the divisor : if both the expressions are of the same degree, either may be used as the divisor. Then take the remainder as a new divisor and the former divisor as dividend, and continue the process until there is no remainder. The last divisor will be the H.C.F. of the given expressions.

EXAMPLE 1. *Find the H.C.F. of* $4a^2+3a-10$ *and* $4a^3+7a^2-3a-15$.

Since the first expression is of the second degree and the second is of the third degree, therefore dividing the second by the first,

$$\begin{array}{r} 4a^2+3a-10)\,4a^3+7a^2-3a-15\,(a+1 \\ \underline{4a^3+3a^2-10a} \\ 4a^2+7a-15 \\ \underline{4a^2+3a-10} \\ 4a\ \ -5 \end{array}$$

Now dividing the divisor $4a^2+3a-10$ by the remainder $4a-5$,

$$\begin{array}{r} 4a-5)\,4a^2+3a-10\,(a+2 \\ \underline{4a^2-5a} \\ 8a-10 \\ \underline{8a-10} \end{array}$$

There is no remainder, hence the H.C.F. is $4a-5$.

EXAMPLE 2. *Find the H.C.F. of* x^3+5x^2+7x+3 *and* $x^3-x^2-10x-8$.

Since both the expressions are of the same degree, therefore dividing the second by the first,

$$\begin{array}{r} x^3+5x^2+7x+3)\,x^3-x^2-10x-8\,(1 \\ x^3+5x^2+7x+3 \\ \hline -6x^2-17x-11 \end{array}$$

Since the first term of the remainder is negative and -1 is not common to the two expressions, and if we divide the remainder by -1, the result will not be affected and the remainder will become $6x^2+17x+11$. If the divisor x^3+5x^2+7x+3 is divided by the remainder $6x^2+17x+11$, the quotient will be fractional. To avoid the inconvenience of fractions we multiply the first divisor, which now becomes the dividend, by 6. By doing so the result will not be affected as no additional common factor has been introduced.

$$\begin{array}{r} 6x^2+17x+11)\,x^3+5x^2+7x+3\,(x \\ 6 \\ \hline 6x^3+30x^2+42x+18 \\ 6x^3+17x^2+11x \\ \hline 13x^2+31x+18 \end{array}$$

Since both the divisor and the remainder are of the second degree and the first term of the remainder is not exactly divisible by the first term of the divisor, we multiply the remainder again by 6.

$$\begin{array}{r} 13x^2+31x+18 \\ 6 \\ \hline 6x^2+17x+11)\,78x^2+186x+108\,(13 \\ 78x^2+221x+143 \\ \hline -35x-35 \end{array}$$

Now we observe that -35 is a common factor of the remainder, so we divide the remainder by -35, and get $x+1$. This will not affect the result, because the two original expressions have no common simple factors, and therefore in rejecting -35 we are not rejecting any common factor of the two expressions. Continuing the process, we have

$$\begin{array}{r} x+1)\,6x^2+17x+11\,(6x+11 \\ 6x^2+6x \\ \hline 11x+11 \\ 11x+11 \\ \hline \end{array}$$

Since there is no remainder, hence the H.C.F. is $x+1$.

The working may be done more compactly if the division of the remainder into the preceding divisor be performed alternately from left to right, and from right to left, thus avoiding the necessity of copying down the new dividend at each stage, thus :

	x^3+5x^2+7x+3	$x^3-x^2-10x-8$	1
		x^3+5x^2+7x+3	
	6	$-1)-6x^2-17x-11$	
x	$6x^3+30x^2+42x+18$	$6x^2+17x+11$	$6x$
	$6x^3+17x^2+11x$		
	$13x^2+31x+18$		
	6		
13	$78x^2+186x+108$		
	$78x^2+221x+143$		
	$-35)-35x-35$		
	$x+1$	$6x^2+6x$	
		$11x+11$	11
		$11x+11$	

The working may be done still more briefly if we use *the method of detached coefficients* thus :

	$1+5+7+3$	$1-1-10-8$	1
		$1+5+7+3$	
	6	$-1)-6-17-11$	
1	$6+30+42+18$	$6+17+11$	6
	$6+17+11$		
	$13+31+18$		
	6		
13	$78+186+108$		
	$78+221+143$		
	$-35)-35-35$		
	$1+1$	$6+6$	
		$11+11$	11
		$11+11$	

From the above example we observe :

(i) *If at the beginning or at any stage, there occurs a common factor in all the terms of any of the expressions, it can be removed, but if it is a factor of both the expressions it will be the common factor of the H.C.F.*

(ii) *If at any stage the first term of the dividend is not divisible by the first term of the divisor, multiply every term of the dividend by a numerical factor to avoid fractional coefficients.*

(iii) *If the first term of the remainder is negative, multiply all its terms by -1, i.e., change the sign of all the terms.*

NOTE. If remainders continually occur, until we come to a remainder which does not contain the common letter, the given expressions will have no H.C.F.

EXAMPLE 3. *Find the H.C.F. of* $4x^5-2x^4-2x^3-2x^2-6x$ *and* $6x^5-15x^4+3x^3+15x^2-9x$.

Here we see that $2x$ and $3x$ are common factors of the two expressions,

$$\therefore \quad 4x^5-2x^4-2x^3-2x^2-6x=2x(2x^4-x^3-x^2-x-3),$$

$$\text{and} \quad 6x^5-15x^4+3x^3+15x^2-9x=3x(2x^4-5x^3+x^2+5x-3).$$

Since x is a common factor of $2x$ and $3x$, therefore x is a factor of their H.C.F. Now removing $3x$ and $2x$ from the two expressions we proceed thus :

x^2	$2x^4-5x^3+x^2+5x-3$	$2x^4-x^3-x^2-x-3$	1
		$2x^4-5x^3+x^2+5x-3$	
		$2x)4x^3-2x^2-6x$	
	$2x^4-x^3-3x^2$	$2x^2-x-3$	
$-2x$	$-4x^3+4x^2+5x-3$		
	$-4x^3+2x^2+6x$		
1	$2x^2-x-3$		
	$2x^2-x-3$		

Hence the H.C.F. is $x(2x^2-x-3)$.

130. To find the H.C.F. of more than two expressions, we first find the H.C.F. of any two of them, and then find the H.C.F. of that H.C.F. and the third expression, and so on.

EXAMPLES LXIX

Find the H.C.F. of

1. x^2+4x+3, x^3-3x^2-9x-5.

2. x^2+x-2, $x^3+x^2-14x-24$.

3. x^3+7x^2-x-7, x^3+5x^2+7x+3.

4. x^3+3x^2+3x+2, x^3-2x^2-2x-3.

5. x^3+3x^2+7x+5, x^3-x^2-x-15.

6. $2x^3+2x^2-9x+9$, $4x^3-6x^2+2x+3$.

7. x^3+x-2, x^3-2x^2+3x-2.

8. $x^3-11x^2+32x-28$, $3x^3-25x^2+54x-32$.

9. $4x^4-14x^3+16x^2-8x$, $6x^3-6x^2-11x-2$.

10. $x^4+3x^3+x^2-3x-2$, $2x^4-x^3-9x^2+4x+4$.

11. $4x^4-10x^2+12x$, $4x^4+x^3-12x^2+4x$.

12. $x^4-2x^3+x^2-8x+8$, $4x^4-12x^3+9x^2-x$.

13. $6x^4-13x^3-8x^2+17x+6$, $10x^4-9x^3-17x^2+8x+6$.

14. $4x^4+12x^3-16x^2-48x$, $6x^4+12x^3-54x^2-108x$.

15. $4x^4-6x^2-28$, $6x^4+10x^3-17x^2-35x-14$.

16. $x^3-7xy^2-6y^3$, $x^3+3x^2y+3xy^2+y^3$.

17. $x^4+x^3y-8x^2y^2-9xy^3-9y^4$, $x^4+x^2y^2+y^4$.

18. $x^3-6x^2+11x-6$, x^3+x^2-9x-9, $x^3-6x^2+5x+12$.

19. $3x^3+10x^2+9x+2$, $3x^3+x^2-3x-1$, $6x^3-13x^2+x+2$.

20. $2x^4+7x^3+13x^2+11x+6$, $2x^4+3x^3+3x^2-x+6$, $6x^4+13x^3+13x^2-6x$.

CHAPTER XXII

LEAST COMMON MULTIPLE

L.C.M. by Factors

131. When an expression is exactly divisible by two or more expressions, it is called a **common multiple** of those expressions.

Thus a^3b^3, a^2b^2 are common multiples of a^2b and ab^2.

In the above example, a^3b^3 and a^2b^2 are both multiples of a^2b and ab^2 but a^2b^2 is of a lower degree than a^3b^3. If we take ab, an expression of a lower degree than a^2b^2, we see that it is neither divisible by a^2b nor by ab^2, therefore it is not the common multiple of a^2b and ab^2. Similarly a^2b is not divisible by ab^2, and ab^2 is not divisible by a^2b. Hence, the common multiple of the lowest degree which is divisible by both a^2b and ab^2 is a^2b^2. It is, therefore, called their **Lowest Common Multiple**. Hence, *the Lowest Common Multiple of two or more simple expressions is the expression of the lowest degree which is exactly divisible by all of them.*

NOTE. For brevity, the Lowest Common Multiple is often written L.C.M.

Similarly, the L.C.M. of x, x^5, x^3 and x^3 is x^5, for the highest power of x which occurs in either of the expressions is x^5 and no lower power of x than x^5 can contain x^5; and the L.C.M. of $2x^2$, $3x^4$, $6x^5$ and $4x^7$ is $12x^7$, for the L.C.M. of 2, 3, 6 and 4 is 12 and of x^2, x^4, x^5 and x^7 is x^7.

EXAMPLE 1. *Find the L.C.M. of* $3x^2y^2z^4$, $2xy^3z^3$ *and* $4x^3y^2z^3$.

The L.C.M. of 3, 2, 4 is 12,

,, ,, ,, x^2, x, x^3 is x^3,

,, ,, ,, y^2, y^3, y^2 is y^3,

and ,, ,, ,, z^4, z^3, z^3 is z^4.

Hence the L.C.M. is $12x^3y^3z^4$.

From the above example it is clear that *the Lowest Common Multiple of two or more simple expressions is the product of the L.C.M. of their numerical coefficients and the highest powers of all the letters which occur in the expressions.*

EXAMPLE 2. *Find the L.C.M. of* $3x^2yz^3$, $4xy^5$ *and* $6x^3y^2z^2$.

The L.C.M. of 3, 4, 6 is 12,

the highest power of x in x^2, x, x^3 is x^3,

,, ,, ,, ,, y ,, y, y^5, y^2 is y^5,

and ,, ,, ,, ,, z ,, z^3, z^2 is z^3.

Hence the L.C.M. is $12x^3y^5z^3$.

NOTE. From the above examples it is evident that the Lowest Common Multiple of simple expressions can be written down by inspection.

EXAMPLES LXX

(*Oral*)

Write down the L.C.M. of

1. a^2b^2, a^2b. **2.** a^2b^3, a^3b^2. **3.** $2a^2b$, $3ab^2$.

4. $3x^2y$, $4xy^2$. **5.** $2xyz$, $4xy^2$. **6.** $3x^2y^3z^4$, $9xy^3z^3$.

7. $5l^3mn^2$, $10l^2m^3n$. **8.** x, x^3, x^5. **9.** a^2, a^5, a^7.

10. $2x^2$, $4x^3$, $8x^4$. **11.** xy, xy^2, x^2y. **12.** a^2b^2, a^2b^3, a^3b^3.

13. ab, bc, ca. **14.** a^3, a^2b^2, b^3. **15.** ax, bx^2, cx^3.

16. ax, by, cz. **17.** p, $2p$, $3p$, $4p$, $5p$. **18.** pq^2, qr^2, rp^2.

19. a^2bc, ab^2c, abc^2. **20.** $5a^2b^2c^3$, $3a^2b^3c^2$, $10a^3b^2c^2$.

21. $7ab^2x^3y^4$, $2a^2b^3x^4y$, $21a^3b^4xy^2$, $6a^4bx^2y^3$.

22. $49l^4m^6n^2p^8$, $21lm^7n^8p^5$, $56l^6mn^5p$, $70l^2m^3n^4p^7$.

132. The method of finding the L.C.M. of compound expressions whose factors are known or can be easily found, is similar to that of finding the L.C.M. of simple expressions. This has been explained in the previous article. *The L.C.M. is the product of the highest powers of all the factors which occur in the expressions.*

Take, for example, the expressions $(a+b)^3(a-2b)^3$ and $(a+b)^4(a-2b)^2$. It is clear that the least common multiple must be divisible by $(a+b)^3$ as well as $(a+b)^4$, therefore it must contain $(a+b)^4$ as a factor; similarly it must also contain $(a-2b)^3$ as a factor. Hence, the L.C.M. is $(a+b)^4(a-2b)^3$.

EXAMPLE 1. *Find the L.C.M. of* x^2-y^2 *and* $x^2+2xy+y^2$.

$$\text{Since } x^2-y^2=(x+y)(x-y),$$
$$\text{and } x^2+2xy+y^2=(x+y)^2.$$

Hence the L.C.M. is $(x+y)^2(x-y)$.

EXAMPLE 2. *Find the L.C.M. of* $x^2-7x+10$ *and* x^2+x-6.

$$\text{Since } x^2-7x+10=(x-2)(x-5),$$
$$\text{and } x^2+x-6=(x-2)(x+3).$$

Hence the L.C.M. is $(x-2)(x+3)(x-5)$.

EXAMPLE 3. *Find the L.C.M. of* $5x^5y^2-5x^2y^5$, $10x^6-10x^4y^2$ *and* $15(x^5y+x^3y^3+xy^5)$.

$$\text{Since } 5x^5y^2-5x^2y^5=5x^2y^2(x^3-y^3)=5x^2y^2(x-y)(x^2+xy+y^2)$$
$$10x^6-10x^4y^2=10x^4(x^2-y^2)=10x^4(x-y)(x+y),$$
$$\text{and } 15(x^5y+x^3y^3+xy^5)=15xy(x^4+x^2y^2+y^4)$$
$$=15xy(x^2+xy+y^2)(x^2-xy+y^2).$$

Hence the L.C.M. is $30x^4y^2(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$.

EXAMPLES LXXI

Find the L.C.M. of

1. x^2-y^2, $x-y$.

2. a^2-b^2, a^3-b^3.

3. a^3-a^2b, a^2-b^2.

4. a^2-b^2, $(a-b)^2$.

5. $(a-b)(a-2b)$, $(a-2b)(a-3b)$.

6. $6a^2b(a+b)^2$, $4ab^2(a^2-b^2)$.

7. abc, a^2c, $ab+bd$.

8. x^3y+xy^3, $x^4y^2-x^2y^4$.

9. a^2-b^2, $(a-b)^2$, $(a+b)^2$.

10. $a+b$, a^2-b^2, $a^2+2ab+b^2$.

11. $(a+2b)^2$, $(a-2b)^2$, a^2-4b^2.

12. a^2-b^2, a^3-b^3, a^3+b^3.

13. x^2y-xy^2, x^3-xy^2, xy^2+y^3.

14. x, $x-3y$, x^2-9y^2.

15. y, $x-y$, x^2y-y^3.

16. x^3y-xy^3, x^3y+xy^3, x^5y-xy^5.

17. $6(x^3-1)$, $4(x^2+x+1)$, $10(x-1)$.

18. $4(x^2-y^2)$, $6(x^2+xy)$, $12(x^3-x^2y)$.

19. $3(x^2-xy)^2$, $(6x^3y^2-6xy^4)$, $24(x^3y^3-y^6)$.

20. x^2+3x, x^2+4x+3, x^2+3x+2.
21. $x^2-10x+24$, $x^2-7x+12$, $x^2-9x+18$.
22. x^2+x-2, x^2-x-6, x^2-4x+3.
23. x^2-25, $x^2-2x-35$, $x^2-12x+35$.
24. x^2-1, x^3+1, x^3-1, x^6-1.
25. x^2-1, x^3+x^2+x+1, x^3-x^2+x-1.
26. x^2-4, x^3+2x^2+4x+8, x^3-2x^2+4x-8.
27. $(x+1)(x+2)$, $(x+2)(x-3)$, $(x-3)(x-4)$, $(x-4)(x+1)$.
28. x^4+x, x^4-x, $x^6+x^4+x^2$.
29. $x+3$, x^2-9, x^3-27, x^2+3x+9.
30. $5(a^4-ab^3)$, $3(a^3-ab^2)$, $6(a^3+2ab^2-2a^2b-b^3)$.
31. $2x-y$, $4x^2-y^2$, $4x^2+2xy+y^2$, $8x^3-y^3$.
32. $2(x^2-y^2)$, $4(x^3-y^3)$, $6(x^4-y^4)$.
33. $7a^4b^2c+7a^3b^2cd$, $6a^3bc^2-6a^2bc^2d$, $14a^2b^3c-14b^3cd^2$.
34. x^2-a^2, $x^2+(a+b)x+ab$, $x^2-(a-b)x-ab$.
35. $a+b+c$, $a^2+b^2+c^2+2ab+2bc+2ca$, $a^3+b^3+c^3-3abc$.

L.C.M. by General Method

133. When expressions cannot be easily resolved into factors, their L.C.M. can be determined by the following method :

Let A and B be any two expressions, and let H be their H.C.F. and L their L.C.M., then

$$A=a\text{H}, \text{ and } B=b\text{H} ;$$

where a and b have *no* common factor, since all the common factors of A and B are contained in H.

$$L=ab\text{H}=\frac{a\text{H}\times b\text{H}}{\text{H}}=\frac{\text{AB}}{\text{H}} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{and also} \quad \text{LH}=\text{AB} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Now from (1) we see that *the L.C.M. of two expressions is obtained by multiplying the two expressions and dividing the product by their H.C.F., or by dividing one of the expressions by their H.C.F. and multiplying the quotient by the other expression.*

And from (2) we see that *the product of two expressions is equal to the product of their H.C.F. and L.C.M.*

EXAMPLE. *Find the L.C.M. of* x^3+4x^2+7x+6 *and* x^3-x^2-3x-9.

The H.C.F. of the two expressions, by the method of Article 129, is found to be x^2+2x+3.

Dividing the first expression by x^2+2x+3, the quotient is $x+2$.

$$\therefore \quad \text{L.C.M.} = (x+2)(x^3-x^2-3x-9)$$
$$= x^4+x^3-5x^2-15x-18.$$

134. To find the L.C.M. of more than two expressions, we first find the L.C.M. of any two of them, and then find the L.C.M. of that L.C.M. and the third expression, and so on.

EXAMPLES LXXII

Find the L.C.M. of

1. x^3+x^2-4x-4, $x^3+6x^2+11x+6$.

2. $x^3+4x^2-8x+24$, x^4-x^3+8x-8.

3. $2x^3-x^2+x+4$, $2x^3-5x^2+7x-4$.

4. $2x^3+2x^2-9x+9$, $4x^3-6x^2+2x+3$.

5. $9x^4-x^2-2x$, $3x^3-10x^2-7x-4$.

6. $x^2+2x-35$, x^3-6x^2+6x-5, x^3+6x^2-6x+7.

7. x^3+x^2-5x+3, x^3+3x^2-x-3.

8. $x^4-2x^3-36x^2-88x-64$, $x^4+2x^3-12x^2-40x-32$.

CHAPTER XXIII

FRACTIONS

Reduction of Fractions to Lowest Terms

135. In Arithmetic we have

$$\frac{35}{40}=\frac{5\times 7}{5\times 8}=\frac{7}{8},$$

similarly in Algebra we have

$$\frac{ab}{ac}=\frac{a\times b}{a\times c}=\frac{b}{c},$$

and $\dfrac{12a^2x^3y^2}{18a^2xy^3}=\dfrac{12}{18}\times\dfrac{a^2}{a^2}\times\dfrac{x^3}{x}\times\dfrac{y^2}{y^3}=\dfrac{2}{3}\times\dfrac{1}{1}\times\dfrac{x^2}{1}\times\dfrac{1}{y}=\dfrac{2x^2}{3y}.$

Hence, we see that *to simplify a fraction or to reduce it to its lowest terms, we divide both the numerator and denominator by all the factors common to both i.e., by their H.C.F.*

EXAMPLE 1. *Reduce* $\dfrac{36a^4c^3d^2}{45a^3b^3c^3}$ *to its lowest terms.*

The H.C.F. of the numerator and denominator is $9a^3c^3$.

$$\therefore \quad \frac{36a^4c^3d^2}{45a^3b^3c^3}=\frac{36a^4c^3d^2\div 9a^3c^3}{45a^3b^3c^3\div 9a^3c^3}=\frac{4ad^2}{5b^3}.$$

EXAMPLE 2. *Simplify* $\dfrac{a^2+ab}{a^2-b^2}$.

$$\frac{a^2+ab}{a^2-b^2}=\frac{a(a+b)}{(a-b)(a+b)}=\frac{a}{a-b}.$$

NOTE. It is a common mistake with beginners to cancel terms of the expressions from the numerator and the denominator. They should note that if the factors of the numerator and the denominator are compound expressions, they are taken as a whole and that a term cannot be removed unless it divides both the numerator and the denominator exactly. For example, a^2 or b cannot be removed from the numerator and the denominator of the fraction $\dfrac{a^2+ab}{a^2-b^2}$ as they do not divide them exactly.

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EXAMPLE 3. *Simplify* $\dfrac{x^4y-xy^4}{x^4+x^2y^2+y^4}$.

$$\frac{x^4y-xy^4}{x^4+x^2y^2+y^4}=\frac{xy(x-y)(x^2+xy+y^2)}{(x^2-xy+y^2)(x^2+xy+y^2)}=\frac{xy(x-y)}{x^2-xy+y^2}.$$

EXAMPLE 4. *Simplify* $\dfrac{b^2-ab}{a^2-b^2}$.

$$\frac{b^2-ab}{a^2-b^2}=\frac{b(b-a)}{(a-b)(a+b)}.$$

Since $b-a=-a+b=-(a-b)$,

$$\text{the fraction } =\frac{-b(a-b)}{(a-b)(a+b)}=\frac{-b}{a+b}.$$

NOTE 1. Multiplying both the numerator and the denominator of the fraction $\dfrac{-b}{a+b}$ by -1, we have

$$\frac{-b}{a+b}=\frac{b}{-a-b}=\frac{b}{-(a+b)},$$

that is, *the value of a fraction is not altered if the signs of the numerator and the denominator are interchanged.*

NOTE 2. Both $\dfrac{-b}{a+b}$ and $\dfrac{b}{-(a+b)}$ are usually written in the form $-\dfrac{b}{a+b}$.

EXAMPLES LXXIII

(*Examples* 1 *to* 16 *may be taken orally*)

Simplify

1. $\dfrac{4a^2}{8a^3}$. 2. $\dfrac{a^2b}{ab^2}$. 3. $\dfrac{a^3b^3}{a^2b^3}$. 4. $\dfrac{a^6b^3}{a^4b^2}$.

5. $\dfrac{2a^2b^3c^4}{3ab^4c^3}$. 6. $\dfrac{6ax}{10x}$. 7. $\dfrac{5xyz}{15x^2y^2z^2}$. 8. $\dfrac{3x^4y^6z^2}{12x^2y^4z^6}$.

9. $\dfrac{12}{2x}$. 10. $\dfrac{15a^3b^2c^4d^5}{25ab^4c^2d^1}$. 11. $\dfrac{50ab^2c^3d^4}{125a^4b^3c^2d}$.

12. $\dfrac{3a^3b^4c^5xy^2}{7b^4c^2xy^5}$. 13. $\dfrac{21ab^3c^5x^2y^4z^6}{35a^2b^4c^6xy^3z^5}$. 14. $-\dfrac{3^3x^5}{27x^4y}$.

15. $\dfrac{(-2)^3ab^3c^5}{4^2a^2b^3c^4}$. 16. $\dfrac{(-3)^5a^2b^2x^2}{9^2a^3c^2x^2y}$. 17. $\dfrac{a^2}{a^2+ab}$.

18. $\frac{ab}{a^2-ab}$. 19. $\frac{x^2y^2}{x^2+x^2y^2}$. 20. $\frac{3xy}{3x^2y-6xy^2}$.

21. $\frac{a^2-ab}{a^2+ab}$. 22. $\frac{a^2-ab}{a^2-b^2}$. 23. $\frac{x^2-1}{(x-1)^2}$.

24. $\frac{6a^2-9ab}{10ab-15b^2}$. 25. $\frac{a^2-a^2b^2}{(a+ab)^2}$. 26. $\frac{a^2+3a}{a^2-9}$.

27. $\frac{a^4+a^2}{a^4-1}$. 28. $\frac{5x-30}{x^2-36}$. 29. $\frac{x^3+y^3}{(x+y)^2}$.

30. $\frac{x^4-y^4}{x^4-x^2y^2}$. 31. $\frac{x-3}{9-x^2}$. 32. $\frac{x-5}{25-x^2}$.

33. $\frac{x+7}{49-x^2}$. 34. $\frac{x^2y^2z^2-x^4y^4}{x^4y^4-z^4}$. 35. $\frac{5x^2-20ax}{80a^2-5x^2}$.

36. $\frac{a^4-1}{a^6-1}$. 37. $\frac{(x+2)(x+3)}{(x+1)(x+2)}$. 38. $\frac{(x-1)(x+6)}{(x+6)(x+7)}$.

39. $\frac{(x+3)(x-1)}{x^2-1}$. 40. $\frac{x^2+4x+3}{x^2+5x+4}$. 41. $\frac{x^2-1}{x^2+2x+1}$.

42. $\frac{x^2+9xy+14y^2}{x^2+5xy-14y^2}$. 43. $\frac{1-5x+6x^2}{1-8x+15x^2}$.

44. $\frac{4x^2-1}{1-9x+14x^2}$. 45. $\frac{x^4+x^2y^2+y^4}{x^2+xy+y^2}$.

46. $\frac{x^4-y^4}{(x^3-y^3)(x+y)}$. 47. $\frac{(x^6-y^6)(x^2-y^2)}{(x^3-y^3)(x^4-y^4)}$.

48. $\frac{x^2+(a+b)x+ab}{x^2+(a-b)x-ab}$. 49. $\frac{(a+b)^2-(c+d)^2}{(a+d)^2-(b+c)^2}$.

50. $\frac{(a+b)^2-4c^2}{a^2-(b+2c)^2}$.

136. When we cannot easily find the factors of either the numerator or the denominator of a fraction which we wish to reduce to its lowest terms, we first find their H.C.F. by the rule given in Article 129, and then divide both the numerator and the denominator by their H.C.F.

EXAMPLE. *Reduce* $\frac{2x^3+9x^2+19x+15}{x^4+4x^3+9x^2+8x+5}$ *to its lowest terms.*

The H.C.F. of the numerator and the denominator is x^2+3x+5.

Dividing the numerator and the denominator by x^2+3x+5, the quotients are $2x+3$ and x^2+x+1 respectively.

$$\frac{2x^3+9x^2+19x+15}{x^4+4x^3+9x^2+8x+5}=\frac{(2x+3)(x^2+3x+5)}{(x^2+x+1)(x^2+3x+5)}=\frac{2x+3}{x^2+x+1}.$$

EXAMPLES LXXIV

Reduce the following fractions to lowest terms :

1. $\dfrac{x^3-23x+10}{3x^3-13x^2-4x+4}$.

2. $\dfrac{x^3-3x+2}{2x^4-3x^3+x}$.

3. $\dfrac{3x^3-2x^2-19x-6}{6x^3-13x^2-14x-3}$.

4. $\dfrac{x^3-3x-2}{x^4+2x^3+2x^2+2x+1}$.

5. $\dfrac{x^3+7x^2+7x-15}{x^3-2x^2-13x+110}$.

6. $\dfrac{x^4+x^3-x^2+3x-12}{x^4+2x^3+x^2-16}$.

Multiplication and Division of Fractions

137. Multiplication of Fractions. To multiply two or more fractions, *multiply together the numerators of all the fractions, and the denominators of all the fractions, and cancel factors common to the product of the numerators and denominators.*

Thus, to find the product of $\dfrac{ab}{2x}$ and $\dfrac{x^2}{a^2b}$, we have

$$\frac{ab}{2x}\times\frac{x^2}{a^2b}=\frac{abx^2}{2xa^2b}=\frac{1}{2}\times\frac{a}{a^2}\times\frac{b}{b}\times\frac{x^2}{x}=\frac{1}{2}\times\frac{1}{a}\times\frac{1}{1}\times\frac{x}{1}=\frac{x}{2a}.$$

EXAMPLE 1. *Simplify* $\dfrac{3a^3y^2}{4bc^2x^3}\times\dfrac{8b^3xy}{10a^2cy^2}\times\dfrac{5c^3x^2y}{9a^2b^3}$.

$$\text{The expression }=\frac{3\times8\times5}{4\times10\times9}\times\frac{a^3}{a^2\times a^2}\times\frac{b^3}{b\times b^3}\times\frac{c^3}{c^2\times c}\times\frac{x\times x^2}{x^3}\times\frac{y^2\times y\times y}{y^2}$$

$$=\frac{1}{3}\times\frac{1}{a}\times\frac{1}{b}\times\frac{1}{1}\times\frac{1}{1}\times\frac{y^2}{1}=\frac{y^2}{3ab}.$$

EXAMPLE 2. *Simplify* $\dfrac{a^2-b^2}{a-c}\times\dfrac{a^2-c^2}{ab+b^2}\times\dfrac{a^2}{a-b}$.

$$\text{The expression }=\frac{(a+b)(a-b)}{(a-c)}\times\frac{(a+c)(a-c)}{b(a+b)}\times\frac{a^2}{(a-b)}=\frac{a^2(a+c)}{b}.$$

138. Division of Fractions. To divide one fraction by another, *first invert the divisor, i.e., interchange the numerator and the denominator of the divisor, and then proceed as in multiplication.*

Thus, in dividing $\frac{3ab}{4xy}$ by $\frac{9ab^2}{8x^2y}$, we have

$$\frac{3ab}{4xy}\div\frac{9ab^2}{8x^2y}=\frac{3ab}{4xy}\times\frac{8x^2y}{9ab^2}=\frac{3\times 8}{4\times 9}\times\frac{a}{a}\times\frac{b}{b^2}\times\frac{x^2}{x}\times\frac{y}{y}$$
$$=\frac{2}{3}\times\frac{1}{1}\times\frac{1}{b}\times\frac{x}{1}\times\frac{1}{1}=\frac{2x}{3b}.$$

EXAMPLE 1. *Simplify* $\frac{5a^3b^2x}{6ay^2z^3}\div\frac{x^3z}{9b^2y^2}\times\frac{2x^2z^4}{15a^2b^4}$.

$$\text{The expression}=\frac{5a^3b^2x}{6ay^2z^3}\times\frac{9b^2y^2}{x^3z}\times\frac{2x^2z^4}{15a^2b^4}$$
$$=\frac{5\times 9\times 2}{6\times 15}\times\frac{x\times x^2}{x^3}\times\frac{y^2}{y^2}\times\frac{z^4}{z^3\times z}\times\frac{a^3}{a\times a^2}\times\frac{b^2\times b^2}{b^4}$$
$$=\frac{1}{1}\times\frac{1}{1}\times\frac{1}{1}\times\frac{1}{1}\times\frac{1}{1}\times\frac{1}{1}$$
$$=1.$$

NOTE. When all the factors of the numerator and the denominator cancel, the result is 1 and *not* zero.

EXAMPLE 2. *Simplify* $\frac{(a-b)^2-c^2}{ab-b^2+bc}\times\frac{abc}{a^2-ab-ac}\div\frac{ac+bc+c^2}{a^2-(b+c)^2}$.

$$\text{The expression}=\frac{(a-b+c)(a-b-c)}{b(a-b+c)}\times\frac{abc}{a(a-b-c)}\times\frac{(a+b+c)(a-b-c)}{c(a+b+c)}$$
$$=a-b-c.$$

EXAMPLES LXXV

Simplify

1. $\frac{ab}{3x}\times\frac{x^2}{a^2}$.
2. $\frac{ab^2}{a^2b}\times\frac{bc^2}{b^2}$
3. $\frac{2a^2}{3c}\times\frac{9c^2}{4a}$.
4. $\frac{5a^2}{3bc}\times\frac{3b^2}{5ac}$.
5. $\frac{xy}{ab}\div\frac{x^2}{b^2}$.
6. $\frac{xyz}{abc}\div\frac{xyz}{abc}$.
7. $\frac{2a^2}{bc}\div\frac{3ab}{c}$.
8. $\frac{a}{b}\div\frac{c}{d}$.
9. $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}$.
10. $\frac{a^2}{x^2}\times\frac{x^2}{b^2}\times\frac{b^2}{a^2}$.
11. $\frac{a^2}{bc}\times\frac{b^2}{ca}\times\frac{c^2}{ab}$.
12. $\frac{2x}{3y}\div\frac{4a}{5b}\times\frac{6a}{7x}$.
13. $\frac{3z^6}{zx}\times\frac{x^4}{z^2}\div\frac{z^4x^2}{6}$.
14. $2\times\frac{x^2}{y^3}\times\frac{y^4}{2x}\times\frac{x}{3x^2}$.

15. $\frac{7ab^2c^3}{8xy^2z^2}\times\frac{-8x^3y^2z}{7a^3b^2c}$.

16. $\frac{5ab^2c^3}{9xy^2z^3}\div\frac{-10a^3b^2c}{6x^3y^2z}$.

17. $\frac{-3lm^3n^5}{4p^2q^2r^2}\times\frac{-8pqr}{9lmn}$.

18. $\frac{-12x^3y^5z^7}{27x^5y^5z^5}\div\frac{-x^2y^2z^2}{3xy^3z^5}$.

19. $\frac{-2kl^3}{3lm^2}\times\frac{klm}{m^3n^3}\times\frac{m^2n^4}{-4k^2l}$.

20. $\frac{x-y}{x^2+xy}\times\frac{x+y}{x^2-xy}$.

21. $\frac{x+y}{x^3-x^2y}\times\frac{xy-y^2}{xy+x^2}$.

22. $\frac{4x^2-1}{4y^2-1}\times\frac{2y+1}{2x-1}$.

23. $\frac{x^2+5x+6}{x^2-25}\div\frac{x-3}{x-5}$.

24. $\frac{x+1}{x+2}\times\frac{x+2}{x+3}\times\frac{x+3}{x+4}$.

25. $\frac{x^2-3x+2}{x^2-5x+4}\div\frac{x^2-5x+6}{x^2-7x+12}$.

26. $\frac{x^3-y^3}{x^2-4y^2}\div\frac{x-y}{x+2y}$.

27. $\frac{x^4-y^4}{x^2-2xy+y^2}\times\frac{xy-y^2}{x^2+y^2}$.

28. $\frac{x^2-x-6}{x(x-3)}\times\frac{x^2+x-2}{x(x-1)}$.

29. $\frac{a^3-b^3}{a^3+b^3}\times\frac{a^2-b^2}{(a-b)^2}$.

30. $\frac{a+b}{(a-b)^2}\div\frac{a^2+b^2}{a^2-b^2}\times\frac{a^4-b^4}{(a+b)^3}$.

31. $\frac{x+4}{x^2-9}\div\frac{x^2+2x-8}{x^2+x-6}$.

32. $\frac{x^2-x-12}{x^3-64}\times\frac{x^2+4x+16}{x^2+x-6}$.

33. $\frac{x^2-2x-24}{x^2-16}\div\frac{x^2+6x+9}{x^2-x-12}$.

34. $\frac{(a-b)^2-c^2}{(a-c)^2-b^2}\times\frac{b^2-(c-a)^2}{c^2-(a-b)^2}$.

35. $\frac{a^2-(b+c)^2}{a^2-(b-c)^2}\div\frac{b^2-(a+c)^2}{b^2-(a-c)^2}$.

36. $\frac{x+1}{x-1}\times\frac{x^2-x-6}{x^2+2x+1}\times\frac{x^2+x-6}{x^4-13x^2+36}$.

37. $\frac{a^2-a-2}{a^2-2a}\times\frac{a^2-5a}{a^2-1}\div\frac{a^2-2a-8}{a^2+a-2}$.

38. $\frac{x^8+x^4+1}{x^6+1}\div\frac{x^2-1}{x^4-1}\times\frac{x-1}{x^3-1}$.

39. $\frac{x^2-3x+2}{x^2+7x+12}\div\frac{x^2-4x-5}{x^2+5x+4}\times\frac{x^2-8x+15}{x^2-5x+6}$.

40. $\frac{x^3-8}{x+5}\div\left(\frac{x-2}{4x^2}\times\frac{2x^3}{x^2+5x}\right)$.

41. $\frac{a^6-b^6}{a^4+2a^2b^2+b^4}\times\frac{a^2+b^2}{a^3-b^3}\div\frac{a^2-ab+b^2}{a+b}$.

42. $\frac{a^6+b^6}{a^6-b^6}\times\frac{a^4+a^2b^2+b^4}{a^4-a^2b^2+b^4}\times\frac{a-b}{a+b}$.

43. $\frac{a^2-2ac+c^2-b^2}{b^2-2bc+c^2-a^2}\div\frac{a^2-b^2-c^2-2bc}{a^2+2ab+b^2-c^2}$.

44. $\frac{x^2-(y-z)^2}{(y+z)^2-x^2}\div\frac{(z+x)^2-y^2}{y^2-(z-x)^2}\times\frac{z^2-(x-y)^2}{(x+y)^2-z^2}$.

Addition and Subtraction of Fractions

139. The method of adding or subtracting fractions in Algebra is precisely the same as that in Arithmetic. As we have seen in Arithmetic, when we have to add or subtract fractions which have different denominators, we first reduce all the fractions to equivalent fractions having the same denominator. For this purpose we reduce all the fractions to their lowest common denominator which is the L.C.M. of the denominators, then divide this L.C.M. by the denominator of each fraction and multiply the quotient by the corresponding numerator, and proceed as in Arithmetic.

Thus, to simplify $\frac{6}{7}+\frac{11}{14}-\frac{4}{5}$, we first take the L.C.M. of the denominators 7, 14 and 5, which is 70. Now to reduce the denominators of the fractions to 70, we divide 70 by the denominators 7, 14 and 5 respectively and get the quotients 10, 5 and 14. Then multiplying both the numerators and denominators of each fraction by 10, 5 and 14, we have

$$\frac{6}{7}+\frac{11}{14}-\frac{4}{5}=\frac{6\times 10}{7\times 10}+\frac{11\times 5}{14\times 5}-\frac{4\times 14}{5\times 14}=\frac{60}{70}+\frac{55}{70}-\frac{56}{70}$$

$$=\frac{60+55-56}{70}=\frac{59}{70}.$$

EXAMPLE 1. *Simplify* $\frac{a}{x}+\frac{b}{2x}-\frac{c}{3x}$.

The L.C.M. of the denominators is $6x$. Dividing $6x$ by x, $2x$ and $3x$, the quotients are 6, 3 and 2. Now multiplying the numerators and the denominators of the fractions by these, we have

$$\frac{a}{x}+\frac{b}{2x}-\frac{c}{3x}=\frac{6\times a}{6\times x}+\frac{3\times b}{3\times 2x}-\frac{2\times c}{2\times 3x}=\frac{6a}{6x}+\frac{3b}{6x}-\frac{2c}{6x}=\frac{6a+3b-2c}{6x}.$$

EXAMPLE 2. *Simplify* $\frac{x-y}{x^2y}-\frac{x-2y}{2xy^2}-\frac{3}{4xy}$.

The L.C.M. of the denominators is $4x^2y^2$.

$$\text{the expression}=\frac{4y(x-y)}{4y\times x^2y}-\frac{2x(x-2y)}{2x\times 2xy^2}-\frac{xy\times 3}{xy\times 4xy}$$

$$=\frac{4y(x-y)}{4x^2y^2}-\frac{2x(x-2y)}{4x^2y^2}-\frac{3xy}{4x^2y^2}=\frac{4y(x-y)-2x(x-2y)-3xy}{4x^2y^2}$$

$$=\frac{4xy-4y^2-2x^2+4xy-3xy}{4x^2y^2}=\frac{5xy-2x^2-4y^2}{4x^2y^2}.$$

EXAMPLE 3. *Simplify* $a-\frac{a}{b}$.

NOTE. When no denominator is given, the denominator 1 may be understood.

$$a-\frac{a}{b}=\frac{a}{1}-\frac{a}{b}=\frac{a\times b}{1\times b}-\frac{a}{b}=\frac{ab}{b}-\frac{a}{b}=\frac{ab-a}{b}=\frac{a(b-1)}{b}.$$

EXAMPLE 4. *Simplify* $\frac{x-y}{xy}+\frac{y-z}{yz}+\frac{z-x}{zx}$.

The expression $=\frac{z\times(x-y)}{z\times xy}+\frac{x\times(y-z)}{x\times yz}+\frac{y\times(z-x)}{y\times zx}$

$$=\frac{xz-yz}{xyz}+\frac{xy-xz}{xyz}+\frac{yz-xy}{xyz}=\frac{xz-yz+xy-xz+yz-xy}{xyz}$$

$$=\frac{0}{xyz}=0.$$

NOTE. If the numerator of a fraction is zero, the fraction is equal to zero.

EXAMPLES LXXVI

Simplify

1. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.
2. $\frac{x^2}{2}-\frac{x^2}{4}+\frac{x^2}{8}$.
3. $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}$.
4. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.
5. $\frac{a}{b}-\frac{a}{2b}+\frac{a}{3b}$.
6. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$.
7. $\frac{a}{x}-\frac{b}{y}-\frac{c}{z}$.
8. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}$.
9. $\frac{a-1}{2}+\frac{a-2}{3}+\frac{a-3}{5}$.
10. $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$.
11. $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}$.
12. $\frac{x}{a^2b}+\frac{y}{ab^2}+\frac{z}{a^2b^2}$.
13. $\frac{x}{4}-\frac{x+1}{6}+\frac{2x}{9}$.
14. $\frac{a-b}{b}-\frac{a-c}{c}$.
15. $\frac{2x-y}{xy}-\frac{y-2x}{yz}$.
16. $\frac{x-5}{2x}-\frac{x-6}{3x}+\frac{5}{6}$.
17. $1+\frac{a}{b}$.
18. $1-\frac{a}{b}$.
19. $a+\frac{a}{b}$.
20. $a-\frac{a}{b}$.
21. $a-\frac{b^2}{a}$.
22. $a+\frac{a^2}{b}+\frac{b^2}{a}$.
23. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.
24. $\frac{a^2-b^2}{a^2b^2}+\frac{b^2-c^2}{b^2c^2}+\frac{c^2-a^2}{c^2a^2}$.
25. $\frac{a-b}{ab}+\frac{a+b}{b^2}+\frac{b}{a^2}+\frac{a^2-b^2}{4ab^2}$.

140. When such factors as $a-b$ and $b-a$ both appear in the denominators of fractions which are to be added or subtracted, much labour is saved if only one is included in the common denominator, for $b-a=-a+b=-(a-b)$.

EXAMPLE 1. *Simplify* $\frac{a}{a-b}+\frac{b}{b-a}$.

The expression $=\frac{a}{(a-b)}+\frac{b}{-(a-b)}=\frac{a}{(a-b)}-\frac{b}{(a-b)}=\frac{a-b}{a-b}=1.$

NOTE. When the numerator and the denominator of a fraction are the same, the fraction is equal to 1.

EXAMPLE 2. *Simplify* $\frac{1}{a+b}+\frac{1}{a-b}$.

The L.C.M. of $a+b$ and $a-b$ is $(a+b)(a-b)$.

$$\frac{1}{a+b}+\frac{1}{a-b}=\frac{(a-b)}{(a+b)(a-b)}+\frac{(a+b)}{(a+b)(a-b)}=\frac{(a-b)+(a+b)}{(a+b)(a-b)}$$

$$=\frac{a-b+a+b}{(a+b)(a-b)}=\frac{2a}{a^2-b^2}.$$

141. It is sometimes advisable to arrange the denominators of fractions which are to be added or subtracted, according to descending or ascending powers of some particular letter, before proceeding with the simplification.

EXAMPLE 1. *Simplify* $\frac{x^2-2x}{x^2-1}-\frac{x+3}{x+1}-\frac{4x}{1-x}$.

Here the denominators of the first two fractions are arranged in descending powers of x, therefore if the terms of the denominator of the third fraction are also arranged in descending powers of x by altering the sign, the expression

$$=\frac{x^2-2x}{x^2-1}-\frac{x+3}{x+1}-\frac{4x}{-(x-1)}$$

$$=\frac{x(x-2)}{(x+1)(x-1)}-\frac{(x+3)}{(x+1)}+\frac{4x}{(x-1)}$$

$$=\frac{x(x-2)}{(x+1)(x-1)}-\frac{(x+3)(x-1)}{(x+1)(x-1)}+\frac{4x(x+1)}{(x+1)(x-1)}$$

$$=\frac{x(x-2)-(x+3)(x-1)+4x(x+1)}{(x+1)(x-1)}$$

$$=\frac{x^2-2x-x^2-2x+3+4x^2+4x}{(x+1)(x-1)}$$

$$=\frac{4x^2+3}{x^2-1}.$$

NOTE. It should be noted here that in the fourth line, the common denominator is written only once. When this is done, *the multipliers of the numerators must be placed within brackets*, otherwise a wrong sign may creep in. The student should also note that the line of a fraction is really a bracket.

EXAMPLE 2. *Simplify* $\frac{1}{x^2+3x+2}+\frac{1}{x^2-x-6}+\frac{1}{x^2-2x-3}$.

$$\text{The expression}=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x-3)}+\frac{1}{(x-3)(x+1)}$$

$$=\frac{(x-3)+(x+1)+(x+2)}{(x+1)(x+2)(x-3)}$$

$$=\frac{3x}{(x+1)(x+2)(x-3)}.$$

EXAMPLES LXXVII

Simplify

1. $\frac{1}{x-1}-\frac{1}{1-x}$.
2. $1+\frac{x-y}{x+y}$.
3. $\frac{x-y}{x+y}-1$.
4. $\frac{x}{x+1}+\frac{x}{x+2}$.
5. $\frac{x}{x+z}-\frac{y}{y+z}$.
6. $\frac{a}{a-b}-\frac{a}{a+b}$.
7. $\frac{1}{a(a-b)}+\frac{1}{a(a+b)}$.
8. $\frac{2}{3(a-b)}+\frac{3}{2(b-a)}$.
9. $\frac{x+y}{x-y}+\frac{y-x}{y+x}$.
10. $\frac{x+5}{x-5}-\frac{x-5}{x+5}$.
11. $\frac{a+3b}{a-3b}-\frac{a-3b}{a+3b}$.
12. $\frac{a^2+b^2}{a^2-b^2}-\frac{a+b}{a-b}$.
13. $\frac{x}{(x-3)^2}-\frac{1}{x-3}$.
14. $\frac{x}{x^2-9y^2}+\frac{1}{3y-x}$.
15. $\frac{1}{x^3-1}+\frac{x+1}{x^2+x+1}$.
16. $\frac{x^2+xy+y^2}{x+y}+\frac{y^2}{x-y}$.
17. $\frac{a}{a-2x}+\frac{2ax}{(a-2x)^2}$.
18. $\frac{x}{(x+y)^2}-\frac{y}{x^2-y^2}$.
19. $\frac{a^3-b^3}{a^2+ab+b^2}-\frac{a^3+b^3}{a^2-ab+b^2}$.
20. $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4ab}{a^2-b^2}$.
21. $\frac{1}{x-y}+\frac{2}{x+y}-\frac{3x-y}{x^2-y^2}$.
22. $\frac{2(a+b)}{a^2-ab}-\frac{1}{3(a+b)}+\frac{2a}{3(a^2-b^2)}$.
23. $\frac{x^2+xy+y^2}{x+y}+\frac{x^2-xy+y^2}{x-y}$.
24. $\frac{1}{x^3+2x^2}+\frac{1}{x^3-2x^2}-\frac{2}{x^3-4x}$.

25. $\frac{x^3y+xy^3}{x^6-y^6}-\frac{y}{y^3-x^3}-\frac{x}{x^3+y^3}.$

26. $\frac{x}{x-2}+\frac{2x}{x+3}-\frac{1}{x^2+x-6}.$

27. $\frac{2x}{3-x}-\frac{3x}{x+3}+\frac{5}{x^2-9}.$

28. $1+\frac{2}{x+y}-\frac{1}{x-y}.$

29. $\frac{(x+a)^2}{x^2-a^2}+\frac{x^2-a^2}{ax-a^2}.$

30. $\frac{x}{x-1}-1+\frac{1}{x(1-x)}.$

31. $1-\frac{a}{a+x}-\frac{ax}{(a+x)^2}.$

32. $\frac{x+y}{x-y}+2+\frac{x-y}{x+y}.$

33. $\frac{a+b}{a^2+ab+b^2}+\frac{1}{a-b}-\frac{a(2a+b)}{a^3-b^3}.$

34. $\frac{1}{x(x-1)}-\frac{2}{x^2-1}+\frac{3}{x(x+1)}.$

35. $\frac{x}{x-y}+\frac{y}{x+y}+\frac{x}{x+y}-\frac{y}{x-y}.$

36. $\frac{x^2-1}{x^2+1}-\frac{x^2+1}{x^2-1}+\frac{4x^2}{x^4-1}.$

37. $\frac{2}{x^2-1}-\frac{3x}{x+1}-\frac{4x}{1-x}.$

38. $\frac{2a+5x}{2a-5x}-\frac{2a-5x}{2a+5x}+\frac{20ax}{25x^2-4a^2}.$

39. $\frac{x+7y}{x-7y}-\frac{x-7y}{x+7y}+2\frac{x^2+49y^2}{x^2-49y^2}.$

40. $\frac{x+3}{(x-2)(x-3)}-\frac{x+2}{(x-2)(x-7)}+\frac{4}{(x-7)(x-3)}.$

41. $\frac{1}{(x-1)(x-3)}+\frac{2}{(3-x)(x-5)}-\frac{3}{(5-x)(x-1)}.$

42. $\frac{2}{x^2-4x+3}-\frac{1}{x^2-3x+2}+\frac{2}{x^2-5x+6}.$

43. $\frac{1}{x^2+2x+1}+\frac{1}{x^2+5x+6}+\frac{1}{x^2+4x+3}.$

44. $\frac{1}{x^2-2x+1}-\frac{1}{x^2-5x+6}+\frac{1}{x^2-4x+3}.$

45. $\frac{x-1}{x^2+7x+10}-\frac{2x+4}{x^2+4x-5}+\frac{x+5}{x^2+x-2}.$

46. $1+\frac{1}{x+a}+\frac{1}{x-a}-\frac{1}{a^2-x^2}.$

Fractions in Cyclic Order

142. In simplifying the fractions which follow, it is best to preserve the *cyclic order* of the letters from the beginning and adhere to it throughout simplification.

EXAMPLE 1. *Simplify* $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$.

Changing the signs of the factors of the denominators so as to put them in cyclic order, the expression

$$=\frac{a}{-(a-b)(c-a)}+\frac{b}{-(b-c)(a-b)}+\frac{c}{-(c-a)(b-c)}$$

$$=-\frac{a}{(a-b)(c-a)}-\frac{b}{(b-c)(a-b)}-\frac{c}{(c-a)(b-c)}$$

$$=-\left\{\frac{a}{(a-b)(c-a)}+\frac{b}{(b-c)(a-b)}+\frac{c}{(c-a)(b-c)}\right\}$$

$$=-\frac{a(b-c)+b(c-a)+c(a-b)}{(a-b)(b-c)(c-a)}$$

$$=-\frac{ab-ac+bc-ab+ac-bc}{(a-b)(b-c)(c-a)}$$

$$=\frac{0}{(a-b)(b-c)(c-a)}$$

$$=0.$$

EXAMPLE 2. *Simplify* $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$.

The expression

$$=-\frac{a^2}{(a-b)(c-a)}-\frac{b^2}{(b-c)(a-b)}-\frac{c^2}{(c-a)(b-c)}$$

$$=-\left\{\frac{a^2}{(a-b)(c-a)}+\frac{b^2}{(b-c)(a-b)}+\frac{c^2}{(c-a)(b-c)}\right\}$$

$$=-\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(b-c)(c-a)}$$

$$=-\frac{-(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}$$ [See Example 1 of Article 126]

$$=1.$$

EXAMPLE 3. *Simplify* $\frac{a^2-a+1}{(a-b)(a-c)}+\frac{b^2-b+1}{(b-a)(b-c)}+\frac{c^2-c+1}{(c-a)(c-b)}$.

The expression

$$=\left\{\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}\right\}$$

$$-\left\{\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}\right\}$$

$$+\left\{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}\right\}$$

$$=1-0+0$$ [See Examples 1 and 2.]

$$=1.$$

EXAMPLE 4. *Simplify*

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

The expression

$$=-\left\{\frac{a}{(a-b)(c-a)(x-a)}+\frac{b}{(a-b)(b-c)(x-b)}+\frac{c}{(c-a)(b-c)(x-c)}\right\}$$

$$=-\frac{a(b-c)(x-b)(x-c)+b(c-a)(x-a)(x-c)+c(a-b)(x-a)(x-b)}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}.$$

The numerator

$=a(b-c)(x-b)(x-c)+b(c-a)(x-a)(x-c)+c(a-b)(x-a)(x-b)$

[Removing the brackets and collecting the coefficients of x^2 and x]

$=x^2(ab-ac+bc-ab+ac-bc)$

$\quad+x(-ab^2+abc-abc+ac^2-abc+a^2b-bc^2+abc-a^2c+abc-abc$

$\qquad+b^2c)+(ab^2c-abc^2+abc^2-a^2bc+a^2bc-ab^2c)$

$=x^2(0)+x(a^2b-a^2c-ab^2+ac^2+b^2c-bc^2)+0$

$=x\{a^2(b-c)-a(b^2-c^2)+bc(b-c)\}$

$=x(b-c)(a^2-ab-ac+bc)$

$=x(b-c)(a-c)(a-b)$

$=-x(b-c)(c-a)(a-b).$

$$\text{the expression}=-\frac{-x(b-c)(c-a)(a-b)}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}$$

$$=\frac{x}{(x-a)(x-b)(x-c)}.$$

EXAMPLES LXXVIII

Simplify

1. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}.$

2. $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-c)(b-a)}+\dfrac{a+b}{(c-a)(c-b)}.$

3. $\dfrac{b-c}{(a-b)(a-c)}+\dfrac{c-a}{(b-c)(b-a)}+\dfrac{a-b}{(c-a)(c-b)}.$

4. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}.$

5. $\frac{a^2+bc}{(a-b)(a-c)}+\frac{b^2+ca}{(b-c)(b-a)}+\frac{c^2+ab}{(c-a)(c-b)}.$

6. $\frac{x+a}{(a-b)(a-c)}+\frac{x+b}{(b-c)(b-a)}+\frac{x+c}{(c-a)(c-b)}.$

7. $\frac{a-b-c}{(a-b)(a-c)}+\frac{b-c-a}{(b-c)(b-a)}+\frac{c-a-b}{(c-a)(c-b)}.$

8. $\frac{a^2+1}{(a-b)(a-c)}+\frac{b^2+1}{(b-c)(b-a)}+\frac{c^2+1}{(c-a)(c-b)}.$

9. $\frac{a^3-1}{(a-b)(a-c)}+\frac{b^3-1}{(b-c)(b-a)}+\frac{c^3-1}{(c-a)(c-b)}.$

10. $\frac{ax}{(a-b)(a-c)}+\frac{bx}{(b-c)(b-a)}+\frac{cx}{(c-a)(c-b)}.$

11. $\frac{a^2-bc}{(a+b)(a+c)}+\frac{b^2-ac}{(b+c)(b+a)}+\frac{c^2-ab}{(c+a)(c+b)}.$

12. $\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}.$

13. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}.$

14. $\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-c)(b-a)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}.$

15. $\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-c)(b-a)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}.$

Complex Fractions

143. A fraction which has a fraction in its numerator or denominator or in both, is called a **Complex Fraction.** Thus

$$\frac{\frac{a}{b}}{c},\quad \frac{a}{\frac{c}{d}},\quad \frac{\frac{a}{b}}{\frac{c}{d}}$$

are complex fractions. To simplify such fractions, we proceed as in Arithmetic, *i.e.*, we simplify the numerator and the denominator separately, and then divide the numerator by the denominator.

EXAMPLE 1. *Simplify* $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$.

The numerator $=\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$.

The denominator $=\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$.

$$\text{the fraction } =\dfrac{\frac{ad+bc}{bd}}{\frac{ad-bc}{bd}}=\frac{ad+bc}{bd}\div\frac{ad-bc}{bd}$$

$$=\frac{ad+bc}{bd}\times\frac{bd}{ad-bc}=\frac{ad+bc}{ad-bc}.$$

EXAMPLE 2. *Simplify* $\dfrac{x+3-\frac{2}{x+4}}{x-3+\frac{10}{x+4}}$.

The fraction

$$=\dfrac{(x+3)-\frac{2}{(x+4)}}{(x-3)+\frac{10}{(x+4)}}=\dfrac{\frac{(x+3)(x+4)-2}{(x+4)}}{\frac{(x-3)(x+4)+10}{(x+4)}}=\dfrac{\frac{x^2+7x+12-2}{x+4}}{\frac{x^2+x-12+10}{x+4}}$$

$$=\frac{x^2+7x+10}{x+4}\div\frac{x^2+x-2}{x+4}=\frac{(x+2)(x+5)}{(x+4)}\times\frac{(x+4)}{(x+2)(x-1)}$$

$$=\frac{x+5}{x-1}.$$

EXAMPLES LXXIX

Simplify

1. $\dfrac{\frac{1}{a}}{b-\frac{c}{a}}$.

2. $\dfrac{1+\frac{1}{a}}{1-\frac{1}{a}}$.

3. $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$.

4. $\dfrac{2a+\frac{1}{2}}{2a-\frac{1}{3}}$.

5. $\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$.

6. $\dfrac{\frac{3}{2x}-\frac{2x}{3}}{\frac{1}{2x}-\frac{1}{3}}$.

7. $\dfrac{x^2-\dfrac{3x}{2}+2}{2x+\dfrac{4}{x}-3}$.

8. $\dfrac{\dfrac{a^2-b^2}{b^2}}{\dfrac{a-b}{b^3}}$.

9. $\dfrac{\dfrac{a}{a+1}+\dfrac{1}{a-1}}{\dfrac{a}{a-1}-\dfrac{1}{a+1}}$.

10. $\dfrac{\dfrac{3x}{1+3x}+\dfrac{1-3x}{3x}}{\dfrac{3x}{1+3x}-\dfrac{1-3x}{3x}}$.

11. $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$.

12. $\dfrac{\left(\dfrac{a+b}{a-b}\right)^2-\left(\dfrac{a-b}{a+b}\right)^2}{\left(\dfrac{a+b}{a-b}\right)^2+\left(\dfrac{a-b}{a+b}\right)^2}$.

13. $\dfrac{\dfrac{x^2+4y^2}{2x^2}-\dfrac{8y^2}{x^2+4y^2}}{\dfrac{x^2+4y^2}{8y^2}-\dfrac{2x^2}{x^2+4y^2}}$.

14. $\dfrac{x-2-\dfrac{30}{x-3}}{x+4+\dfrac{6}{x-3}}$.

15. $\dfrac{2x-1+\dfrac{6}{2(x-3)}}{2(x-1)+\dfrac{3}{2(x-3)}}$.

16. $\dfrac{\dfrac{1}{a}-1}{\dfrac{1}{a}+a+1}+\dfrac{1-a-\dfrac{1}{a}}{\dfrac{1}{a}-a^2}$.

17. $\dfrac{\dfrac{1}{x}+\dfrac{1}{2y}}{\dfrac{1}{x}-\dfrac{1}{3z}}\times\dfrac{\dfrac{1}{x+2y}-\dfrac{1}{3z}}{\dfrac{1}{x-3z}+\dfrac{1}{2y}}$.

Continued Fractions

144. The method of simplifying *Continued Fractions* in Algebra is precisely the same as that in Arithmetic, *i.e.*, we begin the work at the bottom and simplify step by step upwards.

EXAMPLE. *Simplify* $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{a}{x}}}}$.

The fraction $=\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{\dfrac{x+a}{x}}}}=\dfrac{1}{1+\dfrac{1}{1+\dfrac{x}{x+a}}}=\dfrac{1}{1+\dfrac{1}{\dfrac{2x+a}{x+a}}}=\dfrac{1}{1+\dfrac{x+a}{2x+a}}$

$=\dfrac{1}{\dfrac{3x+2a}{2x+a}}=\dfrac{2x+a}{3x+2a}$.

EXAMPLES LXXX

Simplify

1. $\dfrac{1}{a-\dfrac{b}{c}}$.

2. $1+\dfrac{1}{1+\dfrac{1}{a}}$.

3. $a+\dfrac{1}{a+\dfrac{1}{a}}$.

4. $\dfrac{a}{a+\dfrac{1}{a-\dfrac{1}{a}}}$.

5. $1-\dfrac{a-b}{a-b+\dfrac{b^2}{a+b}}$.

6. $\dfrac{x+1}{x+1-\dfrac{1}{x+1+\dfrac{x+1}{x}}}$.

7. $1+\dfrac{2}{3-\dfrac{4}{5+\dfrac{6}{7-\dfrac{8}{x}}}}$.

8. $a-\dfrac{1}{a+2+\dfrac{1}{a+2+\dfrac{1}{a}}}$.

9. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{a}}}-\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{a}}}$.

10. $\dfrac{x}{1-\dfrac{x}{1+\dfrac{x}{1-x}}}-\dfrac{x}{1+\dfrac{x}{1-\dfrac{x}{1+x}}}$.

Harder Cases of Fractions

145. When there are several fractions to be added or subtracted, it is best not always to combine all of them at once, but begin by combining *two* only, if they happen to correspond.

EXAMPLE 1. *Simplify* $\dfrac{1}{a^2-3}+\dfrac{3}{a^2+1}-\dfrac{3}{a^2-1}-\dfrac{1}{a^2+3}$.

$$\text{The expression}=\left(\frac{1}{a^2-3}-\frac{1}{a^2+3}\right)+\left(\frac{3}{a^2+1}-\frac{3}{a^2-1}\right)$$

$$=\frac{(a^2+3)-(a^2-3)}{(a^2-3)(a^2+3)}+\frac{3(a^2-1)-3(a^2+1)}{(a^2+1)(a^2-1)}$$

$$=\frac{6}{a^4-9}+\frac{-6}{a^4-1}=\frac{6a^4-6-6a^4+54}{(a^4-9)(a^4-1)}=\frac{48}{(a^4-9)(a^4-1)}.$$

EXAMPLE 2. *Simplify* $\frac{1}{x+1}-\frac{1}{x-1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}$.

The expression

$$=\left(\frac{1}{x+1}-\frac{1}{x-1}\right)+\frac{2}{x^2+1}+\frac{4}{x^4+1}$$

$$=\frac{-2}{x^2-1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}=\left(\frac{-2}{x^2-1}+\frac{2}{x^2+1}\right)+\frac{4}{x^4+1}$$

$$=\frac{-2(x^2+1)+2(x^2-1)}{(x^2-1)(x^2+1)}+\frac{4}{x^4+1}=\frac{-4}{x^4-1}+\frac{4}{x^4+1}$$

$$=\frac{-4(x^4+1)+4(x^4-1)}{(x^4-1)(x^4+1)}=\frac{-8}{x^8-1}=-\frac{8}{x^8-1}=\frac{8}{-(x^8-1)}$$

$$=\frac{8}{1-x^8}.$$

EXAMPLE 3. *Simplify* $\dfrac{\dfrac{2x+3y}{2x-3y}-\dfrac{2x-3y}{2x+3y}}{\dfrac{2x+3y}{2x-3y}+\dfrac{2x-3y}{2x+3y}}\div\dfrac{\dfrac{2x}{3y}-\dfrac{3y}{2x}}{\dfrac{2x}{3y}+\dfrac{3y}{2x}}$.

The expression

$$=\frac{\dfrac{(2x+3y)^2-(2x-3y)^2}{(2x-3y)(2x+3y)}}{\dfrac{(2x+3y)^2+(2x-3y)^2}{(2x-3y)(2x+3y)}}\div\frac{\dfrac{4x^2-9y^2}{6xy}}{\dfrac{4x^2+9y^2}{6xy}}$$

$$=\frac{\dfrac{4x^2+12xy+9y^2-4x^2+12xy-9y^2}{4x^2-9y^2}}{\dfrac{4x^2+12xy+9y^2+4x^2-12xy+9y^2}{4x^2-9y^2}}\div\frac{4x^2-9y^2}{6xy}\times\frac{6xy}{4x^2+9y^2}$$

$$=\frac{\dfrac{24xy}{4x^2-9y^2}}{\dfrac{2(4x^2+9y^2)}{4x^2-9y^2}}\div\frac{4x^2-9y^2}{4x^2+9y^2}=\frac{24xy}{4x^2-9y^2}\times\frac{4x^2-9y^2}{2(4x^2+9y^2)}\times\frac{4x^2+9y^2}{4x^2-9y^2}$$

$$=\frac{12xy}{4x^2-9y^2}.$$

EXAMPLE 4. *Simplify* $\dfrac{a^3-\dfrac{1}{a^3}}{\left(a-\dfrac{1}{a}\right)\left(a-1+\dfrac{1}{a}\right)}\times\dfrac{1-\dfrac{1}{a}}{1+\dfrac{1}{a}+\dfrac{1}{a^2}}$.

The expression

$$=\frac{\dfrac{a^6-1}{a^3}}{\left(\dfrac{a^2-1}{a}\right)\left(\dfrac{a^2-a+1}{a}\right)}\times\frac{\dfrac{a-1}{a}}{\dfrac{a^2+a+1}{a^2}}$$

$$=\frac{\dfrac{(a+1)(a-1)(a^2+a+1)(a^2-a+1)}{a^3}}{\dfrac{(a+1)(a-1)(a^2-a+1)}{a^2}}\times\frac{a-1}{a}\times\frac{a^2}{a^2+a+1}$$

$$=\frac{(a+1)(a-1)(a^2+a+1)(a^2-a+1)a^2}{a^3(a+1)(a-1)(a^2-a+1)}\times\frac{(a-1)a}{a^2+a+1}$$

$$=a-1.$$

EXAMPLES LXXXI

Simplify

1. $\dfrac{1}{1+2x}+\dfrac{1}{1-2x}+\dfrac{2}{1-4x^2}$.

2. $\dfrac{1}{1+3x}+\dfrac{1}{3x-1}+\dfrac{12}{1-9x^2}$

3. $\dfrac{1-a^2}{1+a^2}+\dfrac{a^2+1}{a^2-1}+\dfrac{4a^2}{1-a^4}$.

4. $\dfrac{1}{1-a}-\dfrac{a}{1+a}-\dfrac{1}{1+a^2}+\dfrac{a^2}{a^2-1}$.

5. $\dfrac{1}{9a^2-2}-\dfrac{2}{9a^2-1}+\dfrac{2}{9a^2+1}-\dfrac{1}{9a^2+2}$.

6. $1+\dfrac{a}{b}-\dfrac{b}{a+b}-\dfrac{a^2}{ab-b^2}+\dfrac{2a^2}{a^2-b^2}$.

7. $\dfrac{1}{x-1}+\dfrac{x^2+1}{x^3+1}-\dfrac{1}{x+1}-\dfrac{x^2+1}{x^3-1}$.

8. $\dfrac{1}{1-x+x^2}-\dfrac{1}{1+x+x^2}+\dfrac{1}{1+x^2+x^4}$.

9. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}$.

10. $\frac{1}{x-y}+\frac{1}{x+y}-\frac{1}{x^2-y^2}+\frac{1}{x^4-y^4}.$

11. $\frac{1}{x-y}-\frac{x+y}{x^2-y^2}+\frac{x^2+xy+y^2}{x^3-y^3}-\frac{x^3+x^2y+xy^2+y^3}{x^4-y^4}.$

12. $\frac{a^2-b^2}{a^2+ab+b^2}\times\frac{a^3-b^3}{a^2+4ab+3b^2}\times\frac{a+3b}{a^2-2ab+b^2}.$

13. $\frac{x^2+x+1}{x^2+1}-\frac{x^2-x+1}{x^2-1}+\frac{x-1}{x+1}+\frac{x+1}{x-1}-\frac{2(x^3+1)}{x^4-1}.$

14. $\left(\frac{x^3+x^2y+xy^2+y^3}{x-y}-\frac{x^3-x^2y+xy^2-y^3}{x+y}\right)\times\left(\frac{x^2}{x^2y^2+y^4}-\frac{y^2}{x^4+x^2y^2}\right).$

15. $\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}-\frac{x}{x^2-1}+\frac{3}{x(x^2-1)}.$

16. $\dfrac{\frac{a+b}{a-b}+\frac{b+a}{b-a}}{\frac{a+b}{a-b}-\frac{b+a}{b-a}}.$

17. $\dfrac{\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}}{\frac{x+y}{x-y}-\frac{x^3+y^3}{x^3-y^3}}\div\left(\frac{x^2+xy+y^2}{x+y}\right)^3.$

18. $\left\{x+1-\dfrac{1}{x+1-\frac{x}{x+1}}\right\}\times\frac{x^2+x+1}{x+1}$

19. $\dfrac{\frac{x^2}{y}+y-x}{\frac{1}{y}-\frac{1}{x}}\div\frac{x^3+y^3}{x^2-y^2}.$

20. $\left(\frac{ax^2-ay^2+2bxy}{x^2+y^2}\right)^2+\left(\frac{by^2-bx^2+2axy}{x^2+y^2}\right)^2.$

21. $\frac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}\div\frac{x^4-y^4}{2xy(x-y)}.$

22. $\left(\frac{x^2+y^2}{2xy}-1\right)\div\frac{x^3+y^3}{xy^2}\times\frac{x^2-xy+y^2}{4xy(x+y)}.$

23. $\dfrac{x-2+\frac{1}{x}}{x+2+\frac{1}{x}}\div\left(\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}\right)^2.$

24. $\dfrac{\left\{1+\frac{a}{a+b}+\frac{a^2}{(a+b)^2}\right\}\left\{1-\frac{a^2}{(a+b)^2}\right\}}{\left\{1-\frac{a^3}{(a+b)^3}\right\}\left\{1+\frac{a}{a+b}\right\}}.$

25. $$\frac{1+\dfrac{a^2}{(b+c)^2}+\dfrac{a^4}{(b+c)^4}}{\left\{1-\dfrac{a^3}{(b+c)^3}\right\}\left\{1+\dfrac{a^3}{(b+c)^3}\right\}}\div\frac{1}{1-\dfrac{a^2}{(b+c)^2}}.$$

26. $$\frac{\dfrac{x+y}{1+xy}-\dfrac{x+z}{1+xz}}{1-\dfrac{(x+y)(x+z)}{(1+xy)(1+xz)}}$$

27. $$\left(\frac{x^3-y^3}{x^3+y^3}-\frac{x-y}{x+y}\right)\left(\frac{x^3+y^3}{x^3-y^3}+\frac{x+y}{x-y}\right).$$

28. $$\left\{\left(\frac{x+y}{x-y}\right)^3-\left(\frac{x-y}{x+y}\right)^3\right\}\div\left\{\left(\frac{x+y}{x-y}\right)^2+1+\left(\frac{x-y}{x+y}\right)^2\right\}.$$

29. $$\left(\frac{x}{1-\dfrac{1}{x}}-x-\frac{1}{1-x}\right)\div\left(\frac{x}{1+\dfrac{1}{x}}+x-\frac{1}{1-x}\right).$$

30. $$\frac{x^2\left(\dfrac{1}{y}-\dfrac{1}{z}\right)+y^2\left(\dfrac{1}{z}-\dfrac{1}{x}\right)+z^2\left(\dfrac{1}{x}-\dfrac{1}{y}\right)}{x\left(\dfrac{1}{y}-\dfrac{1}{z}\right)+y\left(\dfrac{1}{z}-\dfrac{1}{x}\right)+z\left(\dfrac{1}{x}-\dfrac{1}{y}\right)}.$$

31. $$\frac{(2a-3b)^2-a^2}{4a^2-(3b+a)^2}+\frac{4a^2-(3b-a)^2}{9(a^2-b^2)}+\frac{9b^2-a^2}{(2a+3b)^2-a^2}.$$

32. $$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}.$$

33. $$\frac{x^2-(y-z)^2}{(z+x)^2-y^2}+\frac{y^2-(z-x)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$$

34. $$\frac{(x-y)^3+(y-z)^3}{z-x}+\frac{(y-z)^3+(z-x)^3}{x-y}+\frac{(z-x)^3+(x-y)^3}{y-z}.$$

35. $$\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2-\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right).$$

CHAPTER XXIV

IDENTITIES

146. As we have seen in Article 55, an **identity** is an algebraic statement, which asserts that the two expressions on either side of the sign of equality are always equal for all values of the letters involved therein.

In proving an *identity* we commence with one of the sides and show by successive transformations that it can be made to assume the form of the other.

EXAMPLE 1. *If* $x+\frac{1}{x}=a$, *prove that* $x^3+\frac{1}{x^3}=a^3-3a$.

First Method.

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3x.\frac{1}{x}.\left(x+\frac{1}{x}\right) \qquad \text{[See Article 115]}$$

$$=\left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right)$$

$$=a^3-3a.$$

Second Method.

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-x.\frac{1}{x}+\frac{1}{x^2}\right) \qquad \text{[See Article 111]}$$

$$=\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}-1\right)$$

$$=\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}+2-3\right)$$

$$=\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}+2.x.\frac{1}{x}-3\right)$$

$$=\left(x+\frac{1}{x}\right)\left\{\left(x+\frac{1}{x}\right)^2-3\right\}$$

$$=a(a^2-3)$$

$$=a^3-3a.$$

EXAMPLE 2. *If* $a+b+c=0$, *show that*

$$\frac{a+b}{(a+c)(b+c)}+\frac{b+c}{(a+b)(a+c)}+\frac{c+a}{(a+b)(b+c)}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Since $a+b+c=0$,

$$a+b=-c,\ b+c=-a,\ c+a=-b.$$

Hence the left side

$$=\frac{-c}{(-b)(-a)}+\frac{-a}{(-c)(-b)}+\frac{-b}{(-c)(-a)}=\frac{-c}{ba}+\frac{-a}{cb}+\frac{-b}{ca}$$

$$=-\left(\frac{c}{ba}+\frac{a}{cb}+\frac{b}{ca}\right)=-\frac{c^2+a^2+b^2}{abc}=-\frac{(a+b+c)^2-2(ab+bc+ca)}{abc}$$

$$=-\frac{-2(ab+bc+ca)}{abc}=2\left(\frac{ab+bc+ca}{abc}\right)=2\left(\frac{ab}{abc}+\frac{bc}{abc}+\frac{ca}{abc}\right)$$

$$=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

EXAMPLE 3. *If* $2s=a+b+c$, *prove that*

$$\frac{s-a}{(s-b)(s-c)}+\frac{s-b}{(s-c)(s-a)}+\frac{s-c}{(s-a)(s-b)}+\frac{s^2}{(s-a)(s-b)(s-c)}$$

$$=\frac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}.$$

The left side

$$=\frac{(s-a)^2+(s-b)^2+(s-c)^2+s^2}{(s-a)(s-b)(s-c)}.$$

The numerator $=s^2-2as+a^2+s^2-2bs+b^2+s^2-2cs+c^2+s^2$

$$=4s^2-2s(a+b+c)+a^2+b^2+c^2$$

$$=4s^2-2s\times 2s+a^2+b^2+c^2$$

$$=a^2+b^2+c^2.$$

Hence, the left side $=\dfrac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}$.

EXAMPLE 4. *Show that*

$$(x+y+z)^3-(x+y-z)^3-(x-y+z)^3-(y+z-x)^3=24xyz.$$

If we put $x=0$ or $y=0$ or $z=0$ in the left hand expression, it vanishes, therefore by Article 126, $(x-0)$, $(y-0)$, $(z-0)$ *i.e.*, x, y, z are the factors of the expression.

Since the expression is of the third degree in x, y, z, it can have no other factors, except some numerical coefficient. Suppose the numerical coefficient is k.

$$\therefore \quad (x+y+z)^3-(x+y-z)^3-(x-y+z)^3-(y+z-x)^3=kxyz.$$

Now to find k, we put $x=y=z=1$ in the above, we have

$$27-1-1-1=k,$$

$$\therefore \quad k=24.$$

Hence, the left side is equal to $24xyz$.

EXAMPLES LXXXII

Show that, if

1. $x+\frac{1}{x}=3$, $\quad x^2+\frac{1}{x^2}=7$.

2. $x-\frac{1}{x}=a$, $\quad x^3-\frac{1}{x^3}=a^3+3a$.

3. $\left(x+\frac{1}{x}\right)^2=3$, $\quad x^3+\frac{1}{x^3}=0$.

If $2s=a+b+c$, prove that

4. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$ $=abc$.

5. $16s(s-a)(s-b)(s-c)=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

6. $(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)+s^2=bc+ca+ab$.

7. $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}$.

8. $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$.

9. $\frac{s(s-a)}{(s-b)(s-c)}=\frac{2bc+(b^2+c^2-a^2)}{2bc-(b^2+c^2-a^2)}$.

10. $\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}=\frac{s}{(s-a)(s-b)(s-c)}$.

11. $\frac{b+c}{(s-b)(s-c)}+\frac{c+a}{(s-c)(s-a)}+\frac{a+b}{(s-a)(s-b)}=\frac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}$.

If $a+b+c=0$, prove that

12. $a^3+b^3+c^3=3abc$.

13. $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2$.

14. $a^2-bc=b^2-ca=c^2-ab$.

15. $(-a+b+c)(a-b+c)(a+b-c)=-8abc$.

16. $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc=0.$

17. $\dfrac{a}{a^2-bc}+\dfrac{b}{b^2-ca}+\dfrac{c}{c^2-ab}=0.$

18. $\dfrac{1}{a^2+b^2-c^2}+\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{c^2+a^2-b^2}=0.$

19. $\dfrac{a^2+b^2+c^2}{a^3+b^3+c^3}+\dfrac{2(ab+bc+ca)}{3abc}=0.$

20. $a\left(\dfrac{b^3-c^3}{b-c}\right)+b\left(\dfrac{c^3-a^3}{c-a}\right)+c\left(\dfrac{a^3-b^3}{a-b}\right)=0.$

Prove that, if

21. $2s=a+b+c,$

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

22. $x^2+y^2+z^2=xy+yz+zx,\ (y-z)^2+(z-x)^2+(x-y)^2=0.$

23. $x=a(b-c),\ y=b(c-a),\ z=c(a-b)\ ;\ \left(\dfrac{x}{a}\right)^3+\left(\dfrac{y}{b}\right)^3+\left(\dfrac{z}{c}\right)^3=\dfrac{3xyz}{abc}.$

24. $x=b+c-a,\ y=c+a-b,\ z=a+b-c\ ;\ \dfrac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=4.$

25. $x=b+c,\ y=c+a,\ z=a+b\ ;\ \dfrac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=2.$

Prove that

26. $\dfrac{2}{a-b}+\dfrac{2}{b-c}+\dfrac{2}{c-a}=\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}.$

27. $\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}=\left(\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}\right)^2.$

28. $\dfrac{x-y}{z}+\dfrac{y-z}{x}+\dfrac{z-x}{y}+\dfrac{x-y}{z}\cdot\dfrac{y-z}{x}\cdot\dfrac{z-x}{y}=0.$

29. $(a-b)^2+(b-c)^2+(c-a)^2$

$$=2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b).$$

30. $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$

$$=3(a+b-2c)(b+c-2a)(c+a-2b).$$

CHAPTER XXV

SIMPLE EQUATIONS INVOLVING FRACTIONS

147. In solving an equation involving fractions we first try to clear the fractions. This is done by multiplying both sides of the equation by the L.C.M. of the denominators.

EXAMPLE 1. *Solve* $\frac{3x}{x+6}-\frac{x}{x+5}=2.$

The L.C.M. of the denominators is $(x+6)(x+5)$. Multiplying both the sides of the equation by this L.C.M. we get rid of the fractions, and have

$$3x(x+5)-x(x+6)=2(x+6)(x+5),$$

$$\therefore\quad 3x^2+15x-x^2-6x=2x^2+22x+60,$$

$$\therefore\quad 3x^2-x^2-2x^2+15x-6x-22x=60,$$

$$\therefore\quad -13x=60,$$

$$x=-\frac{60}{13}=-4\frac{8}{13}.$$

EXAMPLE 2. *Solve* $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}.$

The equation can be written as

$$\frac{1}{x-1}+\frac{2}{x-2}=\frac{2}{x-3}+\frac{1}{x-3}.$$

By transposition, we have

$$\frac{1}{x-1}-\frac{1}{x-3}=\frac{2}{x-3}-\frac{2}{x-2}.$$

Simplifying each side of the equation separately,

$$\frac{(x-3)-(x-1)}{(x-1)(x-3)}=\frac{2(x-2)-2(x-3)}{(x-3)(x-2)},$$

$$\frac{-2}{(x-1)(x-3)}=\frac{2}{(x-3)(x-2)}.$$

Multiplying both the sides by $\frac{(x-3)}{2}$,

$$\frac{-1}{(x-1)}=\frac{1}{x-2},$$
$$-(x-2)=x-1,$$
$$-x+2=x-1,$$
$$-x-x=-1-2,$$
$$\therefore\quad -2x=-3,$$
$$\therefore\quad x=\frac{3}{2}=1\frac{1}{2}.$$

EXAMPLE 3. *Solve* $\frac{1}{x-3}+\frac{1}{x-5}=\frac{1}{x-2}+\frac{1}{x-6}$.

Simplifying each side separately,

$$\frac{(x-5)+(x-3)}{(x-3)(x-5)}=\frac{(x-6)+(x-2)}{(x-2)(x-6)},$$
$$\therefore\quad \frac{2x-8}{(x-5)(x-3)}=\frac{2x-8}{(x-2)(x-6)}.$$

Dividing both the sides by 2,*

$$\frac{x-4}{(x-3)(x-5)}=\frac{x-4}{(x-2)(x-6)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Multiplying both the sides by the L.C.M. of the denominators,

$$\frac{(x-4)(x-3)(x-5)(x-2)(x-6)}{(x-3)(x-5)}=\frac{(x-4)(x-3)(x-5)(x-2)(x-6)}{(x-2)(x-6)},$$
$$\therefore\quad (x-4)(x-2)(x-6)=(x-4)(x-3)(x-5) \quad \ldots\ldots\ldots\ldots(2)$$

Removing brackets, we have

$$x^3-12x^2+44x-48=x^3-12x^2+47x-60,$$
$$\therefore\quad 44x-47x=-60+48,$$
$$\therefore\quad -3x=-12,$$
$$\therefore\quad x=4.$$

Comparing (1) and (2) we see that multiplying both sides of the equation by the L.C.M. of the denominators is the same thing as *multiplying the numerator on the left side by the denominator on the right side, and the numerator on the right side by the denominator on the left side.* This process is called **cross-multiplying**.

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* NOTE. Students should note that they should not divide both sides of an equation by any term or expression involving the unknown quantity. For instance, here we have divided both the sides of the equation by 2, which is a constant factor and not by $2x-8$, which involves the unknown quantity x.

EXAMPLE 4. *Solve* $\frac{x-1}{x-3}-\frac{x-3}{x-5}=\frac{x-5}{x-7}-\frac{x-7}{x-9}$.

Since by division $\frac{x-1}{x-3}=1+\frac{2}{x-3}$, $\frac{x-3}{x-5}=1+\frac{2}{x-5}$, $\frac{x-5}{x-7}=1+\frac{2}{x-7}$, and $\frac{x-7}{x-9}=1+\frac{2}{x-9}$, therefore the equation can be written as

$$\left(1+\frac{2}{x-3}\right)-\left(1+\frac{2}{x-5}\right)=\left(1+\frac{2}{x-7}\right)-\left(1+\frac{2}{x-9}\right),$$

$$1+\frac{2}{x-3}-1-\frac{2}{x-5}=1+\frac{2}{x-7}-1-\frac{2}{x-9},$$

$$\therefore \quad \frac{2}{x-3}-\frac{2}{x-5}=\frac{2}{x-7}-\frac{2}{x-9},$$

$$\therefore \quad \frac{1}{x-3}-\frac{1}{x-5}=\frac{1}{x-7}-\frac{1}{x-9},$$

$$\frac{(x-5)-(x-3)}{(x-3)(x-5)}=\frac{(x-9)-(x-7)}{(x-7)(x-9)},$$

$$\frac{-2}{(x-3)(x-5)}=\frac{-2}{(x-7)(x-9)},$$

$$\frac{1}{(x-3)(x-5)}=\frac{1}{(x-7)(x-9)}.$$

Cross-multiplying, we have

$$(x-7)(x-9)=(x-3)(x-5),$$

$$\therefore \quad x^2-16x+63=x^2-8x+15,$$

$$\therefore \quad -16x+8x=15-63,$$

$$\therefore \quad -8x=-48,$$

$$\therefore \quad x=6.$$

EXAMPLES LXXXIII

Solve

1. $\frac{6}{x-2}=\frac{5}{x-3}$.

2. $\frac{4}{x-3}=\frac{25}{3x+4}$.

3. $\frac{1}{5x-1}=\frac{4}{5x+1}$.

4. $\frac{3x-4}{5x-6}=\frac{16}{27}$.

5. $\frac{x}{2x+3}=\frac{4}{9}$.

6. $\frac{3x-1}{7x+23}=\frac{4}{3}$.

7. $\frac{x-1}{7x-14}=\frac{x-3}{7x-26}$.

8. $\frac{x+\frac{1}{2}}{x-\frac{1}{3}}=\frac{2x-\frac{1}{2}}{2x-2}$.

9. $\frac{x-3}{x-2\frac{2}{3}}=\frac{x-2\frac{1}{2}}{x-2\frac{1}{3}}$.

10. $\frac{1}{2(2x+3)}+\frac{1}{2(3x+2)}=\frac{2}{2x+3}$.

11. $\dfrac{1}{x+3}+\dfrac{3}{4x+1}=\dfrac{1}{4(x+3)}$.

12. $\dfrac{4}{x+1}-\dfrac{3}{x}=\dfrac{1}{x-2}$.

13. $\dfrac{1}{x+3}+\dfrac{1}{x}=\dfrac{2}{x+7}$.

14. $\dfrac{1}{x+4}-\dfrac{1}{x+5}=\dfrac{2}{x+5}-\dfrac{2}{x+6}$.

15. $\dfrac{3}{x-2}-\dfrac{3}{x+3}=\dfrac{5}{x+3}-\dfrac{5}{x-6}$.

16. $\dfrac{2}{x+2}+\dfrac{1}{x+3}=\dfrac{3}{x+1}$.

17. $\dfrac{1}{x+5}+\dfrac{5}{x+8}=\dfrac{6}{x+7}$.

18. $\dfrac{1}{x-1\frac{1}{3}}+\dfrac{1}{x-2}=\dfrac{2}{x+\frac{2}{3}}$.

19. $\dfrac{4}{x+2}-\dfrac{x}{x-2}=\dfrac{x^2}{4-x^2}$.

20. $\dfrac{3x+8}{3x+1}+\dfrac{5+3x}{1-3x}=\dfrac{5}{1-9x^2}$.

21. $\dfrac{6}{x-7}-\dfrac{1}{x-5}=\dfrac{5}{x-8}$.

22. $\dfrac{2x-1}{x-1}+\dfrac{4x-6}{x-2}=\dfrac{6x-15}{x-3}$.

23. $\dfrac{3x+4}{x+1}=\dfrac{4x+5}{4x+4}+\dfrac{6x+4}{3x+1}$.

24. $\dfrac{3x-2}{x-1}+\dfrac{2x-2}{x-2}=\dfrac{5x-12}{x-3}$.

25. $\dfrac{x+2}{x+1}+\dfrac{3x-1}{3x+1}=\dfrac{8x+9}{4x+4}$.

26. $\dfrac{9}{3x+2}+\dfrac{1}{x-3}=\dfrac{8(x+1)}{2x^2+7x+5}$.

27. $\dfrac{1}{2(x-2)}+\dfrac{3}{x-5}=\dfrac{7x+1}{x^2-7x+10}$.

28. $\dfrac{1}{x-2}-\dfrac{1}{x-4}=\dfrac{1}{x-6}-\dfrac{1}{x-8}$.

29. $\dfrac{1}{x-1}-\dfrac{1}{x}=\dfrac{1}{x+3}-\dfrac{1}{x+4}$.

30. $\dfrac{1}{x+2}+\dfrac{1}{x+3}=\dfrac{1}{x+1}+\dfrac{1}{x+4}$.

31. $\dfrac{1}{x+3}+\dfrac{1}{x+7}=\dfrac{1}{x+4}+\dfrac{1}{x+6}$.

32. $\dfrac{1}{x-3}-\dfrac{1}{x+3}=\dfrac{1}{x+2}-\dfrac{1}{x+8}$.

33. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}$.

34. $\dfrac{x}{x+1}+\dfrac{x+4}{x+5}=\dfrac{x+1}{x+2}+\dfrac{x+3}{x+4}$.

35. $\dfrac{x}{x-2}+\dfrac{9-x}{7-x}=\dfrac{x+1}{x-1}+\dfrac{8-x}{6-x}$.

36. $\dfrac{2x+3}{x+1}-\dfrac{2x+5}{x+2}=\dfrac{2x+7}{x+3}-\dfrac{2x+9}{x+4}$.

37. $\dfrac{2x-15}{x-8}-\dfrac{x-8}{x-9}=\dfrac{4x-47}{x-12}+\dfrac{x-4}{x-5}$.

38. $\dfrac{x+2}{x+1}+\dfrac{4x+17}{x+4}=\dfrac{2x+5}{x+2}+\dfrac{3x+10}{x+3}$.

39. $\dfrac{(x+1)(x+9)}{(x+2)(x+4)}=\dfrac{(x+6)(x+10)}{(x+5)(x+7)}$.

40. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}}=0$.

41. $\dfrac{2}{1-\dfrac{2}{1-\dfrac{2}{1-\dfrac{2}{x}}}}=0$.

CHAPTER XXVI

SIMULTANEOUS EQUATIONS WITH THREE UNKNOWN QUANTITIES

148. We have seen in Article 68, that if we are given one equation in *two unknown quantities*, we can find an indefinite number of solutions. Similarly, if we are given one equation in *three unknown quantities* or two equations in *three unknown quantities*, we can also find an indefinite number of solutions. As there should be *two independent equations* in two unknown quantities to determine the values of those unknown quantities, similarly there should be *three independent equations* in three unknown quantities to determine the values of three unknown quantities.

The method of solution of equations involving three unknowns is precisely the same as that applied in the solution of equations involving two unknowns. As we have seen in Article 69, in solving two simultaneous equations in two unknowns, we *eliminate* one of the unknowns and get one equation in one unknown. Similarly, in solving three simultaneous equations in three unknowns we first *eliminate* one of the unknowns from any pair of the given equations, and then again the same unknown from a different pair of the given equations and thus get two equations involving two unknowns. These two equations, can be solved as in Article 69. The value of the third unknown can be obtained by substituting the values of the two unknowns thus found in any of the original equations.

The above is the general method, but if the student examines the three equations carefully he will find that by a little intelligent variation in the above method, he can obtain the solution much more easily.

EXAMPLE 1. *Solve the equations*

$$x+y+z=7,$$
$$3x+3y-2z=1,$$
$$5x-4y+7z=25.$$

Suppose we wish to eliminate z from the first and the second and then from the first and the third equations, we multiply the first equation by 2,

$$2x+2y+2z=14.$$

Adding this to the second equation,

$$5x+5y=15,$$
$$\therefore \quad x+y=3. \quad \text{...(1)}$$

Again, multiplying the first equation by 7,

$$7x+7y+7z=49.$$

Subtracting the third equation from this,

$$2x+11y=24 \quad \text{...(2)}$$

Now to solve (1) and (2), we multiply (1) by 2,

$$2x+2y=6.$$

Subtracting this from (2),

$$9y=18,$$
$$\therefore \quad y=2.$$

Substituting this value of y in (1)

$$x+2=3,$$
$$\therefore \quad x=1.$$

Now substituting these values of x and y in the first equation,

$$1+2+z=7,$$
$$\therefore \quad z=4.$$

Hence the solution is $x=1$, $y=2$, $z=4$.

NOTE. The value of z can be obtained directly by subtracting (1) from the first equation.

EXAMPLE 2. *Solve* $\dfrac{5x+2y}{4z}=\dfrac{7x-z}{x+2y+2}=\dfrac{5z-4x}{6x}=1.$

The above can be written as

$$\frac{5x+2y}{4z}=1,$$

$$\frac{7x-z}{x+2y+2}=1,$$

$$\frac{5z-4x}{6x}=1.$$

Clearing the fractions by multiplying both the sides of the equations by their denominators,

$$5x+2y=4z,$$
$$7x-\ z=x+2y+2,$$
$$5z-4x=6x.$$

Writing the above as

$$5x+2y-4z=0 \quad \ldots\ldots(1)$$
$$6x-2y-\ z=2 \quad \ldots\ldots(2)$$
$$-10x \quad +5z=0 \quad \ldots\ldots(3)$$

Adding (1) and (2),

$$11x-5z=2.$$

Adding this to (3),

$$x=2.$$

Substituting this value of x in (3),

$$-20+5z=0,$$
$$\therefore\quad z=4.$$

Substituting these values of x and z in (1),

$$5\times2+2y-4\times4=0,$$
$$\therefore\quad 2y=6,$$
$$\therefore\quad y=3.$$

Hence the solution is $x=2$, $y=3$, $z=4$.

EXAMPLE 3. *Solve the equations*

$$\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=3,$$
$$\frac{2}{x}+\frac{3}{y}+\frac{1}{z}=3\frac{5}{6},$$
$$\frac{3}{x}+\frac{1}{y}+\frac{2}{z}=4\frac{1}{6}.$$

Multiplying the first equation by 2,

$$\frac{2}{x}+\frac{4}{y}+\frac{6}{z}=6.$$

Subtracting the second equation from this,

$$\frac{1}{y}+\frac{5}{z}=\frac{13}{6} \quad \ldots\ldots(1)$$

Again, multiplying the first equation by 3,

$$\frac{3}{x}+\frac{6}{y}+\frac{9}{z}=9.$$

Subtracting the third equation from this,

$$\frac{5}{y}+\frac{7}{z}=\frac{29}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Multiplying (1) by 5,

$$\frac{5}{y}+\frac{25}{z}=\frac{65}{6}.$$

Subtracting (2) from this,

$$\frac{18}{z}=\frac{36}{6},$$

$$18\times 6=36\times z,$$

$$\therefore \quad z=3.$$

Substituting this value of z in (1),

$$\frac{1}{y}+\frac{5}{3}=\frac{13}{6},$$

$$\therefore \quad \frac{1}{y}=\frac{13}{6}-\frac{5}{3}=\frac{3}{6}=\frac{1}{2},$$

$$\therefore \quad y=2.$$

Substituting these values of y and z in the first equation,

$$\frac{1}{x}+\frac{2}{2}+\frac{3}{3}=3,$$

$$\frac{1}{x}=1,$$

$$x=1.$$

Hence the solution is $x=1$, $y=2$, $z-3$.

EXAMPLE 4. *Solve the equations*

$$\frac{x+z}{zx}=\frac{7}{10}, \quad \frac{z+y}{yz}=\frac{8}{15}, \quad \frac{y+x}{xy}=\frac{5}{6}.$$

Equations can be written as

$$\frac{x}{zx}+\frac{z}{zx}=\frac{7}{10},$$

$$\frac{z}{yz}+\frac{y}{yz}=\frac{8}{15},$$

$$\frac{y}{xy}+\frac{x}{xy}=\frac{5}{6}.$$

Simplifying,

$$\frac{1}{z}+\frac{1}{x}=\frac{7}{10},$$

$$\frac{1}{y}+\frac{1}{z}=\frac{8}{15},$$

$$\frac{1}{x}+\frac{1}{y}=\frac{5}{6}.$$

Adding the three equations,

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{62}{30},$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{31}{30}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Subtracting the first equation from (1),

$$\frac{1}{y}=\frac{10}{30}=\frac{1}{3},$$

$$\therefore \quad y=3.$$

Subtracting the second equation from (1),

$$\frac{1}{x}=\frac{15}{30}=\frac{1}{2},$$

$$\therefore \quad x=2.$$

Subtracting the third equation from (1),

$$\frac{1}{z}=\frac{6}{30}=\frac{1}{5},$$

$$\therefore \quad z=5.$$

Hence the solution is $x=2$, $y=3$, $z=5$.

EXAMPLES LXXXIV

Solve

1. $x+y+z=7,$
$4x-3y+2z=6,$
$5x+3y-4z=-5.$

2. $x-y+z=2,$
$4x+6y+5z=31,$
$7x-11y+9z=12.$

3. $x+y+z=0,$
$2x+3y+4z=-2,$
$5x+8y-z=6.$

4. $x+2y+z=12,$
$2x-y+z=3,$
$4x-3y+z-1.$

5. $x+2y+3z=20,$
$2x+3y-7z=-15,$
$4x-5y+7z=21.$

6. $3x+2y+2z=8,$
$2x+4y+z=7,$
$7x-4y+4z=15.$

7. $x+2y-4z=7,$
$3x-5y+2z=1,$
$5x-7y-11z=42.$

8. $2x+3y+4z=3,$
$4x+9y+12z=8,$
$5x+7y-13z=1\frac{7}{12}.$

9. $2x+5y+10z=129,$
$5x+8y-z=40,$
$5x+2y+z=30.$

10. $x+2y=3,$
$y+3z=4,$
$z+4x=5.$

11. $x+3y=26,$
$2y+5z=59,$
$3z+7x=62.$

12. $3x+2y+z=72,$
$3y+2z=48+x,$
$3x+2z=60.$

13. $7x+9y+z=40,$
$3x=2y,$
$x=y+z.$

14. $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12,$
$\frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8,$
$\frac{x}{2}+\frac{z}{3}=10.$

15. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=23$
$\frac{x}{3}+\frac{y}{4}+\frac{z}{2}=28,$
$\frac{x}{4}+\frac{y}{2}+\frac{z}{3}=27.$

16. $x+\frac{2y}{3}+\frac{z}{2}=2,$
$\frac{4x}{3}+y-2z=-32,$
$\frac{x}{4}-\frac{y}{2}+3z=42\frac{1}{4}.$

17. $\frac{2x+3y}{z-1}=4,$
$\frac{x+y}{y+z}=\frac{3}{5},$
$\frac{z+4x}{3x-5y}=-1.$

18. $\frac{x-2y}{3(y+z)}=-\frac{1}{3},$
$\frac{4y-z}{5(z+x)}=\frac{1}{5},$
$\frac{z+2x}{7(x+y)}=\frac{1}{7}.$

19. $\frac{1}{x}+\frac{1}{y}=\frac{1}{y}+\frac{1}{z}=\frac{1}{z}+\frac{1}{x}=\frac{2}{3}.$

20. $\frac{1}{x}+\frac{1}{y}=5,$
$\frac{1}{y}+\frac{1}{z}=7,$
$\frac{1}{z}+\frac{1}{x}=6.$

21. $\frac{1}{x}+\frac{2}{y}=10,$
$\frac{4}{y}+\frac{3}{z}=18,$
$\frac{5}{z}+\frac{3}{x}=22.$

22. $\frac{2}{x}+\frac{1}{y}-\frac{3}{z}=0,$
$\frac{2}{y}-\frac{3}{z}=-2,$
$\frac{1}{x}+\frac{1}{z}=\frac{4}{3}.$

23. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9,$
$\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=29,$
$\frac{3}{x}-\frac{2}{y}+\frac{1}{z}=4.$

24. $\frac{x+y}{xy}=1,\quad \frac{y+z}{yz}=\frac{1}{3},\quad \frac{z+x}{zx}=\frac{1}{2}.$

25. $\frac{x+y}{xy}=2,\quad \frac{y+z}{yz}=\frac{1}{2},\quad \frac{z+x}{zx}=\frac{1}{2}.$

CHAPTER XXVII

EQUATIONS WITH LITERAL COEFFICIENTS

149. So far we have been dealing with equations having *numerical* quantities as coefficients. In this chapter we shall take up some equations whose coefficients will be denoted by letters or symbols whose values are supposed to be *known*. Such coefficients are called **literal coefficients.**

The method of solving such equations is the same as that of solving equations with numerical coefficients.

Simple Equations

150. In solving a simple equation with literal coefficients, we first *express the letter which represents the unknown quantity in terms of the letters which represent the coefficients (denoting known quantities), and then proceed as if the coefficients were numerical quantities.* This can be done by collecting together the terms involving the unknown quantity, say x, on one side and the other terms on the other side, and then dividing the other side by the collected coefficients of x.

EXAMPLE 1. *Solve* $\frac{x}{a}-\frac{x}{b}=1.$

Clearing the fractions by multiplying each term by the L.C.M. of the denominators, we have

$$bx-ax=ab,$$
$$x(b-a)=ab,$$
$$x=\frac{ab}{b-a}.$$

EXAMPLE 2. *Solve* $x-\frac{p}{p-q}=\frac{qx}{p+q}.$

Multiplying both sides by p^2-q^2,

$$x(p^2-q^2)-p(p+q)=qx(p-q).$$

Removing brackets,

$$p^2x-q^2x-p^2-pq=pqx-q^2x.$$

Collecting terms involving x to the left side,

$$p^2x-q^2x-pqx+q^2x=p^2+pq.$$

Bracketing the coefficients of x,

$$(p^2-pq)x=p^2+pq,$$

$$\therefore \quad p(p-q)x=p(p+q).$$

Dividing both sides by the coefficients of x,

$$x=\frac{p(p+q)}{p(p-q)}=\frac{p+q}{p-q}.$$

EXAMPLES LXXXV

Solve

1. $\frac{x}{a}+\frac{x}{b}=1.$ **2.** $\frac{x}{l}-\frac{x}{m}=1.$ **3.** $ax+b=bx+a.$

4. $ax-b=cx-d.$ **5.** $ax+b^2=a^2-bx.$ **6.** $ax+b^2=bx+a^2.$

7. $\frac{x}{a}-\frac{x}{b}=\frac{b}{a}-\frac{a}{b}.$ **8.** $\frac{x-a}{a}+\frac{x-4a}{2a}=\frac{x-6a}{3a}.$

9. $\frac{x}{ab}+\frac{x}{bc}+\frac{x}{ca}=a+b+c.$ **10.** $(x-a)(x-b)=(x-c)(x-d).$

11. $(a+b-x)(a-b+x)+(a+x)(b+x)=a^2.$

12. $\frac{a}{x-b}=\frac{b}{x-a}.$ **13.** $\frac{b}{x}=\frac{a}{x-b+a}.$

14. $\frac{a}{bx-c}=\frac{b}{ax-c}.$ **15.** $\frac{x+b}{a-b}=\frac{x-b}{a+b}.$

16. $\frac{a+2b}{x-a}=\frac{a-2b}{x+a}.$ **17.** $\frac{x+a}{x+b}=\frac{x+3a}{x+a+b}.$

18. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x}.$ **19.** $\frac{x+a}{x-b}+\frac{x+b}{x-a}=2.$

20. $\frac{a}{x+b}+\frac{b}{x+a}=\frac{a+b}{x}.$ **21.** $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}.$

22. $\frac{a(x+a)}{x+b}+\frac{b(x+b)}{x+a}=a+b.$

23. $\frac{x-a}{3b+5c}+\frac{x-3b}{5c+a}+\frac{x-5c}{a+3b}=3.$

24. $\frac{x-a}{b+c+2a}+\frac{x-b}{c+a+2b}+\frac{x-c}{a+b+2c}=-3.$

Simultaneous Equations

151. We know that every simultaneous equation of the first degree in x and y can, by collecting together the terms involving x and y, be expressed in the form

$$ax + by = c,$$

where a represents the collected coefficients of x, b the collected coefficients of y, and c the terms independent of x or y.

Hence, in order to solve a pair of simultaneous equations of the first degree in x and y, we first reduce them to the above form and then proceed as if the coefficients were numerical quantities.

EXAMPLE 1. *Solve the equations*

$$ax + by = c,$$
$$dx + ey = f.$$

Here both the equations are given in the standard form, therefore multiplying the first by d and the second by a, we have

$$adx + bdy = cd \quad \text{.................................(1)}$$
$$adx + aey = af \quad \text{.................................(2)}$$

Subtracting (2) from (1),

$$(bd - ae)y = cd - af,$$
$$y = \frac{cd - af}{bd - ae}.$$

Similarly we can obtain the value of x by multiplying the first equation by e and the second by b and then subtracting one from the other. The value of x is $\dfrac{bf - ce}{bd - ae}$.

Hence the solution is $x = \dfrac{bf - ce}{bd - ae}$, $y = \dfrac{cd - af}{bd - ae}$.

EXAMPLE 2. *Solve the equations*

$$\frac{x}{a} + \frac{y}{a+b} = a,$$
$$\frac{x}{a-b} - \frac{y}{b} = -b.$$

Clearing the fractions by multiplying both sides by the L.C.M. of the denominators,

$$(a+b)x+ay=a^2(a+b) \quad \ldots\ldots(1)$$
$$bx-(a-b)y=-b^2(a-b). \quad \ldots\ldots(2)$$

Since both the equations are in the standard form, therefore multiplying (1) by $a-b$ and (2) by a,

$$(a+b)(a-b)x+a(a-b)y=a^2(a+b)(a-b) \quad \ldots\ldots(3)$$
$$abx-a(a-b)y=-ab^2(a-b) \quad \ldots\ldots(4)$$

Adding (3) and (4),

$$(a+b)(a-b)x+abx=a^2(a+b)(a-b)-ab^2(a-b),$$
$$\therefore \quad x(a^2-b^2+ab)=a^4-a^2b^2-a^2b^2+ab^3$$
$$=a^4-2a^2b^2+ab^3$$
$$=a(a^3-2ab^2+b^3)$$
$$=a(a-b)(a^2+ab-b^2),$$
$$x=a(a-b).$$

Substituting this value of x in the second equation,

$$\frac{a(a-b)}{a-b}-\frac{y}{b}=-b,$$
$$a-\frac{y}{b}=-b,$$
$$\frac{y}{b}=a+b,$$
$$y=b(a+b).$$

Hence the solution is $x=a(a-b)$, $y=b(a+b)$.

EXAMPLES LXXXVI

Solve

1. $ax+by=1,$ $lx+my=1.$

2. $ax-by=0,$ $x+y=c.$

3. $x+y=a+b,$ $ax+by=2ab.$

4. $ax+by=2ab,$ $-bx+ay=a^2-b^2.$

5. $lx+my=l(l+m),$ $mx+ly=m(l+m).$

6. $ax-by=a-b,$ $bx+ay=\dfrac{2ab}{a+b}.$

7. $px+qy=p^2-q^2,$ $qx+py=p^2-q^2.$

8. $x+y=p+q,$ $px-qy=q^2-p^2.$

9. $\dfrac{x}{a}+\dfrac{y}{b}=2,$ $ax-by=a^2-b^2.$

10. $\frac{x}{a}+\frac{y}{b}=1,$
$\frac{x}{b}+\frac{y}{a}=1.$

11. $\frac{a}{x}+\frac{b}{y}=1,$
$\frac{b}{x}+\frac{a}{y}=1.$

12. $\frac{m}{x}+\frac{n}{y}=a,$
$\frac{n}{x}+\frac{m}{y}=b.$

13. $(a+b)x+(a-b)y=2ab,$
$(a-b)x+(a+b)y=ab.$

14. $\frac{x}{a}+\frac{y}{b}=1,$
$\frac{x-b}{b}+\frac{y-a}{a}=1.$

15. $\frac{a}{x}+\frac{b}{y}=0,$
$ax+by=1.$

16. $\frac{x}{a}+\frac{y}{b-a}=5c,$
$\frac{x}{b}+\frac{y}{a-b}=7c.$

17. $(a+b)x+(a-b)y=a+b,$
$(a-b)x+(a+b)y=a-b.$

18. $x+y-3z=-a,$
$x-3y+z=-b,$
$3x-y-z=c.$

19. $ax+by=1,$
$by+cz=1,$
$cz+ax=1.$

20. $lx+my=n,$
$my+nz=l,$
$nz+lx=m.$

21. $\frac{x}{a}+\frac{y}{b}=1,$
$\frac{y}{b}+\frac{z}{c}=1,$
$\frac{z}{c}+\frac{x}{a}=1.$

22. $\frac{1}{x}+\frac{1}{y}=\frac{1}{y}+\frac{1}{z}=\frac{1}{z}+\frac{1}{x}=a.$

23. $\frac{1}{x}+\frac{1}{y}=a+b,$
$\frac{1}{y}+\frac{1}{z}=b+c,$
$\frac{1}{z}+\frac{1}{x}=c+a.$

24. $\frac{l}{x}+\frac{m}{y}=1,$
$\frac{m}{y}+\frac{n}{z}=1,$
$\frac{n}{z}+\frac{l}{x}=1.$

25. $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=a+2b+3c,$
$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=a^2+b^2+c^2,$
$\frac{2}{x}-\frac{1}{y}+\frac{3}{z}=2a-b+3c.$

26. $x+y+z=1,$
$ax+by+cz=0,$
$a^3x+b^3y+c^3z=abc.$

27. $x+y+z=a+b+c,$
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3,$
$ax+by+cz=a^2+b^2+c^2.$

Rule of Cross Multiplication

152. Sometimes the following method is very useful in solving certain simultaneous equations.

Take, for instance, the equations

$$a_1x+b_1y+c_1z=0,^*$$
$$a_2x+b_2y+c_2z=0,$$
$$a_3x+b_3y+c_3z=a.$$

Multiplying the first equation by c_2 and the second by c_1,

$$c_2a_1x+b_1c_2y+c_1c_2z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$c_1a_2x+b_2c_1y+c_1c_2z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Subtracting (1) from (2),

$$(c_1a_2-c_2a_1)x+(b_2c_1-b_1c_2)y=0,$$
$$(c_1a_2-c_2a_1)x=-(b_2c_1-b_1c_2)y,$$
$$=(b_1c_2-b_2c_1)y,$$

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{i})$$

Again, multiplying the first equation by a_2 and the second by a_1,

$$a_1a_2x+a_2b_1y+c_1a_2z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$
$$a_1a_2x+a_1b_2y+c_2a_1z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

---

* In these equations coefficients of the same letters are distinguished by a numerical *suffix*, such as a_1, a_2, a_3 ; b_1, b_2, b_3... (read as 'a one', 'a two', 'a three'...). There is, however, no connection between the values of a_1, a_2, a_3 ; or b_1, b_2, b_3 ;...which are quite different as other constants, but this notation only shows that the same letters, though with different suffixes, have the same property. Thus in the present example, a_1, a_2, a_3 are the coefficients of x ; b_1, b_2, b_3, the coefficients of y ; and c_1, c_2, c_3, the coefficients of z.

Sometimes instead of suffixes, we use *dashes*, such as a', a'', a''', b', b'', b''' ;...

Subtracting (3) from (4),

$$(a_1b_2 - a_2b_1)y + (c_2a_1 - c_1a_2)z = 0,$$

$$(a_1b_2 - a_2b_1)y = (c_1a_2 - c_2a_1)z,$$

$$\frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \quad \text{..................(ii)}$$

Hence, from (i) and (ii), we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \quad \text{...........(iii)}$$

Thus we see that when we are given two equations of the form

$$a_1x + b_1y + c_1z = 0$$

$$\text{and } a_2x + b_2y + c_2z = 0,$$

we can write down the ratios $x : y : z$ in terms of the coefficients. Now equating each of these ratios to any constant quantity, say k, we can express the values of x, y and z in terms of k, and then substituting these values of x, y and z in the third equation, we can find the value of k, and when the value of k is known, we can obtain the values of x, y and z.

The result (iii) is very important. It can be written down at once from the given equations thus.

Write down the coefficients of y, z and x from the first two equations, as given below

$$\begin{array}{cccc} \multicolumn{2}{c}{x} & & \\ & \multicolumn{2}{c}{y} & \\ & & \multicolumn{2}{c}{z} \\ b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

To find the denominator of x, multiply b_1 by c_2 and b_2 by c_1, and subtract the second product from the first. Similarly, to find the denominator of y, multiply c_1 by a_2 and c_2 by a_1, and subtract the second product from the first. Again, to find the denominator of z, multiply a_1 by b_2 and a_2 by b_1, and subtract the second product from the first. Thus we obtain the result (iii).

This is called the **Rule of Cross Multiplication**.

153. If we put $z=1$ in the first two equations of the previous article, we have

$$a_1x+b_1y+c_1=0$$
$$\text{and } a_2x+b_2y+c_2=0,$$

and result (iii) becomes

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1} \quad \text{..........(iv)}$$

which gives us the solution of the above equations.

The result (iv) may be regarded as a *formula* and can be used with advantage to find the solution of any pair of equations with *numerical* coefficients of the first degree in x and y.

NOTE. The beginner is advised not to use this formula in solving equations.

EXAMPLE 1. *Solve the equations*

$$5x-3y+2=0,$$
$$7x+4y-30=0.$$

Result (iv) gives

$$\frac{x}{(-3)(-30)-(+4)(+2)}=\frac{y}{(+2)(+7)-(-30)(+5)}=\frac{1}{(+5)(+4)-(+7)(-3)},$$

$$\frac{x}{90-8}=\frac{y}{14+150}=\frac{1}{20+21},$$

$$\therefore \quad \frac{x}{82}=\frac{y}{164}=\frac{1}{41},$$

Multiplying by 41,

$$\frac{x}{2}=\frac{y}{4}=1,$$

$$x=2,\ y=4.$$

EXAMPLE 2. *Solve the equations*

$$2x+y-2z=0,$$
$$9x-13y+5z=0,$$
$$4x+5y+6z=62.$$

From the first two equations, we have

$$\frac{x}{1\times5-(-2)(-13)}=\frac{y}{9(-2)-2\times5}=\frac{z}{2(-13)-9\times1},$$

$$\frac{x}{5-26}=\frac{y}{-18-10}=\frac{z}{-26-9},$$

$$\frac{x}{-21}=\frac{y}{-28}=\frac{z}{-35}.$$

Multiplying by —7,

$$\frac{x}{3}=\frac{y}{4}=\frac{z}{5}.$$

Let each of the ratios be equal to k, then

$$\frac{x}{3}=\frac{y}{4}=\frac{z}{5}=k,$$

$$\therefore\quad x=3k,\ y=4k,\ z=5k \quad\text{......(1)}$$

Substituting these values of x, y and z in the third equation,

$$4\times3k+5\times4k+6\times5k=62,$$

$$\therefore\quad 12k+20k+30k=62,$$

$$\therefore\quad 62k=62,$$

$$\therefore\quad k=1.$$

Substituting this value of k in (1), the solution is

$$x=3,\ y=4,\ z=5.$$

EXAMPLE 3. *Solve the equations*

$$x+y+z=0,$$

$$bcx+cay+abz=0,$$

$$ax+by+cz+(b-c)(c-a)(a-b)=0.$$

From the first two equations, we have

$$\frac{x}{ab-ca}=\frac{y}{bc-ab}=\frac{z}{ca-bc}.$$

Equating each of the ratios to k, we have

$$x=a(b-c)k,\ y=b(c-a)k,\ z=c(a-b)k \quad\text{......(1)}$$

Substituting these values of x, y and z in the third equation,

$$a^2(b-c)k+b^2(c-a)k+c^2(a-b)k+(b-c)(c-a)(a-b)=0,$$

$$\therefore\quad k\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}+(b-c)(c-a)(a-b)=0,$$

$$\therefore\quad -k(a-b)(b-c)(c-a)+(a-b)(b-c)(c-a)=0,$$

[See Example 1 of Article 126]

$$\therefore\quad k=1.$$

Hence the solution is $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

EXAMPLES LXXXVII

Solve

1. $3x+7y+13=0,$
$8x-21y-5=0.$

2. $5x-7y+24=0,$
$7x+5y-70=0.$

3. $4x+3y=0,$
$11x-13y+85=0.$

4. $bx+ay-2ab=0,$
$ax+by-(a^2+b^2)=0.$

5. $x+y-z=0,$
$9x-6y+z=0,$
$2x+3y+5z=23.$

6. $x-2y+z=0,$
$9x+3y-5z=0,$
$x+y+z+6=0.$

7. $2x-4y+5z=0,$
$3x+y-3z=0,$
$4x+5y+6z=155.$

8. $16x+7y+10z=0,$
$13x-20y+9z=0,$
$5x+7y+11z+28=0.$

9. $13x-2y-3z=0,$
$9x+3y-5z=0,$
$\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=3.$

10. $x+2y+2z=0,$
$8x+9y+10z=0,$
$\frac{1}{x}+\frac{3}{y}+\frac{5}{z}=\frac{2}{7}.$

11. $x+y+z=0,$
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0,$
$\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=1.$

12. $x+y+z=0,$
$(b+c)x+(c+a)y+(a+b)z=0,$
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{1}{abc}.$

13. $x+y+z=0,$
$ax+by+cz=0,$
$bcx+cay+abz+(a-b)(b-c)(c-a)=0.$

14. $ax+by+cz=0,$
$a^2x+b^2y+c^2z=0,$
$x+y+z=(a-b)(b-c)(c-a).$

15. $ax+by+cz=0,$
$a^2x+b^2y+c^2z=0,$
$a^3x+b^3y+c^3z=abc(a+b+c).$

16. $x+y+z=0,$
$a^2x+b^2y+c^2z=0,$
$\frac{x}{b^2-c^2}+\frac{y}{c^2-a^2}+\frac{z}{a^2-b^2}=3.$

CHAPTER XXVIII

HARDER PROBLEMS

154. The method of solving problems has already been explained in Chapter XIII. In this chapter we shall give examples of harder problems. Students should remember that from the conditions of the problem he has to find as many independent equations as there are unknown quantities.

EXAMPLE 1. *The denominator of a fraction exceeds its numerator by* 17. *If its numerator be increased by* 3, *or denominator be diminished by* 6, *the fractions thus formed are equal. Find the fraction.*

Let x be the numerator, then $x+17$ is the denominator. If the numerator be increased by 3, the new fraction is $\frac{x+3}{x+17}$. Similarly, if the denominator be diminished by 6, the fraction is $\frac{x}{x+11}$. Since these fractions are equal,

$$\frac{x+3}{x+17}=\frac{x}{x+11}.$$

By cross-multiplying,

$$(x+3)(x+11)=x(x+17),$$

$$\therefore\quad x^2+14x+33=x^2+17x,$$

$$\therefore\quad x^2-x^2+14x-17x=-33,$$

$$\therefore\quad -3x=-33,$$

$$\therefore\quad x=11.$$

Hence, the numerator is 11 and the denominator is 11+17 *i.e.*, 28; therefore the fraction is $\frac{11}{28}$.

EXAMPLE 2. *If the numerator of a fraction be increased by* 1 *and the denominator be diminished by* 1, *it becomes equal to* $\frac{1}{2}$; *and if the numerator be diminished by* 1 *and the denominator be increased by* 1, *it becomes equal to* $\frac{1}{5}$. *Find the fraction.*

Let x be the numerator, and y the denominator. Then the fraction is $\frac{x}{y}$.

Now from the first relation we obtain the equation,

$$\frac{x+1}{y-1}=\frac{1}{3},$$

and from the second relation we obtain the equation,

$$\frac{x-1}{y+1}=\frac{1}{5}.$$

By cross-multiplying, we have

$$3(x+1)=(y-1),$$

and $5(x-1)=(y+1)$.

Simplifying,

$$3x-y=-4 \quad \text{.....................}(1)$$

$$5x-y=6 \quad \text{.....................}(2)$$

Subtracting (1) from (2),

$$2x=10,$$

$$\therefore \quad x=5.$$

Substituting this value of x in one of the equations, we get $y=19$.

Hence the fraction is $\frac{5}{19}$.

EXAMPLE 3. *How many pounds of tea at* Re. 1. 8a. *per lb. be mixed with* 16 *lbs. of tea at* Re. 1. 3a. *per lb., so that the mixture may cost* Re. 1. 6a. *per lb.* ?

Let x denote the number of pounds of tea at Re. 1. 8a. per lb. which is mixed with 16 lbs. of tea at Re 1. 3a. per lb.

The price of 16 lbs. of tea at Re. 1. 3a. per lb. is Rs. $16\times\frac{19}{16}$ or Rs. 19.
And ,, ,, ,, x ,, ,, ,, ,, Re. 1. 8a. ,, ,, ,, Rs. $\frac{3}{2}x$.

$\therefore$,, ,, ,,$(16+x)$ lbs. of mixed tea is Rs. $(19+\frac{3}{2}x)$.

$\therefore$,, ,, ,, 1 ,, ,, ,, ,, ,, Rs. $\frac{19+\frac{3}{2}x}{16+x}$.

But this is equal to Re. 1. 6a. or Rs. $\frac{11}{8}$.

$$\frac{19+\frac{3}{2}x}{16+x}=\frac{11}{8},$$

$$8(19+\tfrac{3}{2}x)=11(16+x),$$

$$152+12x=176+11x,$$

$$12x-11x=176-152,$$

$$x=24.$$

Hence the required number of lbs. of tea to be mixed is 24.

EXAMPLE 4. *Two passengers going to the same place together have* 1 *maund and* 20 *seers of luggage with them and are charged* 10a. *and* Re. 1. 14a. *as freight charges for the excess above the weight allowed free. If the luggage had belonged to one person, he would have been charged* Rs. 5. *What amount of luggage is each passenger allowed free of charge and how much luggage had each ?*

Let x seers be the luggage which each passenger is allowed free of charge.

Then the two passengers can take $2x$ seers of luggage free of charge. Therefore 10a. + Re. 1. 14a. or Rs. $\frac{5}{2}$ is the charge on 1 md. 20 srs. *minus* $2x$ seers or $(60-2x)$ seers of luggage.

$$\text{the charge of 1 sr. of luggage is Rs. } \frac{\frac{5}{2}}{60-2x} \text{ or Rs. } \frac{5}{4(30-x)}.$$

Again, if the luggage had belonged to one person, only x seers of luggage would have been allowed free of charge, therefore Rs. 5 is the charge on 1 md. 20 srs. *minus* x seers or $(60-x)$ seers of luggage.

$$\text{the charge of 1 seer of luggage is Rs. } \frac{5}{60-x}.$$

$$\frac{5}{4(30-x)}=\frac{5}{60-x},$$
$$5(60-x)=20(30-x),$$
$$\therefore \quad 60-x=120-4x,$$
$$\therefore \quad 3x=60,$$
$$\therefore \quad x=20.$$

Hence each passenger is allowed 20 seers of luggage free of charge.

Now since the charges of 1 seer of luggage are Re. $\frac{5}{60-x}$ = Re. $\frac{5}{60-20}$ = Re. $\frac{5}{40}$ = Re. $\frac{1}{8}$ = 2a, and the first passenger is charged 10a. and the second Re 1. 14a. or 30a., therefore the weight of the first passenger's luggage which he carries over and above that allowed free is $\frac{10}{2}$ srs. or 5 srs., and that of the second passenger is $\frac{30}{2}$ srs. or 15 srs. Hence the two passengers have (20+5) srs. or 25 srs. and (20+15) srs. or 35 srs. of luggage with them.

EXAMPLE 5. *A sum of money is to be distributed equally among a certain number of men. If there was* 1 *man less, each would receive exactly* Rs. 400, *and if there were* 2 *more, each would get only* Rs. 280. *How much money is there for distribution and among how many men is it to be distributed ?*

Let x be the number of rupees to be distributed equally among y men.

Then each man will get Rs. $\frac{x}{y}$. If there was 1 man less, each would get Rs. $\frac{x}{y-1}$; and if there were 2 more, each would get Rs. $\frac{x}{y+2}$.

Hence from the first relation, we have

$$\frac{x}{y-1}=400,$$

and from the second relation, we have

$$\frac{x}{y+2}=280.$$

The equations can be written as

$$x=400(y-1) \quad \ldots\ldots(1)$$

and

$$x=280(y+2) \quad \ldots\ldots(2)$$

Subtracting (2) from (1),

$$0=400(y-1)-280(y+2),$$

$$\therefore \quad 0=10(y-1)-7(y+2),$$

$$-10y+7y=-10-14,$$

$$\therefore \quad -3y=-24,$$

$$y=8.$$

Hence the number of men is 8.

Substituting this value of y in (1),

$$x=400(8-1)=400\times 7=2800.$$

Hence the sum of money is Rs. 2800.

EXAMPLE 6. *A number consists of three digits whose sum is 9. The sum of the digits in the tens and hundreds places exceeds the digit in the units place by 1, and the digit in the tens place equals half the sum of the digits in the units and hundreds places. Find the number.*

This problem has three unknown quantities *i.e.*, the digit in the units place, the digit in the tens place and the digit in the hundreds place, therefore to find the value of these, we should have three independent equations.

Let x be the digit in the hundreds place, y the digit in the tens place and z the digit in the units place.

$$\therefore \quad x+y+z=9 \quad \ldots\ldots(1)$$

Since the sum of the digits in the hundreds and tens places *i.e.*, $x+y$ exceeds the digit in the units place *i.e.*, z by 1,

$$\therefore \quad x+y-z=1 \quad \ldots\ldots(2)$$

Again, since the digit in the tens place y equals half the sum of the digits in the hundreds and units places,

$$\therefore \quad y=\tfrac{1}{2}(x+z),$$
$$\therefore \quad 2y=x+z,$$
$$x-2y+z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now solving (1), (2) and (3) for x, y and z, we get $x=2$, $y=3$ and $z=4$.

Hence the number is 234.

EXAMPLES LXXXVIII

1. The denominator of a fraction exceeds its numerator by 3. If both the numerator and the denominator be increased by 1, or if the numerator be diminished by 1 and the denominator by 3, the fractions thus formed are equal. Find the fraction.

2. The denominator of a fraction exceeds its numerator by 8. If the numerator be diminished by 1 and the denominator be increased by 3, or if the numerator be diminished by 3 and the denominator by 1, the fractions thus formed are equal. Find the fraction.

3. A fraction becomes equal to $\frac{1}{2}$ if 2 be added to the numerator, and equal to $\frac{1}{4}$ if 2 be added to the denominator ; find the fraction.

4. If the numerator of a fraction be diminished by 5 and the denominator be increased by 4, it becomes equal to $\frac{4}{9}$; if the numerator be increased by 5 and the denominator be diminished by 1, it becomes equal to $\frac{4}{3}$. Find the fraction.

5. The denominator of a fraction exceeds twice its numerator by 1. If the numerator be increased by 2 and the denominator be diminished by 1, the fraction becomes equal to 1. Find the fraction.

6. The sum of the numerator and the denominator of a fraction is equal to 12. If 1 be subtracted from both the numerator and the denominator, the fraction thus formed becomes equal to $\frac{2}{3}$. Find the fraction.

7. If the numerator of a fraction be multiplied by 4 and the denominator increased by 2, the fraction thus formed becomes double the original fraction, and if the denominator be multiplied by 4 and the numerator increased by 1, the fraction becomes half the original fraction ; find the original fraction.

8. How many pounds of tea at Re. 1. 10a. per lb. must be mixed with 24 lbs. of tea at Re. 1. 4a. per lb., so that the mixture may cost Re. 1. 7a. 9p. per lb. ?

9. In what proportion must two kinds of grain at 12 srs. per rupee and 16 srs. per rupee be mixed, so that the mixture may cost $14\frac{2}{7}$ srs. per rupee ?

10. In what proportion must a shop-keeper mix two kinds of oil at 12a. a seer and Re 1. 2a. a seer, so that by selling the mixture at Re. 1. 4a. a seer he may gain 20% on his outlay ?

11. If a seers of water are added to b seers of milk,

(i) What fraction of mixture is water ?

(ii) What fraction of mixture is milk ?

(iii) How much water is there in x seers of mixture ?

(iv) How much milk is there in x seers of mixture ?

12. Two passengers going to the same place together have 1 maund and 9 seers of luggage with them and are charged 10a. and Re. 1. 12a. as freight charges for the excess above the weight allowed free. If the luggage had belonged to only one person, he would have been charged Rs. 4. 4a. What amount of luggage is each passenger allowed free of charge and how much luggage had each ?

13. Two passengers going to the same place together have 2 maunds and 12 seers of luggage with them and are charged Re. 1. 2a. and Re. 1. 14a. as freight charges for the excess above the weight allowed free. If the luggage had belonged to only one person, he would have been charged Rs. 5. 13a. What amount of luggage is each passenger allowed free of charge and how much luggage had each ?

14. A sum of money is to be distributed equally among a certain number of men. If there were 3 men more, each would receive exactly Rs. 100, and if there were 2 men less, each would get Rs. 200. How much money is there for distribution, and among how many men is it to be distributed ?

15. A sum of money is to be distributed equally among a certain number of men. If there were 4 men more, each would get Re. 1. less, and if there were 5 men less, each would get Rs. 2 more. How much money is there for distribution and among how many men is it to be distributed ?

16. A man bought some oranges. If each orange had cost 6 pies more, he would have got 6 oranges less for the same sum, and if each had cost 6 pies less, he would have got 12 more. How many oranges did he buy and what price did he pay for each ?

17. The price of 3 seers of wheat equals 4 seers of gram, and in spending Rs. 10, 30 seers more of gram is purchased than wheat ; find the price of each.

18. A shop-keeper sold 3 mds. of wheat and 4 mds. of barley for Rs. 13. 8a. to one man and 5 mds. of wheat and 3 mds. of barley for Rs. 17 to another at the same rate ; find the price of each.

19. A dealer sold 9 horses and 7 cows for Rs. 3600 to one man and 6 horses and 13 cows at the same rate and for the same price to another ; find the price of a horse and a cow.

20. A number consists of three digits whose sum is 8. The sum of the digits in the units and hundreds places exceeds the digit in the tens place by 2, and the digit in the hundreds place equals the sum of the digits in the units and the tens places. Find the number.

21. A number consists of three digits. The digit in the units place exceeds the sum of the digits in the tens and hundreds places by 1, three times the digit in the tens place exceeds the sum of the digits in the units and hundreds places by 3, and the digit in the hundreds place is one-third the digit in the units place. Find the number.

22. A number consits of three digits whose sum is 10. The middle digit equals the sum of the other two, and if the digits be reversed the number is increased by 99. Find the number.

23. The sum of the digits of a number lying between 100 and 1000 is 12, and the sum of the digits in the units and tens places is twice the digits in the hundreds place. If 99 be added to the number, the digits are reversed. Find the number.

24. A, B and C together have a certain sum of money. If A gets half as much more as B and C together have, A will have Rs. 1150 ; if B gets one-third as much more as A and C together have, B will have Rs. 1000 ; and if C gets one-fourth as much more as A and B together have, C will have Rs. 975. Find the amount of each.

25. Rama, Gopal and Krishna together have some mangoes. The sum of Rama's and Gopal's mangoes exceeds Krishna's mangoes by 20 ; the sum of Rama's and Krishna's mangoes exceeds twice Gopal's mangoes by 10 ; and five times the difference of Krishna's and Gopal's mangoes equals three times Rama's mangoes. How many mangoes have each of them.

26. The H.C.F. of two numbers in the ratio of 7 : 2 is 715. Find the numbers.

155. EXAMPLE 1. *If* 2 *men and* 7 *boys can do a piece of work in* 14 *days, and* 3 *men and* 8 *boys can do it in* 11 *days ; in how many days can a man and a boy separately do it ?*

Let x be the number of days in which one man can do the work and y the number of days in which one boy can do it.

Therefore in 1 day 1 man can do $\frac{1}{x}$ of the work, and 1 boy can do $\frac{1}{y}$ of the work, *i.e.*, in 1 day, 2 men can do $\frac{2}{x}$ and 7 boys $\frac{7}{y}$ of the work or they together can do $\left(\frac{2}{x}+\frac{7}{y}\right)$ of the work. But 2 men and 7 boys finish the work in 14 days, therefore they do $\frac{1}{14}$ of the work in 1 day.

$$\frac{2}{x}+\frac{7}{y}=\frac{1}{14} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Similarly 3 men and 8 boys can do $\left(\frac{3}{x}+\frac{8}{y}\right)$ of the work in one day.

But 3 men and 8 boys finish the work in 11 days, therefore they do $\frac{1}{11}$ of the work in 1 day.

$$\frac{3}{x}+\frac{8}{y}=\frac{1}{11} \quad \text{.......................................(2)}$$

Solving (1) and (2) for x and y, we get $x=77$ and $y=154$. Hence one man can finish the work in 77 days and one boy in 154 days.

EXAMPLE 2. *A and B together can do a piece of work in 24 days, B and C in 40 days, and C and A in 30 days ; in how many days can each do the work ?*

Let x, y and z represent the number of days in which A, B and C can do the work, then $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ represent their daily work.

Since A and B can do the work in 24 days, therefore they can do $\frac{1}{24}$ of the work in 1 day.

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{24}.$$

Similarly $$\frac{1}{y}+\frac{1}{z}=\frac{1}{40},$$

and $$\frac{1}{z}+\frac{1}{x}=\frac{1}{30}.$$

Adding the three equations,

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{10}.$$

Dividing by 2,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{20}. \quad \text{.......................................(1)}$$

Subtracting the second, the third and the first equations respectively from (1), we have

$$\frac{1}{x}=\frac{1}{40}, \ \frac{1}{y}=\frac{1}{60} \text{ and } \frac{1}{z}=\frac{1}{120}.$$

$$\therefore \quad x=40, \ y=60 \text{ and } z=120.$$

Hence A can do the work in 40 days, B in 60 days and C in 120 days.

EXAMPLE 3. *A cistern has two supply pipes P and Q, and one waste pipe R. If R be closed, P and Q fill the cistern separately in 10 and 12 hours respectively ; and if all the three pipes be opened together, the cistern will be filled in 15 hours. If the cistern be full and P and Q be closed, in how many hours will R empty it ?*

Let x represent the number of hours which R would take to empty the cistern, then in 1 hour it would empty $\frac{1}{x}$ of the cistern. But P and Q fill $\frac{1}{10}$ and $\frac{1}{12}$ of the cistern in 1 hour,

$$\therefore \qquad \frac{1}{10}+\frac{1}{12}-\frac{1}{x}=\frac{1}{15},$$

$$\frac{1}{x}=\frac{1}{10}+\frac{1}{12}-\frac{1}{15}=\frac{7}{60},$$

$$\therefore \quad x=\frac{60}{7}=8\frac{4}{7}.$$

Hence R can empty the cistern in $8\frac{4}{7}$ hours.

EXAMPLES LXXXIX

1. If 5 men and 12 boys can do a piece of work in 4 days, and 15 men and 16 boys can do it in 2 days; in how many days can a man and a boy separately do it?

2. If the work of 3 men equals that of 4 women, and 14 men and 7 women can do a piece of work in 10 days; in how many days can 1 man and 1 woman do it?

3. If 5 men or 7 women can do a piece of work in 37 days; in how many days can 7 men and 5 women together do it?

4. If 3 men or 5 women can reap a field in 17 days; in how many days will 7 men and 11 women reap four times that field?

5. Twelve men and fifteen women can do a piece of work in 20 days. If there had been 10 women more, the work would have been finished 5 days earlier; in how many days can 1 man and 1 woman separately do it?

6. If a man can do a piece of work in x days, a woman in twice that time and a boy in thrice that time;

(i) What fraction of the work can 1 man, 1 woman and 1 boy do in 1 day?

(ii) What fraction of the work can 1 man, 2 women and 3 boys do in 1 day?

7. A and B together can do a piece of work in 12 days, B and C in 15 days, and C and A in 20 days; in how many days can each do the work?

8. A and B can do a piece of work in 6 days. Both work at it for 2 days when A is called off and B then finishes it by himself in 12 days. In how many days can each do the work?

9. A man began a work and after working at it for 2 days he left it and then his son completed it in 9 days. If he had left the work after working for 3 days, his son would have finished it in 6 days ; in how many days could the man do it alone ?

10. Mohan takes twice as much time as Sohan and thrice as much as Rohan to do a piece of work ; if they work together they can do it in 2 days ; in what time can each do it ?

11. Jatindra and Nagendra can do a piece of work in 20 days, Nagendra and Satendra can do it in 30 days, and Satendra and Jatindra in 50 days. They work at it for 8 days when Jatindra is called off, and Nagendra and Satendra continue for 10 days more and then Nagendra is called off ; in how many days will Satendra complete the work ?

12. A can do a piece of work in a hours, and B in b hours ;

(i) What fraction of the work can they do in 1 hour when working together ?

(ii) What fraction of the work can they do in x hours when working together ?

(iii) In how many hours can they finish the work when working together ?

13. A can reap a field in a days, B in b days and C in c days ;

(i) What portion of the field can each reap in 1 day ?

(ii) What portion of the field can each reap in x days ?

(iii) What portion of the field can they reap in 1 day when working together ?

(iv) In how many days can they reap the whole field when working together ?

14. A cistern has two pipes P and Q ; P is a supply pipe and Q a waste pipe. If both the pipes be opened, the cistern is filled in 9 minutes, and if Q be opened 1 minute after P, the cistern is filled in 7 minutes ; in how many minutes can P fill the cistern and Q empty it ?

15. A cistern has three pipes P, Q and R ; P and Q can fill it in 4 hours and 6 hours respectively, and R can empty it in 3 hours. If all the three pipes be opened together, in how many hours will the cistern be filled ?

16. Three pipes A, B and C can fill a cistern in 4, $1\frac{2}{3}$ and $4\frac{1}{2}$ hours respectively. They are all turned on together and after $\frac{3}{4}$ hour B and C are turned off. In how many minutes more will A fill the cistern ?

17. Two pipes can fill a cistern in 5 hours and 6 hours respectively. Both the pipes are turned on together ; after how many hours should the first pipe be turned off so that the cistern may be filled in 3 hours ?

18. Two pipes P and Q can fill a cistern in 1 hour and $1\frac{1}{3}$ hours respectively, and a waste pipe R can empty it in 4 hours. If these pipes be opened in succession for 2 minutes each, in how many hours will the cistern be filled ?

19. Two pipes P and Q can fill a cistern in p and q hours respectively ;

(i) What portion of the cistern can P fill in 1 hour ?

(ii) ,, ,, ,, ,, ,, ,, both ,, ,, ,, ,, ?

(iii) ,, ,, ,, ,, ,, ,, ,, ,, ,, x hours ?

(iv) In how many hours can both the pipes together fill the cistern ?

20. A cistern has three pipes A, B and C. A and B can fill it in a and b minutes respectively and C can empty it in c minutes ;

(i) In how many minutes will the cistern be filled if all the three pipes be opened together ?

(ii) In how many minutes will the cistern be filled if A and B be opened together and after 10 minutes C be also opened ?

(iii) In how many minutes will the cistern be emptied if A and B be opened together and after 10 minutes A and B be closed and C be opened ?

156. EXAMPLE 1. *A train travelled a certain distance at a uniform rate. If the speed had been 5 miles an hour more, the journey would have occupied 6 hours less ; and if the speed had been 2 miles an hour less, the journey would have occupied 3 hours more. Find the length of the journey and the speed of the train.*

Let x miles be the distance and y miles an hour the speed of the train.

Since the train goes y miles in 1 hour, therefore it will go x miles in $\frac{x}{y}$ hours.

If the speed had been 5 miles an hour more, *i.e.*, $(y+5)$ miles an hour ; it would have taken $\frac{x}{y+5}$ hours in going x miles. Therefore from the first relation, we have

$$\frac{x}{y}-\frac{x}{y+5}=6.$$

Similarly, if the speed had been 2 miles an hour less, *i.e.*, $(y-2)$ miles an hour, it would have taken $\frac{x}{y-2}$ hours in going x miles. Therefore from the second relation, we have

$$\frac{x}{y-2}-\frac{x}{y}=3.$$

The first equation can be written as

$$\frac{x(y+5)-xy}{y(y+5)}=6,$$

$$\text{or}\quad \frac{5x}{y(y+5)}=6 \quad \dots\dots\dots\dots\dots\dots(1)$$

The second equation can be written as

$$\frac{xy-x(y-2)}{y(y-2)}=3,$$

$$\text{or}\quad \frac{2x}{y(y-2)}=3 \quad \text{.....................................(2)}$$

Dividing (1) by (2),

$$\frac{5x}{y(y+5)}\div\frac{2x}{y(y-2)}=6\div3,$$

$$\frac{5x}{y(y+5)}\times\frac{y(y-2)}{2x}=2,$$

$$\frac{5(y-2)}{2(y+5)}=2,$$

$$5y-10=4y+20,$$

$$\therefore\quad y=30.$$

Hence, the speed of the train is 30 miles an hour.

Substituting this value of y in (1),

$$\frac{5x}{30\times35}=6,$$

$$x=\frac{30\times35\times6}{5}=1260.$$

Hence the distance is 1260 miles.

EXAMPLE 2. *A man rows to a place 30 miles distant and back in $10\frac{1}{2}$ hours. He takes 9 hours in going 40 miles down the river and 20 miles up the river. Find the rate at which the river is flowing.*

Let x miles an hour be the rate of the boat and y miles an hour the rate of the river. Then $(x+y)$ miles an hour is the rate of rowing *down* the river and $(x-y)$ miles an hour *up* the river. Therefore from the first relation, we have

$$\frac{30}{x+y}+\frac{30}{x-y}=\frac{21}{2},$$

and from the second relation, we have

$$\frac{40}{x+y}+\frac{20}{x-y}=9.$$

Multiplying the first equation by 4 and the second by 3,

$$\frac{120}{x+y}+\frac{120}{x-y}=42 \quad \text{.....................................(1)}$$

$$\frac{120}{x+y}+\frac{60}{x-y}=27 \quad \text{.....................................(2)}$$

Subtracting (2) from (1),

$$\frac{60}{x-y}=15,$$

$$\therefore \quad x-y=4 \quad \text{.......................}(3)$$

Again, multiplying the first equation by 2,

$$\frac{60}{x+y}+\frac{60}{x-y}=21 \quad \text{.......................}(4)$$

Subtracting (4) from (2),

$$\frac{60}{x+y}=6,$$

$$\therefore \quad x+y=10 \quad \text{.......................}(5)$$

From (3) and (5), we have $y=3$.

Hence, the rate of the flow of the river is 3 miles an hour.

EXAMPLES XC

1. A train travelled a certain distance at a uniform rate. If the speed had been 4 miles an hour less, the journey would have occupied 5 hours more; and if the speed had been 1 mile an hour more, the journey would have occupied 1 hour less. Find the length of the journey and the rate of the train.

2. A train travelled a certain distance at a uniform rate. If the speed had been 6 miles an hour more, the journey would have occupied 4 hours less; and if the speed had been 6 miles an hour less, the journey would have occupied 6 hours more. Find the length of the journey.

3. A man travelled a certain distance at a uniform rate. If his speed had been 1 mile an hour more, the journey would have occupied 2 hours less, and if his speed had been $\frac{1}{2}$ mile an hour more, the journey would have occupied $\frac{4}{5}$ of the time. Find the distance and the rate of the man.

4. A man is given $7\frac{1}{2}$ hours, including $1\frac{1}{2}$ hours for rest, in going from one place to another and in returning. If he walks 1 mile an hour faster both ways, he gets $2\frac{1}{2}$ hours for rest; find the rate of the man and the length of the journey.

5. A man walks a distance of 35 miles, partly at 4 miles an hour and partly at 5 miles an hour. If he had walked the distance he walked at 4 miles an hour at 5 miles an hour, and the distance he walked at 5 miles an hour at 4 miles an hour, he would have walked 1 mile more at the same time. Find the time he took to walk the distance of 35 miles.

6. A train running from A to B meets with an accident 45 miles from A, after which it moves with $\frac{4}{5}$ths of its original speed and arrives at B 30 minutes late. If the accident had happened 45 minutes further on, it would have been only 12 minutes late. Find the original speed of the train and the distance from A to B.

7. A train after travelling for 4 hours is detained for 40 minutes, after which it moves with $\frac{6}{5}$ths of its original speed and arrives at its destination in time. If the train had been detained for only 20 minutes after travelling for 6 hours and then moved with $\frac{6}{5}$ths of its original speed, it would also have arrived in time. Find the length of the journey and the original speed of the train.

8. Two trains 110 yards and $73\frac{1}{3}$ yards long, are running on parallel rails. The times of their passing one another when running in the same direction and in opposite directions are 30 seconds and 15 seconds respectively. Find their speeds.

9. A boat goes 30 miles up the stream and 44 miles down the stream in 10 hours, and 40 miles up the stream and 55 miles down the stream in 13 hours. Find the rates of the stream and the boat.

10. A man rows 5 miles down the stream and 3 miles up the stream in the same time. If the speed of the stream had been $\frac{1}{2}$ mile an hour, the speed of rowing down the stream would have been double that up the stream. Find the rate at which the man rows in still water and the rate at which the stream flows.

11. A man rows in still water at the rate of a miles an hour and the river flows at the rate of b miles an hour ;

(i) What is the speed of the boat down the river ?

(ii) What is the speed of the boat up the river ?

(iii) In how many hours will the man go x miles down the river ?

(iv) In how many hours will the man go x miles up the river ?

12. The fore-wheel of a carriage makes 3 revolutions more than the hind-wheel in going 60 yards ; if the circumference of the fore-wheel be increased by one-fourth of its original size, and the circumference of the hind-wheel by one-fifth of its original size, the fore-wheel would make 2 revolutions more than the hind-wheel in going 60 yards. Find the circumference of each wheel.

13. A and B run a cycle race of 1040 yards. At the first heat A gives B a start of 120 yards but is beaten by 5 seconds. At the second heat A gives B a start of 5 seconds and beats him by 120 feet. How many seconds will each take to run the race ?

14. A man has to travel a certain distance by car. When he has travelled 60 miles, he increases his speed by 10 miles an hour. If he had travelled the whole journey with this increased speed, he would have arrived 30 minutes earlier ; but if he had continued at his original speed, he would have arrived 20 minutes later. Find the length of the distance.

CHAPTER XXIX

SQUARE ROOT

Square Root by Inspection

157. We have seen in Article 52 that the *square root* of an expression is that expression whose square is equal to the original expression.

Thus, the square root of $36a^2$ is $6a$ or $-6a$, for

$$(6a)^2 = 6a \times 6a = 36a^2,$$

$$\text{and} \quad (-6a)^2 = (-6a) \times (-6a) = 36a^2.$$

Hence we see that *the square root of a quantity can be positive as well as negative, while the square of a quantity is always positive.*

NOTE. In this chapter we will deal only with the positive root.

EXAMPLE. *Find the square root of* $100a^2b^2$.

$$\sqrt{100a^2b^2} = \sqrt{10^2a^2b^2} = 10ab.$$

158. We have seen in Articles 43 and 45 that

$$(a+b)^2 = a^2 + 2ab + b^2,$$

$$\text{and} \quad (a-b)^2 = a^2 - 2ab + b^2.$$

Therefore the square root of $a^2 + 2ab + b^2$ is $a + b$ and the square root of $a^2 - 2ab + b^2$ is $a - b$. Hence, if a trinomial can be expressed in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$, its square root can be written down at once by inspection.

EXAMPLE 1. *Find the square root of* a^2+4a+4.

Since $a^2+4a+4 = a^2+2.a.2+2^2 = (a+2)^2$.

Hence the square root is $a+2$.

EXAMPLE 2. *Find the square root of* $(a+b)^2 - 6(a+b)c + 9c^2$.

Since the expression $= (a+b)^2 - 2(a+b) \times (3c) + (3c)^2$

$= [(a+b) - 3c]^2$

$= (a+b-3c)^2$.

Hence the square root is $a+b-3c$.

EXAMPLE 3. *Find the square root of* $\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)$.

The expression $=\left(a^2+2+\frac{1}{a^2}\right)-4\left(a-\frac{1}{a}\right)$

$$=\left(a^2-2+\frac{1}{a^2}\right)+4-4\left(a-\frac{1}{a}\right)$$

$$=\left(a-\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)+4$$

$$=\left\{\left(a-\frac{1}{a}\right)-2\right\}^2$$

$$=\left(a-\frac{1}{a}-2\right)^2.$$

Hence the square root is $a-\frac{1}{a}-2$.

159. We have seen in Article 49 that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca.$$

Therefore the square root of $a^2+b^2+c^2+2ab+2bc+2ca$ is $a+b+c$. Hence, if an expression can be expressed in the form $a^2+b^2+c^2+2ab+2bc+2ca$, its square root can be written down at once by inspection.

EXAMPLE. *Find the square root of* $x^2+9y^2+z^2+6xy-6yz-2zx$.

The expression $=(x)^2+(3y)^2+(-z)^2+2(x)(3y)+2(3y)(-z)+2(-z)(x)$

$$=(x+3y-z)^2.$$

Hence the square root is $x+3y-z$.

EXAMPLES XCI

(Examples 1 to 12 may be taken orally)

Find the square root of

1. x^2y^4. **2.** $9x^6y^4$. **3.** $16x^2y^4z^6$.

4. $\frac{25x^4y^8z^{10}}{64a^2b^4c^6}$. **5.** $\frac{121l^4m^2n^8}{9a^2b^2c^2}$. **6.** $\frac{49a^{14}b^{18}c^{10}}{81x^{10}y^{12}z^{14}}$.

7. x^2+6x+9. **8.** $4x^2-4x+1$. **9.** $a^2+14ab+49b^2$.

10. $4a^2-12ax+9x^2$. **11.** $1-6a+9a^2$.

12. $\frac{a^2}{b^2}+22\frac{a}{b}+121$. **13.** $\frac{a^2}{16}-\frac{ab}{2}+b^2$.

14. $\dfrac{a^2}{b^2}+2+\dfrac{b^2}{a^2}$.

15. $\dfrac{4x^2}{9y^2}-2+\dfrac{9y^2}{4x^2}$.

16. $\dfrac{a^8}{b^8}-2+\dfrac{b^8}{a^8}$.

17. $(a+b)^2+2(a+b)+1$.

18. $(a+1)^2-4(a+1)x+4x^2$.

19. $100x^4-20x^2(y+z)+(y+z)^2$.

20. $\left(\dfrac{a+b}{a-b}\right)^2-2+\left(\dfrac{a-b}{a+b}\right)^2$.

21. $(a+3b)^2-(a+3b)+\dfrac{1}{4}$.

22. $\left(\dfrac{a}{b}-1\right)^2-4\left(\dfrac{a}{b}-1\right)+4$.

23. $(a+b)^2+2(a+b)(a-b)+(a-b)^2$.

24. $(x+2y)^2-6(x+2y)(x-3y)+9(x-3y)^2$.

25. $(a-b)^4+2(a^2-b^2)^2+(a+b)^4$.

26. $a^2+b^2+c^2-2ab-2bc+2ca$.

27. $a^2+b^2+2ab+2a+2b+1$.

28. $x^2+y^2+4z^2+2xy-4yz-4zx$.

29. $x^2+4y^2+9z^2+4xy-12yz-6zx$.

30. $x^2+4y^2+\frac{1}{4}z^2-4xy-2yz+zx$.

31. $\left(x+\dfrac{1}{x}\right)^2+8\left(x+\dfrac{1}{x}\right)+16$.

32. $x^2+\dfrac{1}{x^2}+10\left(x+\dfrac{1}{x}\right)+27$.

33. $\left(x-\dfrac{1}{x}\right)^2+4\left(x+\dfrac{1}{x}\right)+8$.

General Method

160. The method of finding the square root of compound expressions in Algebra is similar to that of finding the square root of those quantities in Arithmetic whose factors cannot be easily obtained. For instance, to find the square root of 441 in Arithmetic we proceed thus :

$$\begin{array}{rl} & 441)20+1=21 \\ & \underline{400} \\ 40+1=41) & 41 \\ & \underline{41} \end{array}$$

We first find the square root of 400 which is 20. We write 20 to the right and subtract its square 400 from 441 leaving a remainder 41. We write twice 20 *i.e.*, 40 to the left of 41 and divide 41 by 40. The quotient is 1. We add this 1 to 40, the sum is $40+1$ *i.e.*, 41. Now we divide 41 by 41, the quotient is 1 leaving no remainder. We add this quotient 1 to 20. The sum is $20+1$ *i.e.* 21, which is the *required square root.*

Now since $441=400+40+1=4\times10^2+4\times10+1$, the above work can be arranged thus :

$$\begin{array}{l} \quad 4\times10^2+4\times10+1(2\times10+1 \\ \quad 4\times10^2 \\ \hline 4\times10+1\ )4\times10+1 \\ \qquad\qquad\quad 4\times10+1 \\ \hline \end{array}$$

That is, we first find the square root of 4×10^2 which is 2×10. We write 2×10 to the right and subtract its square 4×10^2 from $4\times10^2+4\times10+1$ leaving a remainder $4\times10+1$. We write twice 2×10 *i.e.*, 4×10 to the left of $4\times10+1$ and divide $4\times10+1$ by 4×10. The quotient is 1. We add this 1 to 4×10, the sum is $4\times10+1$. Now we divide $4\times10+1$ by $4\times10+1$, the quotient is 1 leaving no remainder. We add this quotient 1 to 2×10. The sum is $2\times10+1$, which is the required square root.

Now if we replace 10 by x in the above, the work may be arranged thus :

$$\begin{array}{l} \quad 4x^2+4x+1(2x+1 \\ \quad 4x^2 \\ \hline 4x+1\ )4x+1 \\ \qquad\qquad 4x+1 \\ \hline \end{array}$$

That is, we first find the square root of $4x^2$ which is $2x$. We write $2x$ to the right and subtract its square $4x^2$ from $4x^2+4x+1$ leaving a remainder $4x+1$. We write twice $2x$ *i.e.*, $4x$ to the left of $4x+1$ and divide $4x+1$ by $4x$. The quotient is 1. We add this 1 to $4x$, the sum is $4x+1$. Now we divide $4x+1$ by $4x+1$, the quotient is 1 leaving no remainder. We add this quotient 1 to $2x$. The sum is $2x+1$, which is the required square root.

From the above we see that in order to find the square root of a compound expression, *first arrange the expression according to descending (or ascending) powers of some letter and then find the square root of the first term. This will be the first term of the required root. Now subtract its square from the expression, and divide the first term of the remainder by twice the first term of the root ; the quotient is the second term of the required root. Complete the divisor by adding this quotient with its proper sign to twice the first term of the root. Multiply this sum by the second term of the root and subtract this product from the first remainder. Now again divide the first term of the second remainder by twice the first term of the root ; the quotient is the third term of the required root. Repeat the process till there is no remainder.*

NOTE. If there is any remainder, the square root is not exact.

EXAMPLE 1. *Find the square root of* $4a^4 - 12a^3b + 29a^2b^2 - 30ab^3 + 25b^4$.

$$4a^4 - 12a^3b + 29a^2b^2 - 30ab^3 + 25b^4 \,(2a^2 - 3ab + 5b^2$$
$$4a^4$$
$$4a^2 - 3ab) - 12a^3b + 29a^2b^2 - 30ab^3 + 25b^4$$
$$-12a^3b + 9a^2b^2$$
$$4a^2 - 6ab + 5b^2)\,20a^2b^2 - 30ab^3 + 25b^4$$
$$20a^2b^2 - 30ab^3 + 25b^4$$

EXPLANATION. The expression is already arranged according to descending powers of a. The square root of the first term $4a^4$ is $2a^2$, and this is the *first* term of the root. Subtracting the square of this *i.e.*, $4a^4$ from the given expression, the remainder is $-12a^3b + 29a^2b^2 - 30ab^3 + 25b^4$. Now doubling the first term of the root we obtain $4a^2$. We set this $4a^2$, as a trial divisor, on the left ; and dividing $-12a^3b$ the first term of the remainder by this trial divisor we get $-3ab$, which is the *second* term of the root. Now we complete the divisor by annexing $-3ab$ to it, and multiply the completed divisor by $-3ab$, and subtract the product from the first remainder, the second remainder is $20a^2b^2 - 33ab^3 + 25b^4$. Now doubling the terms of the root already found, we set the result as a second trial divisor to the left. Dividing $20a^2b^2$, the first term of the second remainder by $4a^2$, the first term of the second divisor, we get $5b^2$, which is the *third* term of the root and which we annex to the divisor. We now multiply the completed divisor by $5b^2$ and subtract. Since there is no remainder, hence the square root is $2a^2 - 3ab + 5b^2$.

EXAMPLE 2. *What must be added to* $4x^4-20x^3+37x^2-31x+10$ *to make it a perfect square?*

$$\begin{array}{rl} & 4x^4-20x^3+37x^2-31x+10(2x^2-5x+3 \\ & 4x^4 \\ \hline 4x^2-5x) & -20x^3+37x^2-31x+10 \\ & -20x^3+25x^2 \\ \hline 4x^2-10x+3) & 12x^2-31x+10 \\ & 12x^2-30x+\ 9 \\ \hline & -x+\ 1 \end{array}$$

In order that the expression be a perfect square, the remainder must be zero. Hence, $x-1$ must be added to the given expression to make it a perfect square.

EXAMPLE 3. *What value of* a *will make* $x^4-10x^3+33x^2-ax+16$ *a perfect square?*

$$\begin{array}{rl} & x^4-10x^3+33x^2-ax+16(x^2-5x+4 \\ & x^4 \\ \hline 2x^2-5x) & -10x^3+33x^2-ax+16 \\ & -10x^3+25x^2 \\ \hline 2x^2-10x+4) & 8x^2-\ ax+16 \\ & 8x^2-40x+16 \\ \hline & 40x-ax \end{array}$$

The expression is a perfect square if $40x-ax=0$, *i.e.*, if $a=40$.

161. When an expression contains powers of a certain letter both in the numerator and the denominator, for instance the expression $\frac{9x^2}{16}+\frac{4}{81x^2}-\frac{3x}{4}+\frac{2}{9x}-\frac{1}{12}$, it is arranged in descending powers of x thus:

$$\frac{9x^2}{16}-\frac{3x}{4}-\frac{1}{12}+\frac{2}{9x}+\frac{4}{81x^2}.$$

The student will learn the reasoning for this arrangement in Chapter xxxvi. At this stage all that he has to remember is that in order to arrange an expression involving a certain letter which occurs both in the numerator and the denominator in descending powers of that letter, he should first arrange those terms in descending powers of that letter which contain that letter in the numerator, and then put the term, if any, which is independent of that letter, and then arrange the terms in ascending powers of that letter in the denominator.

EXAMPLE. *Find the square root of* $\frac{x^2}{y^2}+\frac{4y^2}{9x^2}+\frac{2x}{y}-\frac{4y}{3x}-\frac{1}{3}$.

First arranging the terms in descending powers of x, we have

$$\frac{x^2}{y^2}+\frac{2x}{y}-\frac{1}{3}-\frac{4y}{3x}+\frac{4y^2}{9x^2}.$$

Now proceeding as before,

$$\begin{array}{r|l}
 & \frac{x^2}{y^2}+\frac{2x}{y}-\frac{1}{3}-\frac{4y}{3x}+\frac{4y^2}{9x^2}\left(\frac{x}{y}+1-\frac{2y}{3x}\right. \\
 & \frac{x^2}{y^2} \\
\hline
\frac{2x}{y}+1 & \frac{2x}{y}-\frac{1}{3}-\frac{4y}{3x}+\frac{4y^2}{9x^2} \\
 & \frac{2x}{y}+1 \\
\hline
\frac{2x}{y}+2-\frac{2y}{3x} & -\frac{4}{3}-\frac{4y}{3x}+\frac{4y^2}{9x^2} \\
 & -\frac{4}{3}-\frac{4y}{3x}+\frac{4y^2}{9x^2} \\
\hline
\end{array}$$

Hence the square root is $\frac{x}{y}+1-\frac{2y}{3x}$.

EXAMPLES XCII

Find the square root of

1. $x^4+2x^3+3x^2+2x+1$.

2. $x^4-4x^3+10x^2-12x+9$.

3. $4x^4+4x^3-3x^2-2x+1$.

4. $4x^4+24x+12x^3+16+25x^2$.

5. $9x^4-29x^2+81-30x^3+90x$.

6. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

7. $x^4-2x^3y-x^2y^2+2xy^3+y^4$.

8. $4a^4+8a^3b-4ab^3+b^4$.

9. $4x^8-4x^7+5x^6-2x^5+x^4$.

10. $x^{10}+4x^6+5x^8+4x^7+2x^9$.

11. $a^6-8a^5+10a^4+28a^3-7a^2-12a+4$.

12. $a^6+28a^2b^4+20a^4b^2-12ab^5-22a^3b^3-8a^5b+9b^6$.

13. $x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2$.

14. $\frac{x^4}{4}+2x^3+7x^2+12x+9$.

15. $x^6+2x^4-x^3+x^2-x+\frac{1}{4}$.

16. $\frac{x^4}{4}-\frac{2x^3}{3}-\frac{11x^2}{36}+x+\frac{9}{16}$.

17. $\frac{x^2}{y^2}-\frac{4x}{y}+5-\frac{2y}{x}+\frac{y^2}{4x^2}$.

18. $\frac{4x^2}{9y^2}+\frac{4x}{3y}-1-\frac{3y}{x}+\frac{9y^2}{4x^2}$.

19. $\frac{9x^2}{16y^2}+\frac{4y^2}{81x^2}-\frac{3x}{4y}+\frac{2y}{9x}-\frac{1}{12}$.

20. $x^4+\frac{9}{x^4}+2x^2+\frac{6}{x^2}+7$.

21. $x^4+\frac{1}{x^4}+10x^3+25x^2+\frac{10}{x}+2$.

22. $\frac{x^4}{y^4}+\frac{y^4}{x^4}+3-\frac{2y^2}{x^2}-\frac{2x^2}{y^2}$.

23. $\frac{1}{9}x^6-2x^5+\frac{31}{3}x^4-\frac{37}{3}x^3+7x^2-2x+\frac{1}{4}$.

24. $x^4+x^2yz+\frac{y^2z^2}{4}-2x^2z^2-yz^3+z^4$.

25. $\frac{4x^2}{9y^2}+\frac{9y^2}{16z^2}+\frac{16x^2}{25z^2}-\frac{16x^2}{15yz}+\frac{6xy}{5z^2}-\frac{x}{z}$.

26. What must be added to $x^4-2x^3+5x^2-4x+2$ to make it a perfect square?

27. What must be added to $9x^4-12x^3+34x^2-23x+24$ to make it a perfect square?

28. What must be added to $9x^4+12x^3y-38x^2y^2-20xy^3+49y^4$ to make it a perfect square?

29. What value of x will make $4x^4+4x^3+5x^2+3x+4$ a perfect square?

30. What value of a will make $9x^4-30x^3+67x^2-70x+a$ a perfect square?

31. What value of a will make $4x^4+12x^3+13x^2+2ax+1$ a perfect square?

CHAPTER XXX

QUADRATIC EQUATIONS

162. An equation which contains the square, and no higher power of the unknown quantity, is called a **Quadratic Equation.**

Thus $2x^2 + 3x + 4 = 0$ is a quadratic equation.

A general quadratic equation in x contains a term in x^2, a term in x, and a term independent of x.

163. In the following examples we will take up the solution of those quadratic equations in which the terms of the first degree are missing.

EXAMPLE 1. *Solve the equation* $x^2 = 25$.

Taking the square root, we have

$$x = \pm 5,$$

for if $x = 5$, then $x^2 = 25$, and if $x = -5$, then again $x^2 = (-5)^2 = 25$.

NOTE. The square root of x^2 is $\pm x$, but it is not necessary to write $\pm x = \pm 5$, for this means

$$+x = +5 \quad \ldots\ldots (1)$$
$$+x = -5 \quad \ldots\ldots (2)$$
$$-x = +5 \quad \ldots\ldots (3)$$
$$-x = -5 \quad \ldots\ldots (4)$$

But (1) and (4) mean the same thing as $x = +5$, and (2) and (3) the same as $x = -5$.

EXAMPLE 2. *Solve the equation* $5x^2 - 8a^2 = x^2 + a^2$.

$$5x^2 - 8a^2 = x^2 + a^2,$$
$$\therefore \quad 5x^2 - x^2 = 8a^2 + a^2,$$
$$\therefore \quad 4x^2 = 9a^2,$$
$$\therefore \quad x^2 = \tfrac{9}{4}a^2,$$
$$\therefore \quad x = \pm \tfrac{3}{2}a.$$

EXAMPLE 3. *Solve the equation* $(x+5)^2 = 7$.

Taking the square root, we have

$$x + 5 = \pm \sqrt{7},$$
$$\therefore \quad x = -5 \pm \sqrt{7}.$$

EXAMPLES XCIII

(Examples 1 to 11 may be taken orally)

Solve

1. $x^2=4$.
2. $x^2-9=0$.
3. $x^2+4=40$.
4. $x^2-10=90$.
5. $4x^2-100=0$.
6. $7x^2=343$.
7. $x^2=4a^2$.
8. $x^2+a^2=10a^2$.
9. $ax^2=a^3$.
10. $\frac{ax^2}{b}=\frac{a^3}{b^3}$.
11. $\frac{lx^2}{m}=\frac{l^5}{4m^3}$.
12. $3x^2-3a^2=x^2-a^2$.
13. $3x^2+b^2=x^2+9b^2$.
14. $4(2x^2-1)=2(x^2+25)$.
15. $2(x^2-25)=(25-x^2)$.
16. $x^2-100=8-\frac{x^2}{3}$.
17. $\frac{x^2}{2}-\frac{1}{3}=\frac{x^2}{8}+1\frac{1}{6}$.
18. $(x-2)^2=3$.
19. $(x+3)^2=11$.
20. $(x+1)^2=16$.
21. $(x-7)^2=49$.

Solution by Factorization

164. It is evident that *if the value of a product be zero, the value of one or other of the factors of that product must be zero.*

Thus, if $a\times b=0$, then either $a=0$, or $b=0$.

Similarly if $abc=0$, then either $a=0$, or $b=0$, or $c=0$.

And if $(x-1)(x-2)=0$, then either $x-1=0$, or $x-2=0$, *i.e.*, the equation $(x-1)(x-2)=0$ will be satisfied if either $x-1=0$, or $x-2=0$, *i.e.* if $x=1$, or $x=2$, so that the roots are 1 and 2.

Similarly the equation $x(x-1)(x-2)=0$ will be satisfied if $x=0$, or $x-1=0$, or $x-2=0$, *i.e.*, if $x=0$, 1 or 2, so that the roots of the equation are 0, 1 and 2.

Hence we see that *the solution of a quadratic equation can be written down by inspection if the equation is given in the form of a product of factors of the first degree equated to zero.*

EXAMPLE 1. *Solve the equation* $(x-1)(x+2)=0$.

The equation is satisfied if $x-1=0$, or if $x+2=0$, *i.e.*, if $x=1$, or $x=-2$.

Hence the roots of the equation are 1 and -2.

EXAMPLE 2. *Solve the equation* $x(2x-3)(3x+4)=0$.

The equation is satisfied if $x=0$, or $2x-3=0$, or $3x+4=0$, *i.e.*, if $x=0$, or $x=\frac{3}{2}$, or $x=-\frac{4}{3}$.

Hence the roots of the equation are 0, $1\frac{1}{2}$, $-1\frac{1}{3}$.

165. Since all the terms of an equation can be transposed to one side and if these terms form an expression whose factors can be easily found, the roots of the equation can be obtained by equating each factor separately to zero.

EXAMPLE 1. *Solve the equation* $x^2=5x$.

Bringing all the terms to the left side, we have

$$x^2-5x=0,$$
$$\therefore \quad x(x-5)=0,$$
$$x=0,\ x-5=0\ ;$$
$$\therefore \quad x=0,\ 5.$$

EXAMPLE 2. *Solve the equation* $x^2-5x+6=0$.

Factorising the left side, we have

$$(x-2)(x-3)=0,$$
$$x-2=0,\ x-3=0\ ;$$
$$\therefore \quad x=2,\ 3.$$

EXAMPLE 3. *Solve the equation* $\frac{6-x}{2}-\frac{2x-11}{x-3}=\frac{x-8}{6}$.

Multiplying both sides by $6(x-3)$, we have

$$3(x-3)(6-x)-6(2x-11)=(x-8)(x-3),$$
$$\therefore \quad -3x^2+27x-54-12x+66=x^2-11x+24,$$
$$-3x^2-x^2+27x-12x+11x-54+66-24=0,$$
$$\therefore \quad -4x^2+26x-12=0,$$
$$\therefore \quad 2x^2-13x+6=0,$$
$$\therefore \quad (x-6)(2x-1)=0,$$
$$\therefore \quad x-6=0,\ 2x-1=0,$$
$$\therefore \quad x=6,\ \tfrac{1}{2}.$$

EXAMPLES XCIV

(*Examples* 1 *to* 29 *may be taken orally*)

Write down the roots of the following equations :

1. $(x-1)(x-3)=0.$
2. $(x-3)(x-4)=0.$
3. $(x-2)(x-3)=0.$
4. $(x+1)(x+2)=0.$
5. $(1+x)(x-1)=0.$
6. $(x-a)(x-b)=0.$
7. $(x-a)(x+b)=0.$
8. $(x+l)(x+m)=0.$
9. $(2x-1)(3x-1)=0.$
10. $(2x+1)(x-5)=0.$
11. $(ax+b)(ax-b)=0.$
12. $\left(\frac{x}{2}-3\right)\left(\frac{x}{3}-5\right)=0.$
13. $\left(\frac{x}{2}+3\right)\left(\frac{x}{3}+4\right)=0.$
14. $2\left(\frac{x}{2}-7\right)\left(\frac{x}{7}+2\right)=0.$
15. $\left(\frac{lx}{m}+1\right)\left(x-\frac{l}{m}\right)=0.$
16. $l\left(\frac{lx}{m}-1\right)\left(mx+\frac{l}{n}\right)=0.$
17. $x(x-2)=0.$
18. $x(2x+1)=0.$
19. $x(x-1)(x-2)=0.$
20. $x(2x-1)(5x+1)=0.$
21. $ax(ax+b)(cx-d)=0.$
22. $(x-1)(x-3)(x+5)=0.$
23. $(x-3)^2=0.$
24. $(2x+a)^2=0.$
25. $ax^2(ax-1)^2=0.$
26. $\{x-(a-b)\}^2=0.$
27. $\{x+(a-b)\}\{x-(a+b)\}=0.$
28. $(x-p)\left(2x-\frac{a-b}{3}\right)=0,$
29. $(x-p+q)\left(\frac{x}{a}+\frac{l+m}{n}\right)=0.$

Solve the following equations :

30. $x^2-3x+2=0.$
31. $x^2-5x+6=0.$
32. $x^2-6x+5=0.$
33. $x^2-x-12=0.$
34. $x^2-9=0.$
35. $x^2+6x-7=0.$
36. $2x^2-5x+2=0.$
37. $2x^2+5x-3=0.$
38. $6x^2-5x+1=0.$
39. $6x^2+5x+1=0.$
40. $6x^2-11x+3=0.$
41. $10x^2+13x-3=0.$
42. $x(x+2)=15.$
43. $x(x-1)=12.$
44. $\frac{x(x-3)}{2}=5.$
45. $\frac{x}{2}+\frac{6}{x}=4.$
46. $\frac{x}{4}-\frac{8}{x}=1.$
47. $x=\frac{1}{6x-1}.$

48. $\frac{x(2x-3)}{7}=5.$

49. $(x-1)(3x+1)=(2x-5)(x+9).$

50. $(x-1)(x-2)=2.$

51. $\frac{x+8}{3x-5}=\frac{x-8}{x+5}.$

52. $\frac{7}{3x-4}=1+\frac{2}{x+2}.$

53. $\frac{x^2-9}{15x}=1-\frac{3}{x}.$

54. $\frac{x}{x+3}+\frac{1}{10}=3-\frac{x+3}{x}.$

55. $\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x-3}=0.$

56. $\frac{2x-5}{x-4}+\frac{3x+5}{2x-1}=5.$

57. $\frac{3x+1}{6}-\frac{3}{3x+2}=\frac{1}{3}.$

58. $\frac{x+\frac{3}{2}}{5x-2}-\frac{\frac{3x}{2}-1}{4x+1}=\frac{1}{13}.$

59. $\frac{\frac{x}{3}-1}{\frac{x}{3}+1}+6\frac{6}{7}=\frac{\frac{x}{3}+1}{\frac{x}{3}-1}.$

60. $\frac{x}{x-4}+\frac{x-\frac{5}{2}}{x-3}=4\frac{1}{6}.$

Solution by Completing the Square

166. When the factors of a quadratic expression are not easily found, we can solve the equation by keeping all the terms involving the variable to one side and then completing the square as explained in Article 112. This method is illustrated in the following examples.

EXAMPLE 1. *Solve the equation* $x^2+6x-7=0.$

Keeping the terms involving x to the left side and taking -7 to the right, we have

$$x^2+6x=7.$$

Adding $(3)^2$ *i.e.* 9, the square of half the coefficient of x to both sides,

$$x^2+6x+9=7+9,$$

$$\therefore \quad (x+3)^2=16,$$

$$\therefore \quad x+3=\pm 4,$$

$$\therefore \quad x=-3\pm 4.$$

Hence the roots are $-3+4$ and $-3-4$ *i.e.*, 1 and -7.

EXAMPLE 2. *Solve the equation* $2x^2-x+1=0$.

The equation can be written as

$$2x^2-x=-1.$$

Dividing both sides by 2, the coefficient of x^2,

$$x^2-\tfrac{1}{2}x=-\tfrac{1}{2}.$$

Adding $(-\frac{1}{4})^2$, the square of half the coefficient of x, to both sides,

$$x^2-\frac{1}{2}x+\left(-\frac{1}{4}\right)^2=\left(-\frac{1}{4}\right)^2-\frac{1}{2},$$

$$\left(x-\frac{1}{4}\right)^2=\frac{1}{16}-\frac{1}{2}=-\frac{7}{16},$$

$$x-\frac{1}{4}=\pm\frac{\sqrt{-7}}{4},$$

$$x=\frac{1}{4}\pm\frac{\sqrt{-7}}{4}=\frac{1\pm\sqrt{-7}}{4}.$$

Hence the roots are $\dfrac{1+\sqrt{-7}}{4}$ and $\dfrac{1-\sqrt{-7}}{4}$.

NOTE. Since there is no quantity, positive or negative, whose square is -7 (or any negative number), it is impossible to find any real quantity to represent $\sqrt{-7}$. Hence $\sqrt{-7}$ is not a *real* quantity and there is no real value of x which satisfies the equation. In such a case the roots of the equation are said to be **imaginary** or **impossible.**

EXAMPLES XCV

Solve

1. $x^2-4x+3=0$.

2. $x^2-2x-15=0$.

3. $x^2-4x-45=0$.

4. $x^2-10x=21$.

5. $x^2-x=2$.

6. $x^2-3x=4$.

7. $x^2+3x-18=0$.

8. $x^2-2x-99=0$.

9. $x^2-x=1$.

10. $x^2-x=72$.

11. $x^2-x=156$.

12. $x^2-7x=8$.

13. $x^2-35x=200$.

14. $3x^2-10x+3=0$.

15. $3x^2+10x-32=0$.

16. $5x^2-4x-1=0$.

17. $6x^2=2-x$.

18. $4x^2=18-21x$.

19. $3x^2=5-4x$.

20. $8x^2+19x-15=0$.

21. $20x^2+x=12.$

22. $6x^2-9x=8\frac{2}{3}.$

23. $3x^2-21x+22\frac{2}{3}=0.$

24. $x^2+4ax=12a^2.$

25. $x^2-5ax=36a^2.$

26. $16x^2-8kx+k^2=0.$

27. $9x^2-7kx-2k^2=0.$

28. $3x(x-3)-x(2x-3)=91.$

29. $x(4x+1)=3x(x-3)+24.$

30. $x^2+5(5x-12)-3(x+5)=0.$

31. $3x(x+4)+2x(x+2)=25.$

32. $x(3-4x)-2x(x+1)=22.$

33. $x-\frac{1}{x}+1=0.$

34. $x-\frac{2}{x}=1.$

35. $9x+\frac{1}{x}=6.$

36. $x+\frac{1}{x}=\frac{5}{2}.$

37. $\frac{x}{2}-\frac{1}{x}=\frac{1}{2}.$

38. $\frac{3x-2}{2x-3}=\frac{3x-8}{x+4}.$

39. $\frac{x+3}{4}-\frac{4}{x+3}=\frac{3}{2}.$

40. $\frac{x}{x+1}+\frac{x+1}{x}=1\frac{3}{4}.$

41. $\frac{3x-1}{4x+7}=\frac{x+1}{x-7}.$

42. $\frac{2x-1}{x-3}=\frac{x+3}{2x-7}.$

43. $\frac{4}{x+1}=\frac{2(x-3)}{x-2}.$

44. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=4\frac{1}{3}.$

45. $\frac{10}{3x-5}+\frac{2}{2x+1}=1\frac{2}{11}.$

46. $\frac{2x}{x-3}=\frac{x+3}{2x-7}-\frac{1}{3-x}.$

47. $\frac{3x}{x-2}=\frac{5x-2}{x+5}-\frac{8}{2-x}.$

48. $\frac{x+1}{x-1}-\frac{1-x}{1+x}=\frac{5x}{x^2-1}.$

49. $\frac{x+2\frac{1}{3}}{x+2}-\frac{3}{x-2}=\frac{1}{x^2-4}.$

50. $\frac{x-1}{x-2}-\frac{x+4}{x+2}=\frac{3}{5}.$

51. $\frac{x+2}{x+1}+\frac{x-2}{x-3}=2\frac{6}{35}.$

52. $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{x+2}-\frac{1}{x+4}.$

Equations with Literal Coefficients

167. EXAMPLE 1. *Solve the equation* $ax^2+bx+c=0.$

Transposing,

$$ax^2+bx=-c.$$

Dividing by a,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Completing the square by adding $\left(\frac{b}{2a}\right)^2$ to both sides,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a},$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$=\frac{b^2-4ac}{4a^2},$$

$$\therefore \quad x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a},$$

$$\therefore \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

NOTE. Since $ax^2+bx+c=0$ is a general quadratic equation, therefore $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ may be regarded as a *formula* from which the roots of any quadratic equation may be obtained by substitution.

EXAMPLE 2. *Solve the equation* $\frac{x}{a-b}=\frac{a}{x-b}$.

Cross-multiplying,

$$x(x-b)=a(a-b),$$

$$x^2-bx=a^2-ab,$$

$$x^2-bx+\left(-\frac{b}{2}\right)^2=a^2-ab+\left(-\frac{b}{2}\right)^2,$$

$$\left(x-\frac{b}{2}\right)^2=a^2-ab+\frac{b^2}{4}$$

$$=\frac{4a^2-4ab+b^2}{4}$$

$$=\frac{(2a-b)^2}{4},$$

$$x-\frac{b}{2}=\pm\frac{2a-b}{2}.$$

$$x=\frac{b}{2}\pm\frac{2a-b}{2}.$$

Hence the roots are $\frac{b}{2}+\frac{2a-b}{2}$ and $\frac{b}{2}-\frac{2a-b}{2}$, *i.e.*, a and $b-a$.

EXAMPLES XCVI

Solve

1. $x^2+(a+b)x+ab=0$.
2. $x^2-(a+b)x+ab=0$.
3. $ax^2+(a^2-1)x-a=0$.
4. $lx^2+mx+n=0$.
5. $x^2+bx+c=0$.
6. $px^2-qx+r=0$.
7. $px^2-qx-r=0$.
8. $x-\frac{1}{x}=c-\frac{1}{c}$.
9. $\frac{x}{l}+\frac{l}{x}=\frac{m}{l}+\frac{l}{m}$.
10. $\frac{x}{a}-\frac{a}{b}=\frac{b}{a}-\frac{a}{x}$.
11. $\frac{a}{a-x}+\frac{a}{a+x}=4$.
12. $abx^2-(b-ac)x-c=0$.
13. $x^2+2(p-q)x-q(2p-q)=0$.
14. $abx^2+(a^2+b^2)x+ab=0$.
15. $(l^2-m^2)(x^2-1)-4lmx=0$.
16. $\frac{x+a}{x-a}-\frac{x-a}{x+a}=\frac{x-b}{x+b}-\frac{x+b}{x-b}$.

Nature of the Roots

168. We have seen in Article 167, that the roots of the equation $ax^2+bx+c=0$ are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a},$$

in which the quantity under the radical sign, b^2-4ac, may be *positive*, *zero* or *negative*. Let us now examine the nature of the roots under these three conditions separately.

(1) If b^2-4ac is positive, the roots are **real** and **unequal.** Moreover,

(i) if b^2-4ac is a perfect square, the roots are **rational** and can be found exactly ;

and (ii) if b^2-4ac is not a perfect square, the roots are **irrational** and cannot be found exactly. Their values can, however, be found correct to a certain number of decimal places if the numerical values of a, b and c are known.

(2) If $b^2-4ac=0$, both the roots become $-\frac{b}{2a}$ and are therefore **real** and **equal.**

(3) If b^2-4ac is negative, the roots are **imaginary**, for the square root of a negative quantity is imaginary. (See solved Example 2, Article 166)

EXAMPLE 1. *Show that the equation* $5x^2-6x+1=0$ *has rational roots.*

Here $a=5$, $b=-6$ and $c=1$;

$\therefore\quad b^2-4ac=(-6)^2-4.5.1=36-20=16.$

Hence the roots are rational.

EXAMPLE 2. *For what value of* k *will the equation* $4x^2-12x+k=0$ *have equal roots?*

The condition for equal roots gives

$$(-12)^2-4.4.k=0,$$

$$\therefore\quad 16k=144,$$

Hence $k=9$.

Relation between Roots and Coefficients

169. If the roots of the equation $ax^2+bx+c=0$ are added, the sum

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=-\frac{2b}{2a}=-\frac{b}{a}.$$

If the roots are multiplied, the product

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}\times\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Hence, we see that

(i) the sum of the roots $= -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$,

and (ii) the product of the roots $= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$.

If the given equation is written in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

the above relations may be stated thus :

If the coefficient of x^2 in a quadratic equation is unity,

(i) *the sum of the roots is equal to the coefficient of* x *with the sign changed* ; and (ii) *the product of the roots is equal to the constant term.*

Hence any quadratic equation may also be expressed in the form

$$x^2 - (\text{sum of the roots})\, x + (\text{product of the roots}) = 0.$$

EXAMPLE 1. *Form the equation whose roots are* 2 *and* -5.

The sum of the roots $= 2 + (-5) = -3$,

and the product of the roots $= 2 \times (-5) = -10$.

Hence the equation is

$$x^2 - (-3)x + (-10) = 0,$$

i.e., $x^2 + 3x - 10 = 0$.

EXAMPLE 2. *Form the equation whose roots are* $\frac{3}{5}$ *and* $-\frac{2}{7}$.

The sum of the roots $= \frac{3}{5} - \frac{2}{7} = \frac{11}{35}$,

and the product of the roots $= (\frac{3}{5})(-\frac{2}{7}) = -\frac{6}{35}$.

Hence the equation is

$$x^2 - \frac{11}{35}x - \frac{6}{35} = 0,$$

i.e., $35x^2 - 11x - 6 = 0$.

EXAMPLES XCVII

Find, without actual solution, the nature of the roots of the following equations :

1. $x^2 - 5x + 6 = 0$.

2. $4x^2 - 4x + 1 = 0$.

3. $x^2 + x + 1 = 0$.

4. $x^2 + x - 1 = 0$:

5. $3x^2 = 2x^2 + 9$.

6. $6x^2 + x - 2 = 0$.

Find, without actual solution, the sum and the product of the roots of the following equations :

7. $x^2-3x+5=0$.
8. $x^2-x+1=0$.
9. $x^2+x-6=0$.
10. $2x^2-4x-7=0$.
11. $x^2+bx+c=0$.
12. $lx^2-mx+n=0$.

Form the equations whose roots are

13. 2, 3.
14. 4, -3.
15. 0, -5.
16. $\frac{1}{2}$, $-\frac{1}{2}$.
17. $a+b$, $a-b$.
18. $\frac{1}{3}p$, $-\frac{2}{3}p$.

Solve the following equations and check your answers by finding the sum and the product of the roots :

19. $x^2-x-6=0$.
20. $x^2+2x-3=0$.
21. $x^2+4x-192=0$.
22. $x^2-19x-780=0$.
23. $x^2-(a+b)x+ab=0$.
24. $x^2-2ax+2ab-b^2=0$.

25. For what values of a will the equation $x^2+ax+9=0$ have equal roots ?

26. Show that the roots of the equation $3x^2-x-1=0$ are real and different.

27. Show that the roots of the equation $x^2-3kx+2k^2=0$ are rational.

28. Show that the equation $x^2+2mx+5m^2=0$ cannot be satisfied by any real values of x.

29. Show that the roots of the equation $(x-a)(x-b)=3$ are always real.

30. Show that the roots of the equation $ax^2-(a+1)x+1=0$ are rational.

Easy Simultaneous Quadratic Equations

170. In this chapter, we shall consider chiefly those simultaneous quadratic equations in which one equation is of the first degree and the other quadratic. In solving such equations, we express one of the unknown quantities in terms of the other and then substitute this value of the first unknown in the second equation, which will become a quadratic equation in terms of the second unknown. Since for each value of this quadratic there is a corresponding value of the first unknown, it follows that such equations have two pairs of solutions. When both the equations are quadratic, no general method can be given for their solution. Different devices are applied in solving different questions.

EXAMPLE 1. *Solve the equations*

$$x+y=5,\ x^2+y^2=13.$$

The first equation can be written as

$$x=5-y.$$

Substituting this value of x in the second equation,

$$(5-y)^2+y^2=13,$$
$$25-10y+y^2+y^2=13,$$
$$\therefore\quad 2y^2-10y+12=0,$$
$$\therefore\quad y^2-5y+6=0,$$
$$\therefore\quad (y-2)(y-3)=0,$$
$$\therefore\quad y=2,\ 3.$$

Substituting the first value of y in the first equation,

$$x+2=5,$$
$$\therefore\quad x=3.$$

Substituting the second value of y in the first equation,

$$x+3=5,$$
$$\therefore\quad x=2.$$

Hence the solution is $x=2$, $y=3$ and $x=3$, $y=2$.

EXAMPLE 2. *Solve the equations*

$$x+y=6,\ xy=8.$$

The first equation can be written as

$$x=6-y.$$

Substituting this value of x in the second equation,

$$(6-y)y=8,$$
$$\therefore\quad 6y-y^2=8,$$
$$\therefore\quad y^2-6y+8=0,$$
$$\therefore\quad (y-2)(y-4)=0,$$
$$\therefore\quad y=2,\ 4.$$

Substituting the first value of y in the first equation,

$$x+2=6,$$
$$\therefore\quad x=4.$$

Substituting the second value of y in the first equation,

$$x+4=6,$$
$$\therefore\quad x=2.$$

Hence the solution is $x=4$, $y=2$ and $x=2$, $y=4$.

EXAMPLE 3. *Solve the equations*

$$x^2+y^2=29,\ xy=10.$$

Multiplying the second equation by 2,

$$2xy=20 \quad \text{...(1)}$$

Adding (1) to the first equation,

$$x^2+2xy+y^2=49,$$
$$\therefore\ (x+y)^2=49,$$
$$\therefore\ x+y=\pm 7 \quad \text{...(2)}$$

Subtracting (1) from the first equation,

$$x^2-2xy+y^2=9,$$
$$\therefore\ (x-y)^2=9,$$
$$\therefore\ x-y=\pm 3 \quad \text{...(3)}$$

Now (2) and (3) can be written as four pairs of equations thus :

(i) $x+y=7$, $x-y=3$. (ii) $x+y=7$, $x-y=-3$. (iii) $x+y=-7$, $x-y=3$. (iv) $x+y=-7$, $x-y=-3$.

Adding and subtracting each pair of equations, the required solution is (i) $x=5$, $y=2$; (ii) $x=2$, $y=5$; (iii) $x=-2$, $y=-5$; (iv) $x=-5$, $y=-2$.

EXAMPLES XCVIII

Solve the following equations :

1. $x+y=3$, $x^2+y^2=5$. **2.** $x+y=7$, $x^2+y^2=25$. **3.** $x-y=2$, $x^2+y^2=20$. **4.** $x-y=1$, $x^2+y^2=41$.

5. $x-y=10$, $x^2+y^2=58$. **6.** $x+y=10$, $x^2-y=-4$. **7.** $x+2y=5$, $x^2+2y^2=9$.

8. $x+3y=10$, $x^2+y^2=10$. **9.** $x+y=3$, $xy=2$. **10.** $x-y=2$, $xy=15$.

11. $x+y=1$, $xy=-2$. **12.** $x^2+y^2=10$, $xy=3$. **13.** $x^2+y^2=5$, $xy=2$.

14. $x^2+y^2=65$, $xy=28$. **15.** $5x-3y+1=0$, $xy=2$. **16.** $2x+3y=7$, $xy=2$.

17. $x^2+y^2=125$, $x+y=15$. **18.** $x^2+y^2=185$, $x-y=3$. **19.** $x^2+y^2=181$, $x-y=1$.

20. $px+qy=2$, $pqxy=1$. **21.** $x^2+xy+y^2=39$, $x+y=7$. **22.** $x^2-xy+y^2=19$, $x-y=-3$.

CHAPTER XXXI

PROBLEMS INVOLVING QUADRATIC EQUATIONS

171. In this chapter we shall give some problems which lead to quadratic equations. We have seen in the previous chapter that a quadratic equation in one variable always has two solutions. In some cases both the solutions satisfy the problem and in others only one. The student is, therefore, advised to see carefully whether both the solutions are applicable or not.

EXAMPLE 1. *Find two consecutive even numbers, the sum of whose squares is* 100.

Let the numbers be x and $x+2$.

$$\therefore \quad x^2+(x+2)^2=100,$$
$$\therefore \quad 2x^2+4x+4=100,$$
$$\therefore \quad x^2+2x-48=0,$$
$$\therefore \quad (x-6)(x+8)=0,$$
$$\therefore \quad x=6 \text{ or } -8.$$

If we consider only positive numbers, then 6 is the only solution, hence the numbers are 6 and 8.

But if we consider negative numbers also, then -8 is also a solution, hence the numbers are also -8 and -6.

EXAMPLE 2. *A number consists of two digits; the digit in the units place exceeds the digit in the tens place by* 2, *and the number is equal to three times the product of the digits. Find the number.*

Let x be the digit in the tens place, then the digit in the units place is $(x+2)$ and the number is $10x+(x+2)$.

$$\therefore \quad 10x+(x+2)=3x(x+2),$$
$$\therefore \quad 10x+x+2=3x^2+6x,$$
$$\therefore \quad 3x^2-5x-2=0,$$
$$\therefore \quad (x-2)(3x+1)=0,$$
$$\therefore \quad x=2, -\tfrac{1}{3}.$$

Since the digits of a number are positive integers, the second value of x is inadmissible. Rejecting this value we have $x=2$, hence the number is 24.

EXAMPLE 3. *Divide a straight line* 10 *inches long into two parts such that the square on one part is equal to twice the square on the other.*

Let the length of one part be x inches, then the length of the other part is $(10-x)$ inches.

$$\therefore \quad (10-x)^2=2x^2,$$
$$100-20x+x^2=2x^2,$$
$$\therefore \quad x^2+20x=100,$$
$$x^2+20x+100=100+100,$$
$$\therefore \quad (x+10)^2=200,$$
$$\therefore \quad x+10=\pm\sqrt{200},$$
$$x=-10\pm\sqrt{200}$$
$$=-10\pm14{\cdot}14\ldots$$
$$=-10+14{\cdot}14\ldots,\ -10-14{\cdot}14\ldots$$
$$=4{\cdot}14\ldots,\ -24{\cdot}14\ldots$$

If we consider only the positive length, then the length of one part is 4·14... inches and of the other is 5·86... inches.

In the figure, if AB is the straight line, AX represents this length. The negative value $-24{\cdot}14\ldots$inches is represented by AX′, which is measured in the direction opposite to AX.

EXAMPLE 4. *The length of a rectangle exceeds its breadth by* 2 *feet, and its area is* 48 *square feet. Find the length of its sides.*

Let the breadth of the rectangle be x feet, then its length is $(x+2)$ feet and area $x(x+2)$ square feet.

$$\therefore \quad x(x+2)=48,$$
$$x^2+2x-48=0,$$
$$(x-6)(x+8)=0,$$
$$\therefore \quad x=6,\ -8.$$

Since the negative value is inadmissible as the length of a rectangle cannot be negative, therefore rejecting it, we have breadth=6 ft., and length=8 ft.

EXAMPLES XCIX

1. What is the number whose square exceeds six times the number by 16 ?

2. What is the number whose square is less than ten times the number by 21 ?

3. What is the number whose square is equal to twice the product of the number and 5 ?

4. A number is such that five times the product of the number and 3 exceeds its square by 36. Find the number.

5. What number is less than the square of the number by 30 ?

6. Divide 7 into two parts such that the sum of their squares is equal to 25.

7. Find two numbers whose difference is 2 and the sum of whose squares is 74.

8. The square of a number exceeds six times the sum of the number and 9 by 1. Find the number.

9. Find two consecutive even numbers the sum of whose squares is 52.

10. Find two consecutive odd numbers the sum of whose squares is 130.

11. Find two consecutive even numbers whose product exceeds their sum by 62.

12. Two numbers differ by 3, and the sum of their reciprocals is $\frac{11}{28}$. Find the numbers.

13. Find two consecutive even numbers the sum of whose reciprocals is $\frac{5}{12}$.

14. One number is three times another number ; if each be increased by 1 the sum of the reciprocals is $\frac{7}{20}$. Find the numbers.

15. The numerator of a fraction is less than its denominator by 1 ; if both the numerator and the denominator be increased by 4, the fraction thus formed exceeds the original fraction by $\frac{1}{48}$. Find the fraction.

16. A number consists of two digits ; the digit in the units place exceeds the digit in the tens place by 3, and the number is twice the product of the digits. Find the number.

17. A number consists of two digits ; the digit in the units place exceeds the digit in the tens place by 1, and the number exceeds twice the product of the digits by 5. Find the number.

18. Divide a straight line 6 inches long into two parts such that the square of one part is equal to the product of the other part and the whole straight line.

19. A straight line AB, 6″ long, is divided internally at X. Find AX, when (i) $AX^2 = 2AB.\,BX$; (ii) $AX^2 = 2BX^2$.

20. Find the length and breadth of a rectangle whose perimeter is 44 feet and whose area is 120 square feet.

21. The length of a rectangle exceeds its breadth by 10 yards, and its area is 1200 square yards. Find the length of its sides.

22. One side of a rectangle exceeds the other by 2 feet and is less than its diagonal by 2 feet. Find the area of the rectangle.

23. 1440 bricks are required to pave the floor of a room 18 ft. long and 10 ft. broad. Find the length of a brick if its length is twice its breadth.

24. The area of a rectangle is 154 square yards. If its length were 2 yards less and its breadth 2 yards more, its area would be 180 square yards. Find the length of its sides.

25. The diagonal of a rectangle is 15 feet, and its area is 108 square feet. Find the length of its sides.

26. The hypotenuse of a right-angled triangle is 13 inches and its area is 30 square inches. Find its sides.

27. How many bricks, each x inches long, y inches broad and z inches thick, will be required to build a wall a yards long, b inches thick and c feet high ?

28. The length of a room exceeds its breadth by 2 feet and its height is three-fourths its breadth. Find the dimensions of the room if the area of the ceiling and the floor is less than the area of the four walls by 132 square feet.

29. The length of a room exceeds its breadth by 5 feet, and 840 square feet of paper are required to cover its four walls. If its height were 2 feet more and its length 5 feet less, the same amount of paper would be required to cover its walls. Find the dimensions of the room.

30. The perimeter of one square exceeds another by 24 feet, and the area of the bigger square exceeds the area of the smaller square by 156 square feet. Find the sides of the squares.

31. A rectangular garden, 50 yards long and 40 yards broad, has within it a road of uniform width running all round it. Find the width of the road if its area is 344 square yards.

32. The length, breadth and height of a room are in the ratio of 7 : 5 : 4. If each is increased by 1 foot, the area of the four walls would increase by 124 square feet. Find the dimensions of the room.

172. EXAMPLE 1. *A motor car travels* 420 *miles with a uniform speed ; if its speed had been* 5 *miles an hour more, it would have taken* 2 *hours less for the journey ; find the speed of the car.*

Let the speed of the car be x miles an hour, then the time occupied is $\frac{420}{x}$ hours.

If its speed were 5 miles an hour more, then the time occupied would be $\frac{420}{x+5}$ hours.

$$\frac{420}{x+5}=\frac{420}{x}-2,$$

$$\therefore \quad 420x=420(x+5)-2x(x+5),$$

$$\therefore \quad 420x=420x+2100-2x^2-10x,$$

$$2x^2+10x-2100=0,$$

$$\therefore \quad x^2+5x-1050=0,$$

$$\therefore \quad (x-30)(x+35)=0,$$

$$\therefore \quad x=30,-35.$$

The negative value is inadmissible, hence the speed of the car is 30 miles an hour.

EXAMPLE 2. *Two pipes running together can fill a cistern in* $11\frac{1}{9}$ *minutes; if one pipe takes* 5 *minutes more than the other to fill the cistern, find the time in which each pipe would fill the cistern.*

Let x minutes be the time taken by the larger pipe to fill the cistern, then the smaller pipe would fill the cistern in $(x+5)$ minutes.

Therefore the larger pipe fills $\frac{1}{x}$ and the smaller $\frac{1}{x+5}$ of the cistern in 1 minute, *i.e.*, they together fill $\left(\frac{1}{x}+\frac{1}{x+5}\right)$ of the cistern in 1 minute, which is equal to $\frac{9}{100}$ of the cistern.

$$\therefore \quad \frac{1}{x}+\frac{1}{x+5}=\frac{9}{100},$$

$$\therefore \quad 100(x+5)+100x=9x(x+5),$$

$$\therefore \quad 100x+500+100x=9x^2+45x,$$

$$\therefore \quad 9x^2-155x-500=0,$$

$$\therefore \quad (x-20)(9x+25)=0,$$

$$\therefore \quad x=20,-\frac{25}{9}.$$

Rejecting the negative value, we see that one pipe would fill the cistern in 20 minutes and the other in 25 minutes.

EXAMPLE 3. *A man by selling a horse for* Rs. 144 *finds that his loss per cent. is one sixteenth of the number of rupees that he paid for the horse; for how much did he purchase the horse?*

Let Rs. x be the purchasing price of the horse,

then his loss on Rs. 100 is Rs. $\frac{x}{16}$,

,, ,, Rs. x is Rs. $\frac{x^2}{1600}$.

the selling price is Rs. $\left(x-\frac{x^2}{1600}\right)$.

Hence $$x-\frac{x^2}{1600}=144,$$

$$\therefore \quad 1600x-x^2=230400,$$

$$\therefore \quad x^2-1600x+230400=0,$$

$$\therefore \quad (x-160)(x-1440)=0,$$

$$\therefore \quad x=160,\ 1440.$$

Since both the values satisfy the conditions of the problem, therefore the purchasing price of the horse is either Rs. 160 or Rs. 1440.

EXAMPLES C

1. A man bought a certain number of goats for Rs. 240. If he had got 2 goats less, each would have cost Rs. 4 more. How many goats did he buy and what was the price of each ?

2. A man bought a certain number of horses for Rs. 4800. If he had got 4 horses more, each would have cost Rs. 40 less. How many horses did he buy ?

3. A boy rode 150 miles on a motor-cycle at a uniform rate. If he had ridden 1 mile an hour faster, he would have taken $\frac{5}{8}$ hour less to complete the journey. Find his rate.

4. Two trains run a distance of 200 miles, one at x miles an hour, and the other at $x+5$ miles an hour. What time will each take to cover the journey ? If the second train covers the journey 2 hours earlier than the first, find the value of x.

5. A goods train, whose speed is 15 miles an hour less than that of a passenger train, takes 6 hours longer than the passenger train to go 280 miles. Find the speed of the two trains.

6. An aeroplane goes from Delhi to Calcutta, a distance of 900 miles, with a uniform speed. If its speed had been 10 miles an hour more, it would have reached Calcutta 3 hours earlier. Find its speed.

7. A man starts from Aligarh to go to Muttra, a distance of 35 miles on horseback, and on reaching Muttra returns to Aligarh on a cycle. Find his speed on horseback, if his speed on the cycle is 2 miles an hour faster and if he takes 2 hours less in returning.

8. Two cars start at the same time from two places 600 miles apart and travelling in opposite directions meet after 6 hours. If one takes 5 hours more to do the journey than the other ; find their speed.

9. Two trains start simultaneously from two stations A and B, 450 miles apart, and travel in opposite directions. The train from A reaches B 8 hours, and the train from B reaches A $12\frac{1}{2}$ hours after they meet ; find the speed of each train.

10. A train starts from a station A to go to a station B, a distance of $137\frac{1}{2}$ miles. An hour later another train starts from A, and in 4 hours comes to a point where the first train had passed 12 minutes earlier. The second train then increases its speed by 5 miles an hour and overtakes the first train just on entering B. Find the speed of each train.

11. The circumference of one wheel is 2 feet more than that of another, and the larger wheel makes 440 revolutions less in a mile than the smaller. Find the circumference of each wheel.

12. If the front wheel of a carriage is x feet in circumference, how many revolutions can it make in 1 mile ? If the hind wheel which is 2 feet greater in circumference makes 132 revolutions less in a mile than the front wheel, find the value of x.

13. If a cycle wheel 8 feet in circumference takes $\frac{1}{22}$ second less to make one revolution, the rate of the rider increases by $\frac{10}{11}$ mile an hour ; at what rate is the rider going ?

14. Mohan and Sohan start from the same place and at the same time to travel a distance of 30 miles. Sohan's speed is less than Mohan's by 2 miles an hour. If Sohan reaches the destination $\frac{1}{2}$ hour later than Mohan, find the speed of each.

15. A boy was sent to buy eggs for Re. 1. 8a., but by breaking 6 eggs the price of each egg was increased by 4 pies. Find the price of a dozen eggs.

16. A man purchased some horses for Rs. 3000. Three of them died, and he sold the rest at Rs. 65 more for each horse than he paid and thus gained 6% on his outlay. How many horses did he buy ?

17. A man purchased some goats for Rs. 800. He kept 20 goats with him and sold the rest for Rs. 720 and gained Rs. 2 per goat. How many goats did he buy ?

18. I bought some yards of woollen cloth for Rs. 125. I kept 5 yards with me and sold the rest at Re. 1. 12a. per yard more than I paid and gained Rs. 10 on my outlay. How many yards of cloth did I buy ?

19. I bought some cricket balls for Rs. 30. If I had paid Re. 1. less for each ball, I would have got 15 balls more for the same money. What is the price of each ball ?

20. A and B together can do a piece of work in 24 days. If B takes 20 days more to do the work than A, in how many days can A do it alone ?

21. Three men and seven women can do a piece of work in 2 days. If a woman takes 3 days more than a man to do the work, in how many days can one man do the work ?

22. Two pipes running together can fill a cistern in 12 minutes ; if one pipe takes 10 minutes more than the other to fill the cistern, in what time would each pipe separately fill the cistern ?

23. A labourer undertakes to do a work for Rs. 40 in a certain number of days. He takes 4 days more to finish the work and thus receives 5a. 4p. less as his daily wages. In how many days did he undertake to finish the work ?

24. Two labourers are employed on different wages. On the completion of the work, one labourer gets Rs. 25 and the other, who worked for 4 days less, gets Rs. 16. If the first labourer had worked for as many days as the second, and the second had worked for as many days as the first, their total wages would have been equal. Find the number of days each worked and the daily wages of each.

25. A tradesman finds that by selling a cycle for Rs. 75, which he had bought for Rs. x, he gains x%. Find the value of x.

26. A beats B by y yards in a race of x yards. If A's speed is z yards per second, find B's speed.

27. Two men have 60 goats between them; they sell them at different prices, but receive the same sum of money. If the first had sold at the second's price, he would have received Rs. 640; and if the second had sold at the first's price, he would have got Rs. 490. How many goats had each?

28. A man finds that by selling a cow for Rs. 56 his percentage of profit is the same as the number of rupees he paid for the cow; for how much did he purchase the cow?

29. A boy by selling an inkpot for 4a. 8p. finds that his gain per cent. is the same as the number of pies he paid for it; for how much did he purchase the inkpot?

30. A man finds that by selling a goat for Rs. 16 his percentage of loss is the same as the number of rupees he paid for the goat; for how much did he purchase the goat?

31. A man by selling a watch for Rs. 72 finds that his gain per cent. is one third of the number of rupees that he paid for the watch; for how much did he purchase the watch?

32. A man sold an article at a gain of 6%. If he had bought it at 4% less and sold it for Re. 1. 3a. more, he would have gained 12%. Find the cost price of the article.

33. I spent z pounds in purchasing oranges at the rate of x oranges for y pence and sold them at the rate of y oranges for x pence. How many pounds did I gain?

34. A man gets Rs. x a week. He worked for $(x-9)$ weeks. If he had received Rs. 5 a week less and worked 4 weeks more, he would have received Rs. 400. Find his weekly wages and the number of weeks he worked.

MISCELLANEOUS EXAMPLES III

A

1. Resolve into factors

$$x^2-7x+12,\ 6x^2+5x-6,\ x^3-8.$$

2. Of which of the expressions 3, $3x$, $6x^3$, $9xy$, $9xy^2$, $3x^4y$, $6x^2y^2$, $12xy^2$ is $3x^2$ a factor? Of which of the expressions $2x$, $3x^2$, $3xy^2$, $6y^2$, $9x^2y^2$ is $6x^2$ a multiple?

3. What values of a will make $9x^2+axy+25y^2$ a perfect square?

4. Fill in the blank in $-\frac{1}{x}-\frac{1}{y}=-\frac{\quad}{xy}$.

5. Solve the equations

$$ax+by=1,\ bx+ay=1.$$

6. A train travels m miles in h hours; how many seconds will it take to travel f feet?

B

1. Resolve into factors

$3x^2-10x-8,\ 81a^4+64b^4,\ 9x^4-82x^2y^2+9y^4.$

2. Write down (i) the square root of $(x+2y)^2-6(x+2y)+9$; and (ii) the square of $x-2y+5$.

3. If $2s=a+b+c$, show that $(s-a)+(s-b)+(s-c)-s=0$.

4. Simplify

$$\frac{a^2+3a+2}{a^2+2a+1}\times\frac{a^2+5a+4}{a^2+7a+10}.$$

5. Solve the equations

$x-2y=5,\ 3y+4z=6,\ 5z+6x=21.$

6. A train goes at the rate of a feet per minute,

(i) How long will it take to go a mile?

(ii) How many yards will it go in an hour?

(iii) How many miles will it go in b hours?

C

1. Resolve into factors

$39x^2-7x-22,\ x^4+2x^2+9,\ x^8-16y^8.$

2. Of which of the pairs of the expressions (i) $6x^2$ and $12x^3$; (ii) $3xy$ and $4x^2y$; (iii) $6x^4y^3$ and $10x^5y^4$ is $2x^2$ a common factor?

3. Find the square root of $\frac{x^2}{y^2}+\frac{y^2}{4x^2}-\frac{x}{y}+\frac{y}{2x}-\frac{3}{4}$.

4. Simplify $\frac{a}{-b}-\frac{-a}{b}+\frac{-a}{-b}$.

5. Solve $2x-\frac{12}{x}=5$.

6. Find a fraction such that if 1 be subtracted from its numerator the value will be $\frac{2}{3}$, and if 6 be added to the denominator the value will be $\frac{1}{2}$.

D

1. Shew that $x-z$ is a factor of $(x-y)^3+(y-z)^3$.

2. Is $2x^2y^3$ a common factor of $2x^2$ and y^3? Why?
Is $2x^2y^2z^2$ a common multiple of $2xy$ and yz? Why?

3. Find the square root of $a^2+\frac{1}{a^2}-2\left(a+\frac{1}{a}\right)+3$.

4. Simplify $\dfrac{\left(1-\frac{b}{a}\right)\left(1+\frac{b}{a}\right)}{\frac{a}{b}+\frac{b}{a}+2}$.

5. Solve $\frac{6}{3x-5}-\frac{1}{x-5}=\frac{2}{2x-5}$.

6. The average of four numbers is a and the average of three numbers is b; what is the fourth number?

E

1. Prove that $(a-b)$, $(b-c)$, $(c-a)$ are factors of $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.

2. If $x+\frac{1}{x}=p$, prove that $x^2+\frac{1}{x^2}=p^2-2$.

3. Simplify $\frac{2+x}{2(x+1)}+\frac{2-x}{2(x-1)}+\frac{x}{x^2+1}$.

4. Solve $\frac{12}{x+2}=6-2\left(\frac{3x+2}{x+1}\right)$.

5. Solve the equations

$$\frac{a}{x}+\frac{b}{y}=0,\ ax+by=c.$$

6. A boy bought a number of oranges for Rs. 2. If he had bought 8 more for the same sum, he would have paid 4 pies less for each. How many did he buy?

F

1. Resolve into factors, and thence find the L.C.M. of $a^2+6ab+5b^2$, $a^3-a^2b-ab^2+b^3$, $a^2+5ab-6b^2$.

2. Find the H.C.F. of $x^3-x^2-8x+12$ and $3x^2-2x-8$.

3. Find the square root of $x(x+1)(x+2)(x+3)+1$.

4. Simplify $\left(\frac{x}{x-y}-\frac{x}{x-z}\right)\div\left(\frac{y}{y-x}-\frac{z}{z-x}\right)$.

5. Solve $\frac{x}{x+1}+\frac{x+1}{x}=2\frac{1}{6}$.

6. A number consists of two digits. When the number is divided by the sum of the digits; the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number.

G

1. Factorise $x^3y^2-x^2y^3+x^2y^2-x+y-1$.

2. Find the L.C.M. of $2x^2-8$, $9x^2-9x+6$ and $6x^2+18x+12$.

3. Prove that $(a-b)^2+(b-c)^2+(c-a)^2+2(a-b)(b-c)+2(b-c)(c-a)+2(c-a)(a-b)=0$.

4. Simplify $\dfrac{a^2}{x(a-x)}+\dfrac{x^2}{a(x-a)}-\dfrac{(a-x)^2}{ax}$.

5. Solve the equation $\dfrac{1}{x-1}-\dfrac{1}{x}+\dfrac{1}{x+4}-\dfrac{1}{x+3}=0$.

6. If m men do as much work as p boys, and n men take a days to finish a piece of work, how long would q boys take ?

H

1. Factorise $a^2x-b^2x-a^2y+b^2y$.

2. Find the L.C.M. of x^3+a^3, x^3-a^3, $x^4+x^2a^2+a^4$, x^2-xa+a^2.

3. Simplify $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x+1}-\dfrac{x}{x^2-1}+\dfrac{1}{x(x^2-1)}$.

4. Solve the equation $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+b}+\dfrac{1}{x}$.

5. Solve
$$x+y-z=3,$$
$$x+z-y=5,$$
$$y+z-x=7.$$

6. A farmer bought an equal number of two kinds of sheep, one kind at Rs. 6 each, the other at Rs. 8 each ; if he had spent his money equally on the two kinds, he would have had three sheep more than he did. How many of each kind did he buy ?

I

1. If $x-\dfrac{1}{x}=1$, prove that $x^3-\dfrac{1}{x^3}=4$.

2. If the product of $x+1$ and $x-6$ is equal to the product of $x+2$ and $x-9$, what is the value of x ?

3. Simplify $\left\{1-\dfrac{4}{x-1}+\dfrac{12}{x-3}\right\}\left\{1+\dfrac{4}{x+1}-\dfrac{12}{x+3}\right\}$.

4. Solve the equation $\frac{3}{5-x}+\frac{2}{4-x}=\frac{8}{x+2}$.

5. Solve
$$\frac{1}{x}+\frac{3}{y}-\frac{2}{z}=6,$$
$$\frac{3}{x}+\frac{1}{z}=5,$$
$$\frac{2}{y}+\frac{5}{z}=16.$$

6. How many seconds will a train travelling at x feet a second take to pass through a station platform y yards long, the length of the train being z yards ?

J

1. Factorise $x^3-xy(x+y)+y^3$.

2. If $y+z=2a$, $z+x=2b$, $x+y=2c$, express x, y, z in terms of a, b, c.

3. Simplify $\frac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}\div\frac{x^4-y^4}{2xy(x-y)}$.

4. Solve the equation $(x-a)(x-b)=ab-x^2$.

5. Solve
$$3x+4y+2z=19,$$
$$7x-3y=15,$$
$$7z-4y=-1.$$

6. A man is paid x annas for each day that he works and is fined y pies for each day that he is absent. He works for 4 days out of 6 in a week. What are his week's wages in annas ?

K

1. Factorise $(a^2-b^2)^2-2(a^2+b^2)+1$.

2. Find the H.C.F. of $x^3+6x^2+11x+6$ and $x^3+9x^2+27x+27$.

3. Simplify $\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}$.

4. Solve the equation $\frac{x-\frac{1}{3}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}$.

5. Solve
$$(a+b)x+(a-b)y=2a,$$
$$(a-b)x+(a+b)y=2b.$$

6. An officer can form his men into a hollow square 5 deep and also into a hollow square 6 deep, but the front in the latter formation contains 4 men less than in the former : find the number of men.

L

1. If $f(x)=x^3-2x^2+3x-4$, find the values of $f(2)$, $f(-3)$ and $f(0)$.

2. Find the H.C.F. of $x^4-15x^2+28x-12$ and $2x^3-15x+14$, and hence reduce the fraction $\dfrac{x^4-15x^2+28x-12}{2x^3-15x+14}$ to its lowest terms.

3. Simplify $\dfrac{x^2}{(x^2-y^2)(x^2-z^2)}+\dfrac{y^2}{(y^2-z^2)(y^2-x^2)}+\dfrac{z^2}{(z^2-x^2)(z^2-y^2)}$.

4. Solve the equation $\dfrac{x-8}{x-10}-\dfrac{x-5}{x-7}=\dfrac{x-7}{x-9}-\dfrac{x-4}{x-6}$.

5. Solve $\dfrac{3}{2x+3y}+\dfrac{4}{2x-3y}=4\dfrac{3}{7}$,

$$\dfrac{7}{2x+3y}+\dfrac{1}{2x-3y}=2.$$

[Let $2x+3y=u$ and $2x-3y=v$, and then solve for u and v.]

6. Two persons started at the same time from A, one rode on horseback at the rate of $7\frac{1}{2}$ miles an hour and arrived at B 30 minutes later than the other who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between A and B.

M

1. Resolve x^3-4x^2+x+6 into factors. Write down the values of x for which the expression vanishes.

2. Find the L.C.M. of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1.

3. Prove that $(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)$

$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

4. Simplify $\dfrac{x(x+1)+1}{(x-y)(x-z)}+\dfrac{y(y+1)+1}{(y-x)(y-z)}+\dfrac{z(z+1)+1}{(z-x)(z-y)}$.

5. Solve $\dfrac{6}{2x+3y}+\dfrac{23}{3x-4y}=\dfrac{5}{2}$,

$$\dfrac{4}{2x+3y}+\dfrac{1}{3x-4y}=1\dfrac{1}{23}.$$

6. The sum of two fractions which are reciprocals of each other is $2\frac{1}{6}$. Find their difference.

N

1. Factorise $a(b^3-c^3)+bc(b^2-c^2)+b^3(b-c)$.

2. Find the H.C.F. of $6x^4+2x^3+19x^2+8x+21$ and $4x^4-2x^3+10x^2+x+15$.

3. Can you find by inspection a root of the equation

$$(2x+3)(x-4)=(x-4)\ ?$$

How do you know this is root ? Solve the equation completely.

4. Show that $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

5. Simplify $\dfrac{a^2-(b-c)^2}{(a-b)(a-c)}+\dfrac{b^2-(c-a)^2}{(b-c)(b-a)}+\dfrac{c^2-(a-b)^2}{(c-a)(c-b)}$.

6. A mixture is made up of a gallons at x rupees per gallon, b gallons at y rupees per gallon and c gallons at z rupees per gallon ; what is the value of the mixture per gallon ?

O

1. Factorise $(a-b)^3+(b-c)^3+(c-a)^3$.

2. Find the H.C.F. and L.C.M. of $x^5+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9 and $x^4+2x^3-5x^2-6x+9$.

3. Prove that

$$(y-z)(y+z-x)+(z-x)(z+x-y)+(x-y)(x+y-z)=0.$$

4. Simplify $\dfrac{a(x-b)(x-c)}{bc(a-b)(a-c)}+\dfrac{b(x-c)(x-a)}{ca(b-c)(b-a)}+\dfrac{c(x-a)(x-b)}{ab(c-a)(c-b)}$.

5. Solve the equation

$$\frac{a+c}{x-2b}-\frac{b+c}{x-2a}=\frac{a-c}{x+2b}-\frac{b-c}{x+2a}.$$

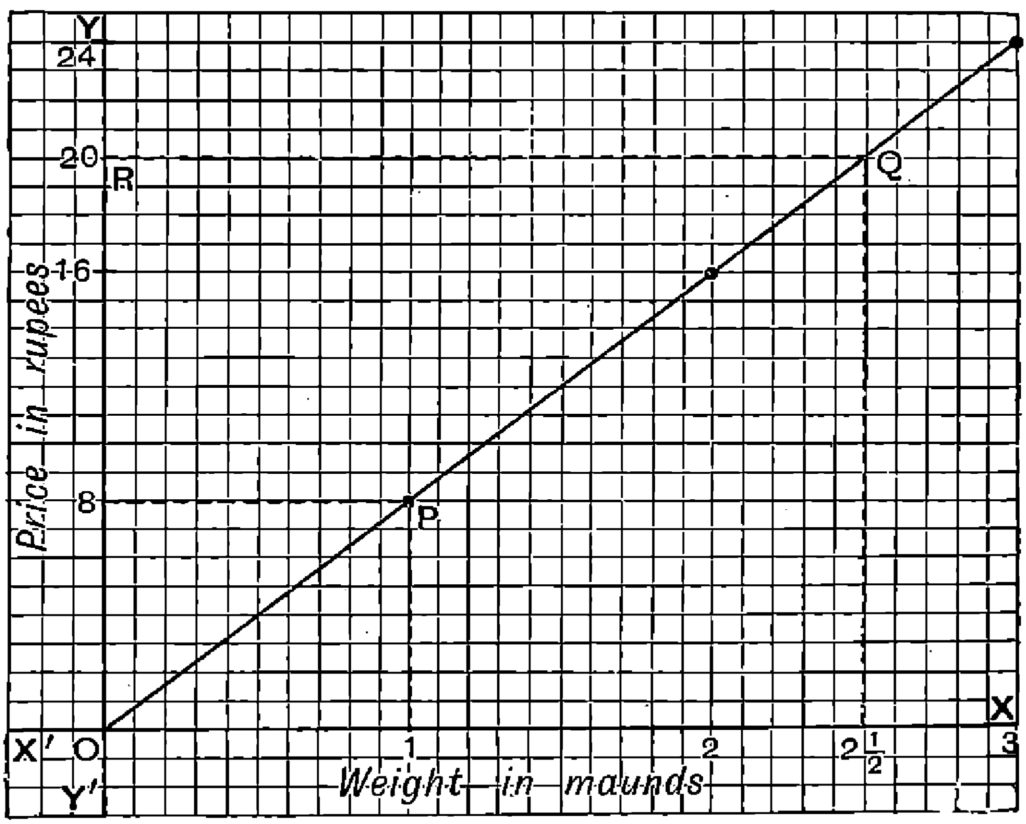
6. A man buys a certain number of oranges at 3 for 2d. and one-third of that number at 2 for 1d. ; at what price must he sell them to get 20% profit ? If his profit be 5s. 4d., find the number bought.

CHAPTER XXXII

APPLICATION OF GRAPHS IN PROBLEMS

173. In Chapter XVII we saw that the relation between two variable quantities x and y given by an equation could be represented graphically by a straight line. Hence, we can conclude that *in problems which involve two variable quantities such that a change in one of them causes a proportional change in the other, the relation between them can always be represented by means of a graph.*

EXAMPLE 1. *If* 1 *maund of rice costs* Rs. 8, *construct a graph from which you can read off the price of any number of maunds and from your graph read off the price of* $2\frac{1}{2}$ *maunds of rice.*



This problem has two variable quantities 'maunds' and 'rupees', such that a change in one produces a corresponding change in the other.

Take XOX′ and YOY′ as the axes of co-ordinates.

Measure 'maunds' along OX, taking 1 inch to represent 1 maund ; and 'rupees' along OY, taking one-tenth of an inch to represent 1 rupee.

Since the price of 1 maund of rice is Rs. 8, first measure along OX a length equal to 1 inch to represent 1 maund and then a distance equal to ·8″ to represent 8 rupees parallel to OY and mark a point P there.

Join OP. Then OP is the required graph. Since the straight line can be produced to any length, we can read off from the graph the price of any number of maunds of rice, though our range of maunds and prices is limited owing to the small size of our diagram.

To find the price of $2\frac{1}{2}$ maunds of rice, measure a length of $2\frac{1}{2}$ inches along OX to represent $2\frac{1}{2}$ maunds and then draw a straight line parallel to OY to meet OP produced in Q. From Q draw QR perpendicular to OY. Then OR represents the required price which is Rs. 20.

NOTE. The graph can also be drawn by first expressing the relation between 'maunds' and 'rupees' in the form of an alegebraic equation thus :

Suppose the price of x maunds of rice$=y$ rupees,

,, 1 maund ,, ,, $=\frac{y}{x}$

$$\frac{y}{x}=8,$$

$$\therefore \quad y=8x.$$

This is the equation of a straight line which passes through the origin. Again, when $x=1$, $y=8$, the point (1, 8) also lies on the graph.

Now taking the scale 1 inch to 1 maund along OX and 1 inch to 10 rupees along OY, we plot the point (1, 8), which is P in the figure. Joining P to the origin we get the required graph.

EXAMPLE 2. *Given that 1 inch=2·54 centimetres, construct a graph to convert any number of inches into centimetres, or centimetres into inches. Express $1\frac{1}{2}$ inches in centimetres and 5 centimetres in inches.*

This problem has two variable quantities 'inches' and 'centimetres' which increase and decrease proportionally.

Take XOX′ and YOY′ as the axes of co-ordinates.

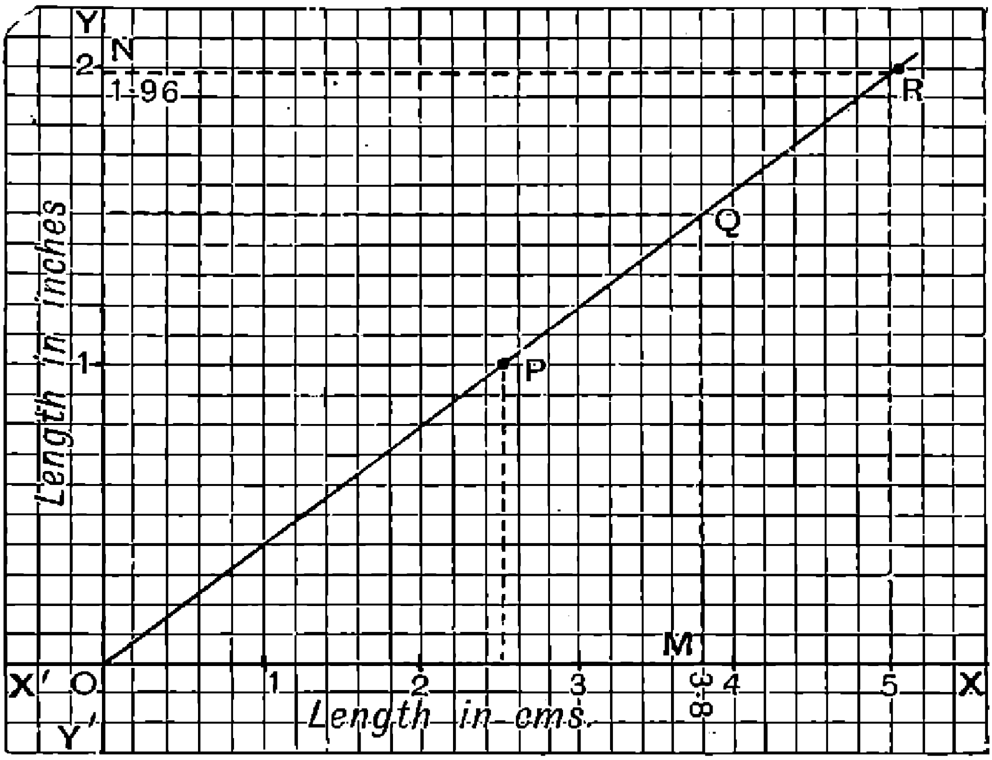
Measure 'centimetres' along OX, taking ·5 inch to represent 1 centimetre ; and 'inches' along OY, taking 1 inch to represent 1 inch.

Since 1 inch=2·54 cm. first measure along OX a length equal to 1·27 in. to represent 2·54 cm. and then a distance equal to 1 in. to represent 1 in. parallel to OY and mark a point P there.

Join OP, then OP is the required graph.

To express $1\frac{1}{2}$ inches in centimetres, measure a length of $1\frac{1}{2}$ in. along OY and then draw a straight line parallel to OX to meet OP in Q.

From Q draw QM perpendicular to OX. Then OM represents the required number of centimetres which is 3·8 cm. approximately.



Again, to express 5 cm. in inches, measure a length of 2·5 in. along OX to represent 5 cm., and then draw a straight line parallel to OY to meet OP in R. From R draw RN perpendicular to OY. Then ON represents the required number of inches which is 1·96 in. approximately.

NOTE. The relation between 'inches' and 'centimetres' can also be expressed in the form of an algebraic equation thus :

$$\text{Suppose } x \text{ cm.} = y \text{ in.},$$

$$1 \text{ cm.} = \frac{y}{x} \text{ in.}$$

$$2·54 \text{ cm.} = \frac{2·54y}{x} \text{ in.}$$

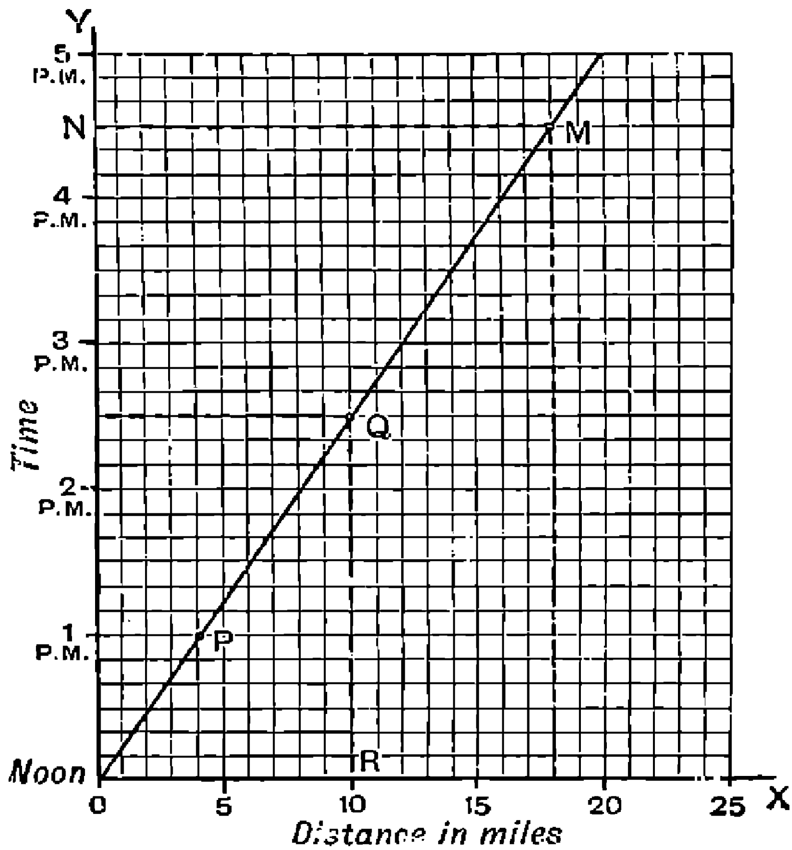
$$\frac{2·54y}{x} = 1,$$

$$y = \frac{1}{2·54} x.$$

This is the equation of a straight line which passes through the origin. Again, when $x=2·54$, $y=1$, the point (2·54, 1) also lies on the graph.

Now taking the scale 1 inch to 2 centimetres along OX and 1 inch to 1 inch along OY, we plot the point (2·54, 1) which is P in the figure, and joining it to the origin we get the required graph.

EXAMPLE 3. *A man starts at noon to walk at the rate of 4 miles an hour. Draw the graph of his motion and from your graph read off his distance at 2.30 p.m., and also find at what time he will be 18 miles from the starting point.*



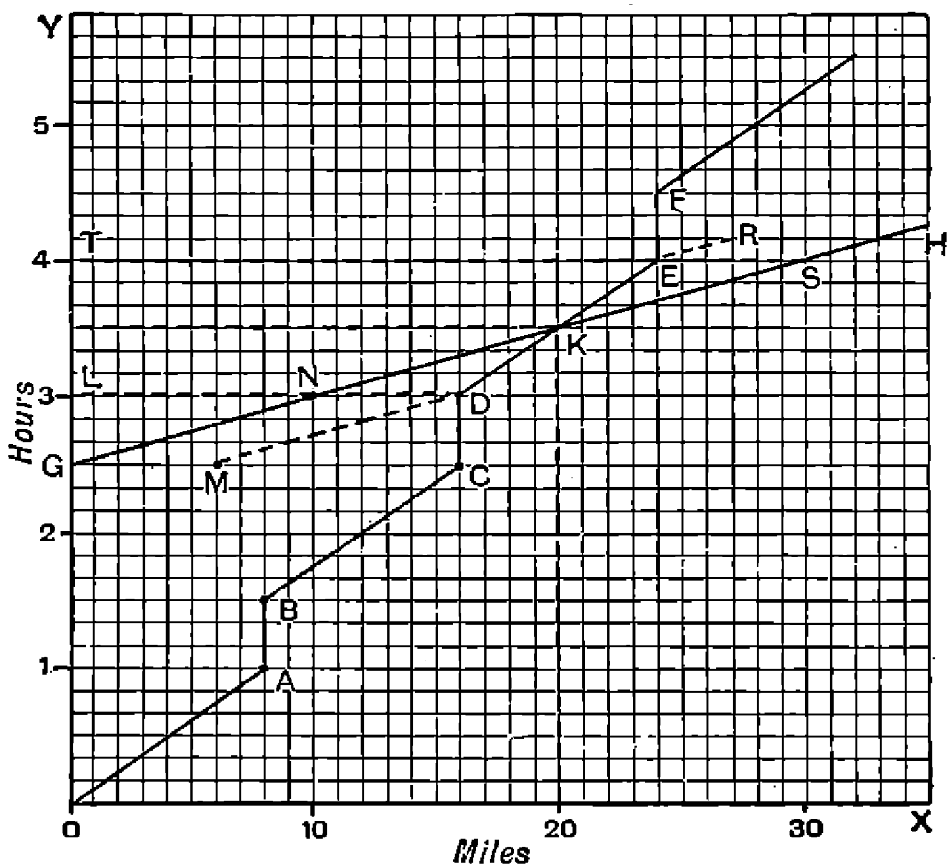
Let OX and OY be the axes of co-ordinates. Along OX take 1 unit to represent 1 mile and along OY 6 units to represent 1 hour.

Since the rate of the man's walk is 4 miles an hour, first take 4 units along OX to represent 4 miles and then 6 units parallel to OY to represent 1 hour and mark a point P there. Join OP. Then OP is the required graph.

To find the distance at 2.30 p.m., first take 15 units along OY to represent 2.30 p.m. and then draw a straight line parallel to OX to meet OP produced in Q. From Q draw QR perpendicular to OX. Then OR represents the required distance which is 10 miles.

Again, to find the time when he will be 18 miles from the starting point O, first take 18 units along OX to represent 18 miles and then draw a straight line parallel to OY to meet OP in M. From M draw MN perpendicular to OY. Then OY represents the required time which is 4.30 p.m.

EXAMPLE 4. *P starts on a cycle at 8 miles an hour stopping for half an hour at the end of every hour; after $2\frac{1}{2}$ hours, Q starts from the same place and motors without stopping at 20 miles an hour. Draw the graphs of their motion and from your graph find (i) when and where Q overtakes P; (ii) at what times they are 6 miles apart.*



Take OX and OY as axes of co-ordinates and measure distances horizontally along OX, taking 1 unit to represent 1 mile and time vertically along OY, taking 6 units to represent 1 hour.

Since P has gone 8 miles in 1 hour, therefore OA is the graph of his motion for the first hour. In the next half hour he makes no advance, therefore AB is the graph of his first stoppage. Similarly BC is the graph after the first stoppage and CD is the graph of his second stoppage, and so on.

NOTE. It should be noted that during a stoppage the time advances, while the distance from the starting place (measured horizontally along OX) remains the same.

Now Q starts $2\frac{1}{2}$ hours after P; we mark a point G along OY to represent this time, and as he motors at the rate of 20 miles an hour without stoppages, the graph of his motion is GH.

Since the two graphs intersect at K, therefore the *abscissa* of K gives the *place* and the *ordinate* gives the *time* of their meeting.

Hence Q overtakes P 20 miles from the starting point, and $3\frac{1}{2}$ hours after the start of P.

To find when they are 6 miles apart, we have to determine when the horizontal distance between the graphs represents 6 miles.

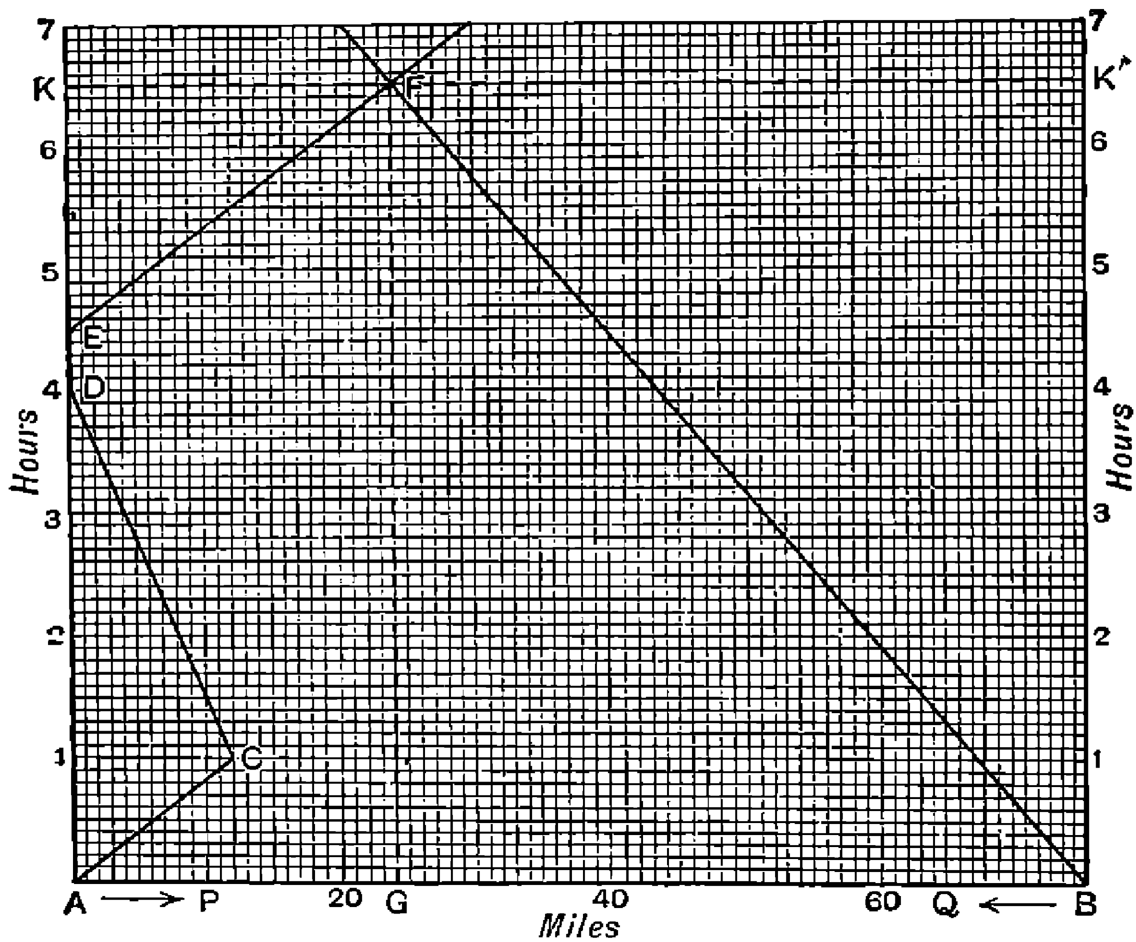
Take GM horizontally equal to 6 miles, draw MD parallel to GK to meet the graph of P's motion at D. From D draw DNL horizontally to meet GK in N and OY in L.

Since ND = GM = 6 miles, therefore OL represents the time which is 3 hours after P's start when they are 6 miles apart.

Again, take a point H on GK where it passes a corner of a square and measure HR horizontally equal to 6 miles. Draw RE parallel to HK to meet the graph of P's motion at E. Draw ES horizontally to meet HK in S and OY in T.

Since ES = RH = 6 miles, therefore OT represents the time, which is 4 hours after P's start, when they are again 6 miles apart.

EXAMPLE 5. *P starts on a cycle at the rate of* 12 *miles an hour from Allahabad to go to Benares, a distance of* 76 *miles. After* 1 *hour, his*



cycle gets punctured ; he then walks back to Allahabad at the rate of 4 *miles an hour. In half an hour he gets the puncture repaired and then cycles again for Benares at the rate of* 12 *miles an hour. Q starts from*

Benares to Allahabad at the same time as P first started from Allahabad and travels on horseback at the rate of 8 miles an hour. Draw the graphs of their motion and find when they will meet.

On squared paper take two points A and B to represent Allahabad and Benares, 7.6 cm. apart, where 1 cm. represents 10 miles. At A and B draw AK and BK′ perpendiculars to AB and take 1 cm. to represent 1 hour along these lines.

Since P starts from A and goes towards B, AC is the graph of his first hour's cycling ; CD is the graph of his walking back to Allahabad ; DE is the graph of his half hour's stay at Allahabad ; and EF is the graph of his cycling again to Benares.

NOTE. It should be noted that when he walks back to Allahabad his direction is reversed.

Now Q starts from B and rides towards A, therefore BF is the graph of his motion.

The two graphs intersect at F. From F draw FG perpendicular to AB, and through F draw KFK′ parallel to AB to meet AK and BK′ in K and K′ respectively. Now AK or BK′ represents the time when they meet and AG and BG the distances from Allahabad and Benares where they meet.

Hence from the diagram, we find that they meet after $6\frac{1}{2}$ hours at a distance of 24 miles from Allahabad or 52 miles from Benares.

EXAMPLES CI

1. If 4 seers of rice cost Re. 1, construct a graph from which you can read off the price of 6 seers of rice and the quantity of rice that can be purchased for Rs. 2. 8a. What is the equation connecting the price of rice in rupees and the number of seers ?

2. If Re. 1 is worth 1s. 6d., construct a graph from which you can read off the value of any number of rupees in shillings and pence. Read off the value of Rs. 3 in shillings and 3s. in rupees.

3. Given that 1 cm. = ·39 in., construct a graph and read off the number of centimetres in 1 inch and the number of inches in $3\frac{1}{2}$ centimetres.

4. If 6 oranges cost 4 annas, construct a graph and from your graph find the number of oranges that can be had for Re. 1, and the price of 20 oranges. What is the equation connecting the number of oranges with their price in annas ?

5. A train travels uniformly at the rate of 40 miles an hour, draw the graph of its motion and from your graph find how far it will go in 15 minutes and the time it will take in going 25 miles.

6. If Krishna runs at the rate of 100 yards in 30 seconds, draw the graph of his motion, and from your graph find how far he will run in 25 seconds and the time he will take in running 110 yards.

7. A labourer gets Rs. 14 a month, construct a graph of his weekly wages and from your graph read off his wages for 5 weeks.

8. If 1 yard of cloth costs 12 annas, construct a graph from which you can read off the price of cloth in feet, and from your graph find the price of 5 feet of cloth and the number of feet of cloth that can be had for Re. 1.

9. If 1 cubic inch is equivalent to 16·4 cubic centimetres, construct a graph and read off the value of 70 cu. cm. in cubic inches and the value of 3 cu. in. in cubic centimetres.

10. A man walks at the rate of 3 miles an hour, draw the graph of his motion and from your graph find the time he will take in walking 12 miles.

11. A boy cycles at the rate of 10 miles an hour, draw the graph of his motion and from your graph find (i) how far he will go in $2\frac{1}{2}$ hours ; and (ii) what time he will take in going 34 miles.

12. A man starts at the rate of $3\frac{1}{2}$ miles an hour. An hour later another man starts from the same place and cycles at the rate of $10\frac{1}{2}$ miles an hour. Draw the graphs of their motion and find when and where the second man will overtake the first.

13. A train goes from Allahabad to Fyzabad, a distance of 98 miles, at the rate of 25 miles an hour ; after half an hour another train leaves Fyzabad for Allahabad at the rate of 32 miles an hour. Draw the graphs of their motion and find when and where they will pass one another.

14. A ship goes from Bombay to Karachi, a distance of 560 miles, at the rate af 31 miles an hour ; at the same time another ship leaves Karachi for Bombay at the rate of 25 miles an hour. Draw the graphs of their motion and find (i) when and where they will pass one another ; and (ii) at what times they are 28 miles apart.

15. A man starts from Bareilly and cycles to Shahjehanpore, a distance of 42 miles, at the rate of 8 miles an hour ; after $\frac{3}{4}$ hour, another man starts from Shahjehanpore and motors to Bareilly at the rate of 32 miles an hour. Draw the graphs of their motion and find (i) when and where they will meet ; and (ii) at what times they are 10 miles apart.

16. *A* starts on horseback at the rate of 16 miles an hour stopping for half an hour at the end of every hour ; after $2\frac{1}{2}$ hours, *B* starts from the same place and follows him in a motor car without stopping at the rate of 40 miles an hour. Draw the graphs of their motion and find when and where *B* will overtake *A*.

17. *A* starts from Lucknow at noon and motors to Roorkee, a distance of 300 miles, at the rate of 50 miles an hour ; after 2 hours he meets with an accident and has to stay for 1 hour, after which he starts again at the rate of 30 miles an hour. At 2.40 p.m., a second man motors from Roorkee to Lucknow at the rate of 40 miles an hour without stopping. Draw the graphs of their motion and find (i) when

and where they will meet ; and (ii) at what time before meeting they will be 35 miles apart.

18. P goes from A to B, a distance of 35 miles, at the rate of 10 miles an hour. On reaching B he rests for half an hour and then starts back for A at the same rate. After one hour he meets Q who had started from A one hour after P had left A. Draw the graphs of their motion and find the rate of Q.

19. P starts from A and cycles to B, a distance of 30 miles, at the rate of 6 miles an hour. After 1 hour his cycle gets punctured ; he then walks back to A at the rate of 3 miles an hour. In half an hour he gets the puncture repaired and then cycles again for B at the rate of 12 miles an hour. Q starts from B to A at the same time when P first started from A and walks at the rate of 4 miles an hour. Draw the graphs of their motion and find (i) when and where they will meet ; and (ii) at what times they will be 8 miles apart.

20. At noon A starts to cycle at the rate of 12 miles an hour from Moradabad to Saharanpore, a distance of 120 miles. After 2 hours his cycle gets punctured. At 2.6 p.m. a motorist going to Moradabad at the rate of $26\frac{2}{3}$ miles an hour picks him up. On reaching Moradabad he starts at once for Saharanpore again on another cycle at the rate of 16 miles an hour. B also starts at noon on a cycle from Saharanpore to Moradabad at the rate of 16 miles an hour. After going 2 hours, he rests for half an hour and then resumes his journey in a motor car at the rate of 48 miles an hour. Draw the graphs of their motion, and find (i) when and where they will meet ; and (ii) at what time before meeting they are 32 miles apart.

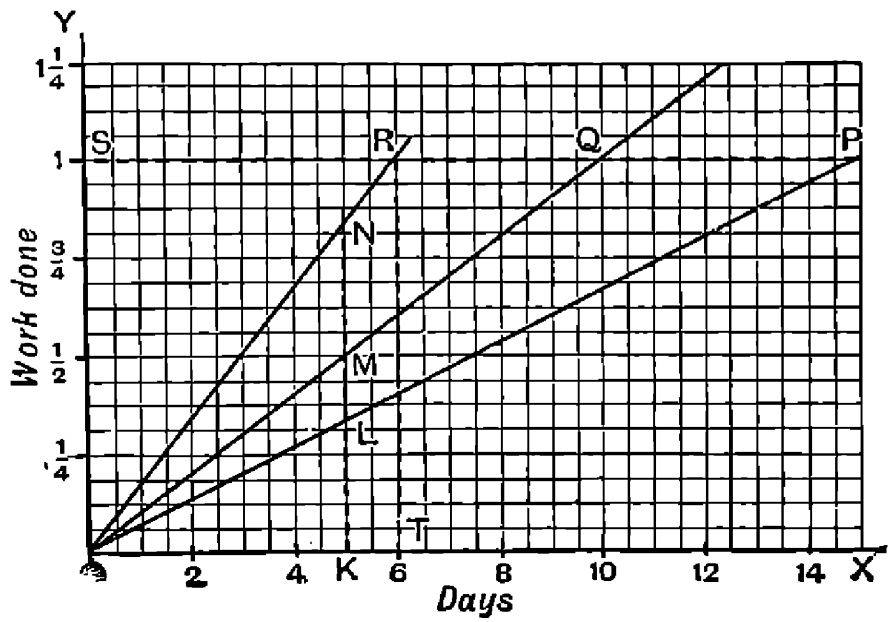
21. The following table shows the distances (in miles) of certain stations from Agra, and the times (in hours and minutes) of two trains, one going from Agra to Tundla and the other from Tundla to Agra. Supposing each run to be made at a constant speed, draw the graphs of their motion and find when and at what distance from Agra they will pass one another :

Distance	7.02	arrive	Agra	depart	6.00
2	6.55	depart ↑	Jumna Bridge	arrive ↓	6.08
	6.50	arrive		depart	6.11
8		depart	Kuberpore	arrive	6.40
		arrive		depart	6.42
12	...	depart	Etmadpore	arrive	6.50
	...	arrive		depart	6.52
15	6.22	depart	Tundla	arrive	7.00

22. The following table shows the distances (in miles) of certain stations from Bareilly, and the times (in hours and minutes) of two trains, one going from Bareilly to Kasgunj and the other from Kasgunj to Bareilly. Supposing each run to be made at a constant speed, draw the graphs of their motion and find when and at what distance from Bareilly they will pass one another :

Distance						
	13.35	arrive	Bareilly	depart	8.45
27	11.30 11.10	depart arrive	Budaun	arrive depart	10.25 10.40
55	8.35 8.30	depart arrive	Soron	arrive depart	12.25 12.30
64	7.30	depart	Kasgunj	arrive	13.15

174. EXAMPLE 1. *A can do a piece of work in 15 days and B can do it in 10 days. Find graphically in how many days both of them will do it together.*



Let OX and OY be the axes of co-ordinates. Along OX take 2 divisions to represent 1 day and along OY take 16 divisions to represent the whole or one work.

From OY measure OS equal to 16 divisions and draw SP parallel to OX. Now SP will represent one work. Since A can do the work in 15 days, OP will represent the graph of A's rate of work and similarly OQ will represent the graph of B's rate of work.

To find the graph of their combined rates of work, take any point L on OP and draw LK perpendicular to OX and produce KL to meet OQ in M. Now in time OK, the amount of work done by A is KL and by B is KM.

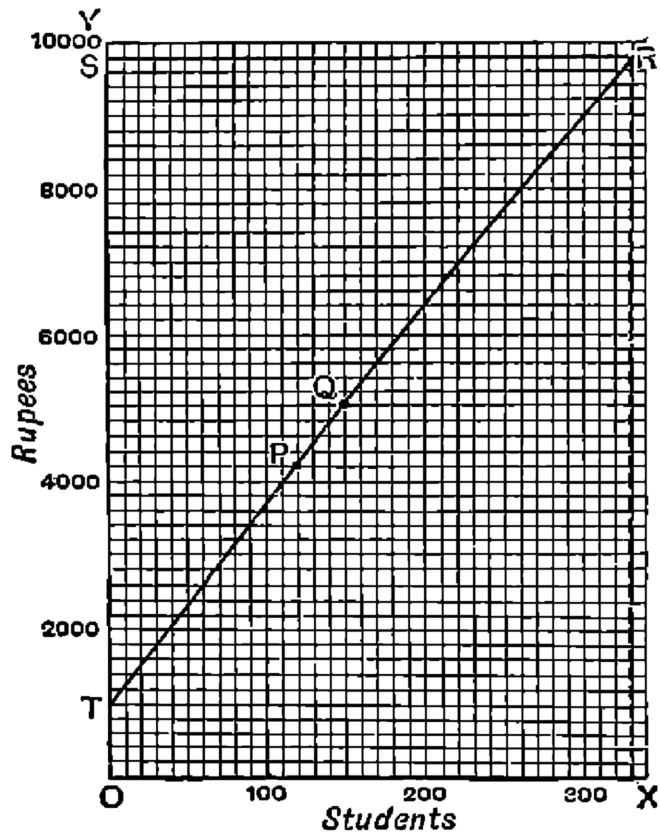
Produce KM to N so that MN=KL. Then KN will represent the combined work of A and B in time OK.

Join ON and produce it to meet SP in R. Then OR is the graph of their combined work.

From R draw RT perpendicular to OX. Since RT represents 6 days, hence the time required for both of them to do the work jointly is 6 days.

EXAMPLE 2. *The expenses of a school are partly constant and partly proportional to the number of students. If the expenses are* Rs. 4200 *for* 120 *students and* Rs. 5000 *for* 150 *students, find graphically the expenses for* 330 *students and also the constant expenses.*

Let OX and OY be the axes of co-ordinates. Along OX take one small division to represent 10 students and along OY one small division to represent Rs. 200.

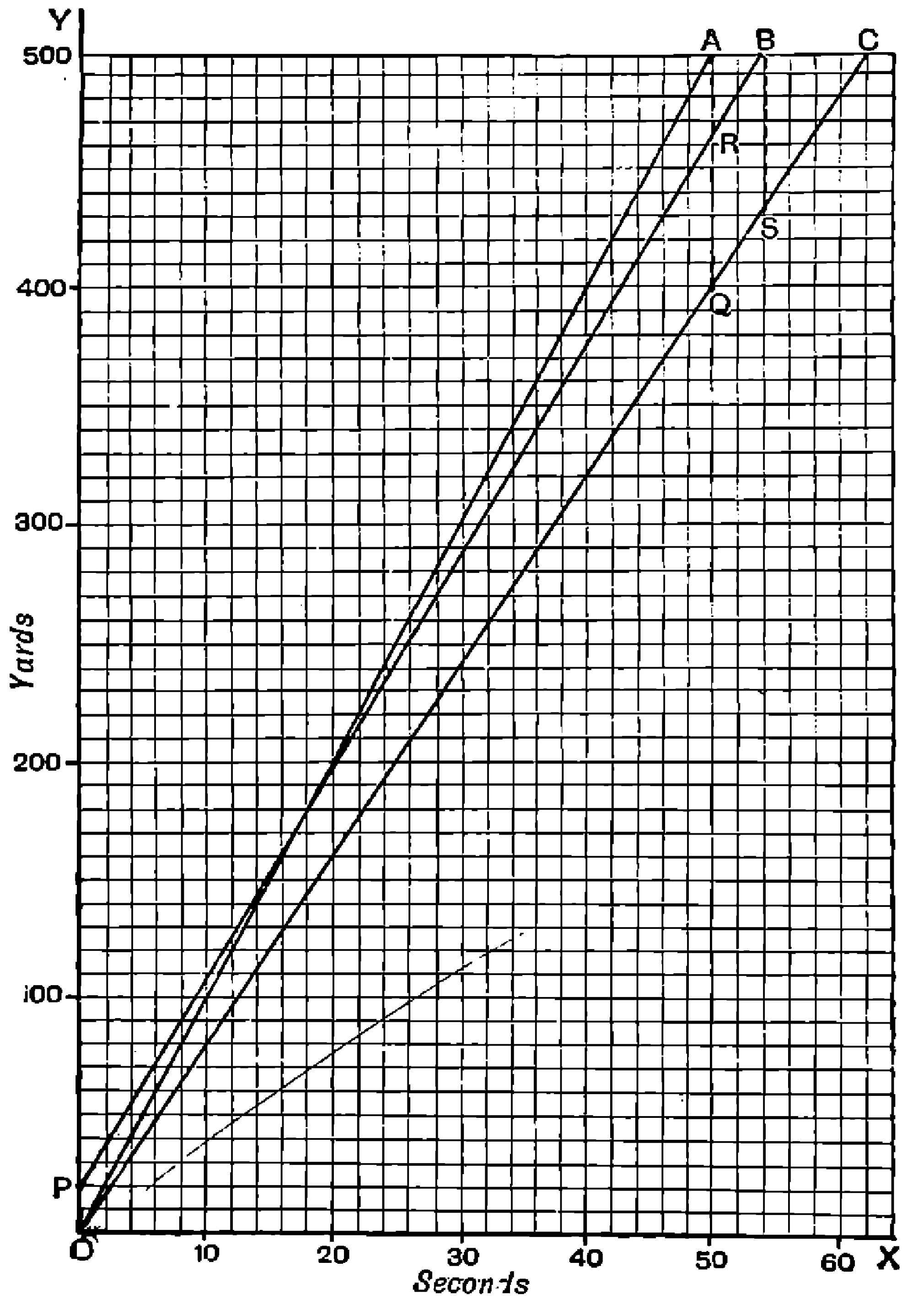


Since the expenses for 120 students are Rs. 4200, first measure a distance equal to 12 divisions along OX to represent 120 students and then vertically upwards a distance equal to 21 divisions to represent Rs. 4200 and mark a point P. Similarly mark Q. Join PQ and produce it both ways.

To find the expenses for 330 students, first count 33 divisions along OX and then draw a straight line vertically upwards to meet PQ produced at R. From R draw RS perpendicular to OY to meet OY in S. Since OS represents Rs. 9800, hence the expenses for 330 students are Rs. 9800.

Again, since PQ meets OY in T and OT represents Rs. 1000, hence the constant expenses are Rs. 1000.

EXAMPLE 3. *A, B and C run a race of* 500 *yards. A and C start from scratch ; A runs the distance in* 50 *seconds and beats C by* 100 *yards. B has* 20 *yards' start and beats A by* 4 *seconds. If they are running* *with constant speed, find graphically by how many yards A beats B and B beats C in the race and how many seconds after A, C reaches the winning post.*



Let OX and OY be the axes of co-ordinates. Along OX take 1 inch to represent 20 seconds and along OY take 1 inch to represent 100 yards.

Let O be the starting point of A and C. Along OY measure OP equal to ·2″ to represent 20 yards. Then P is the starting point of B.

Now measure horizontally along OX a length of 2·5″ to represent 50 seconds and then vertically upwards parallel to OY a distance equal to 5″ to represent 500 yards and mark a point A. Join OA. Then OA is the graph of A's motion.

Measure AQ from A vertically downwards equal to 1″ to represent 100 yards and mark a point Q, then OQ is the graph of C's motion.

Along YA measure a length AB equal to ·2″ to represent 4 seconds. Join PB, then PB is the graph of B's motion.

Let AQ meet PB in R. Now when A reaches the winning post, B will be at R. Since AR measures 3·55″ or 35·5 yards, therefore A beats B by $35\frac{1}{2}$ yards.

Produce OQ to meet YA at C. Since AC measures 6·25″ which represents $12\frac{1}{2}$ seconds, therefore C reaches the winning post $12\frac{1}{2}$ seconds after A.

From B draw BS parallel to YO to meet OC in S. Now when B reaches the winning post, C will be at S. Since BS measures 6·8″ or 68 yards, therefore B beats C by 68 yards.

EXAMPLES CII

1. *A* can do a piece of work in 3 days and *B* can do it in 6 days. Find graphically in how many days both of them will do it together.

2. Two pipes can fill a tank in 10 hours and 30 hours respectively. In how many hours will both of them fill it together ?

3. Two pipes *A* and *B* can fill a cistern in 3 minutes and 5 minutes respectively, and *C* can empty it in $7\frac{1}{2}$ minutes. If all the pipes be opened together, in what time will the cistern be filled ?

4. In a race of 200 yards, *A* beats *B* by 20 yards, and *B* beats *C* by 8 yards. Find graphically by how many yards *A* will beat *C* in a race of 250 yards.

5. *A* runs a mile in 4 minutes, and *B* in 4 minutes and 24 seconds. Find graphically, how many yards' start *A* can give *B* in a race of 220 yards that there may be a dead heat.

6. *A*, *B* and *C* run a race of 400 yards. *A* and *C* start from scratch ; *A* runs the distance in 40 seconds and beats *C* by 80 yards. *B* has 16 yards' start and is beaten by *A* by $3\frac{1}{3}$ seconds. If they are running with constant speed, find graphically (i) by how many yards *A* beats *B* ; (ii) by how many yards *B* beats *C* ; and (iii) how many seconds after *A*, *C* reaches the winning post.

7. *A*, *B* and *C* run a race of 150 yards. *A* and *B* start from scratch ; *A* runs the distance in 20 seconds and beats *B* by 30 yards. *C* has 6 yards' start and beats *A* by 2 seconds. If they are running with constant speed, find graphically the positions of *A* and *B* when *C* reaches the winning post.

8. The expenses of a hospital are partly constant and partly proportional to the number of patients residing therein. If the expenses for 105 patients are Rs. 2600 and for 180 patients Rs. 4100, find graphically the expenses for 150 patients and the constant expenses.

9. The expenses of a hotel are partly constant and partly proportional to the number of persons staying therein. If the expenses for 50 persons are Rs. 1200 and for 85 persons Rs. 1725, find graphically the expenses for 60 persons and the constant expenses.

10. In printing a book it costs a publisher Rs. 600 to prepare the type and Rs. 2 to print each copy. Find graphically (i) the cost of printing 2500 copies ; and (ii) the number of copies costing Rs. 6200.

11. Two clerks are appointed on fixed monthly salaries and they get fixed increments every year. The first clerk gets Rs. 130 p.m. at the beginning of the third year and Rs. 190 p.m. at the beginning of the seventh year of his service. The second clerk gets Rs. 165 p.m. at the beginning of the fifth year and Rs. 195 p.m. at the beginning of the eighth year of his service. Find graphically (i) in what year they will get equal salaries ; and (ii) their starting salaries.

CHAPTER XXXIII

GRAPHS OF STATISTICS

175. We have already seen that if two quantities are so related that a change in one causes a corresponding change in the other, the relation of the two may be set down in the form of a graph. So far we have dealt with only those cases in which this variation between the two quantities was based on some mathematical law and we could find as many points on our graph as we pleased. In this chapter we shall consider those cases in which the values of the two quantities have been obtained by *direct observation* or *experiment*. Thus we shall have to consider only a limited number of corresponding values of the two quantities.

EXAMPLE 1. *The following table gives the average monthly rainfall (in inches) of Meerut. Construct a graph to illustrate the fluctuation in the rainfall from month to month* :

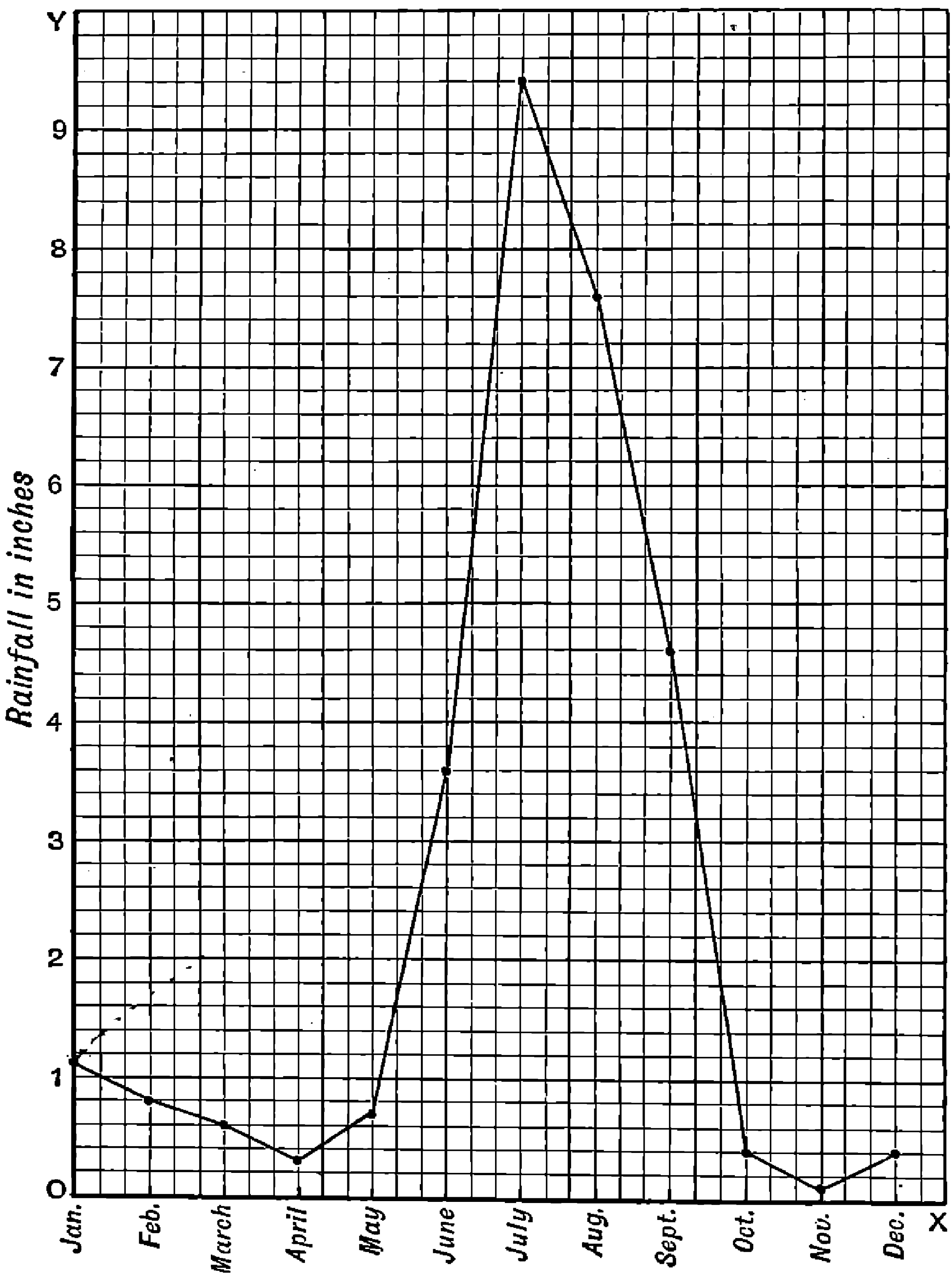
Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1·1	·8	·6	·3	·7	3·6	9·4	7·6	4·6	·4	·1	·4

Let years be represented along OX, taking ·3 inch to represent each interval of one month beginning with January, and let the rainfall be represented along OY, taking 1 inch to represent two inches of rainfall.

Here the left hand vertical line OY represents the month of January and the rainfall in January is 1·1 inches. We show this by marking on it a point at a distance of ·55 inch above OX.

In February the rainfall is ·8 inch. On the fourth vertical line from the left which represents the month of February, mark a point at a distance of ·4 inch above OX.

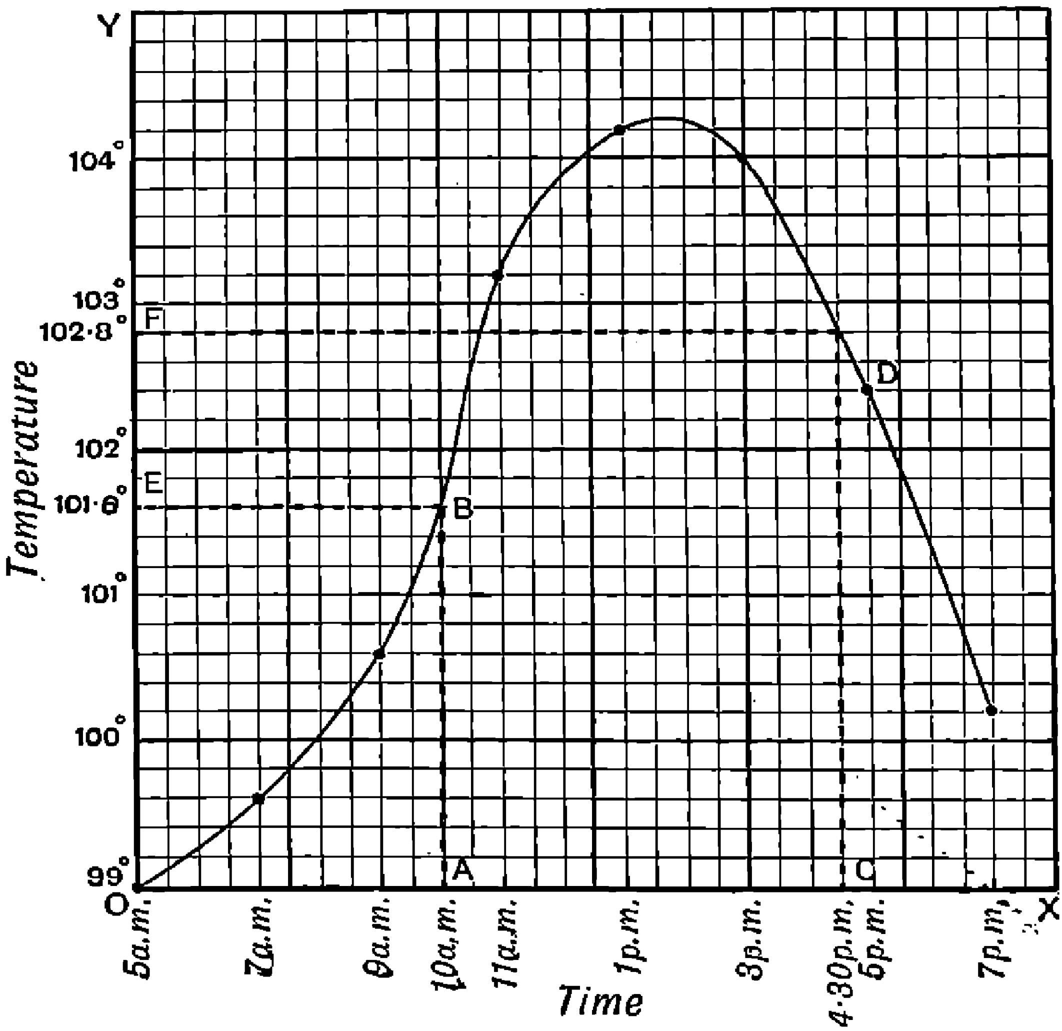
Similarly on the verticals for March, April, May, etc., mark points at distances of ·3 in., ·15 in., ·35 in., etc., above OX. Now join these points consecutively by *straight lines*, the first point being joined to the second, the second to the third, the third to the fourth, and so on. The resulting *broken line* is the required graph, which shows the fluctuation in the rainfall from month to month.


Y
9
8
7
6
5
4
3
2
1
O
Rainfall in inches
Jan.
Feb.
March
April
May
June
July
Aug.
Sept.
Oct.
Nov.
Dec.
X

EXAMPLE 2. In a certain hospital, the temperature of an indoor patient was recorded as follows :

At	5 a.m.	99·0° F.	At	1 p.m.	104·2° F.
	7 a.m.	99·6° F.		3 p.m.	104·0° F.
	9 a.m.	100·6° F.		5 p.m.	102·4° F.
	11 a.m.	103·0° F.		7 p.m.	100·2° F.

Draw a graph showing the temperature of the patient at different hours of the day and from your graph read off the temperature of the patient at 10 a.m. and 4.30 p.m.



Let hours be represented along OX, taking 1 inch to represent 5 hours. *i.e.*, ·2″ to represent 1 hour beginning at 5 a.m., and let the temperature (in degrees) be represented along OY, taking 1 inch to represent 2°, beginning at 99°.

After plotting the points given in the data, we join them by a *continuous curve* which will be the required graph.

NOTE. In the previous example consecutive points were joined by straight lines. In this example, consecutive points are joined by a continuous curve, because a meaning can be given to points intermediate to those that have been marked. For instance, anytime between 5 a.m. and 7 p.m. has a corresponding temperature in degrees. This is not the case in the previous example, for there can be no rainfall figure for any point between May and June, say, since there is no month between May and June.

Students should remember that in drawing observational graphs they should consider whether any meaning can be given to intermediate points ; if there is a meaning the plotted points should be joined by a continuous curve ; if there is no meaning, the points should be joined by a broken line.

We know by actual experience that the changes in the temperature are not sudden or abrupt, but gradual, and moreover it is probable that the maximum temperature would have occurred at some time between 1 p.m. and 3 p.m. and that the temperature would not have remained constant between these hours, hence the most probable graph is the curve as drawn in the figure.

To find the temperatures at 10 a.m. and 4.30 p.m., we take points A and C on OX representing 10 a.m. and 4.30 p.m. and through these points draw vertical lines AB and CD meeting the curve at B and D. From B and D draw straight lines parallel to XO meeting OY in E and F. Now the temperatures required are 101·6° F. and 102·8° F.

EXAMPLES CIII

(*Scales should be so chosen that the graphs should more or less fill up the paper used*)

1. The following scores were made by a student in a series of 8 innings at cricket :

10, 24, 0, 3, 76, 32, 55, 12.

Exhibit these graphically.

2. The average monthly rainfall (in inches) of Simla is given in the following table :

Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
3·2	3·1	2·5	2·3	3·7	7·8	18·4	17·9	6·2	1·2	·4	1·3

Draw a graph to represent the variations.

3. The following table gives the annual rainfall (in inches) of Dehra-Dun from 1915 to 1926 :

1925	1926	1927	1928	1929	1930	1931	1932	1933	1934	1935
81·1	76·4	104·5	70·1	69·1	75·0	67·4	91·9	109·4	105·1	75·1

Draw a graph to represent the variations.

4. The mean monthly temperature (in degrees) of Agra for twelve months is given in the following table :

Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
60·1	64·8	76·8	88·1	94·0	93·4	86·0	84·2	84·2	79·4	68·7	61·2

Represent the variations graphically.

5. The following table gives the temperature in degrees Fahrenheit taken at different hours on a certain day :

7 a.m.	9 a.m.	11 a.m.	1 p.m.	3 p.m.	5 p.m.	7 p.m.
45·0	51·5	59·0	69·5	68·0	60·5	50·0

Exhibit the variations graphically.

6. The premium (in rupees) of life insurance for Rs. 1000 payable at death are given for various ages in the following table :

Age	25	30	35	40	45	50	55	60
Premium	20	23	26	31	36	43	55	71

Exhibit these graphically and find the premium at the ages 32 and 53.

7. The following table gives the average weight (in pounds) of children at different ages :

Age in years	6	8	10	12	14
Weight	43	50	60	76	97

Exhibit these graphically and find the average weight of a child at ages 7 and 13.

8. The following table gives the census returns of the population (in crores) of India in the years specified :

1881	1891	1901	1911	1921	1931
25·4	28·7	29·4	31·5	31·9	35·3

Represent these graphically and from the graph read off the population in the years 1886 and 1906.

9. Exhibit graphically the following scales of charges as quoted by the manager of a hotel :

No. of guests	300	400	500	600	700	800
Charges per head	Rs. 6	Rs. 4. 14a.	Rs. 4. 2a.	Rs. 3. 10a.	Rs. 3. 4a.	Rs. 3.

Estimate the charges per head of 350 and 750 guests.

10. The times of sun-rise (in hours and minutes) of Bombay on the first of every month are given in the following table :

Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
7-12	7-13	6-59	6-33	6-11	6-1	6-5	6-15	6-23	6-29	6-39	6-55

Represent the variations graphically.

11. The average monthly rainfall (in inches) of Allahabad and Patna is given in the following table :

Months	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Allahabad	·8	·5	·4	·1	·3	5·1	12·2	10·9	6·3	2·4	·3	·2
Patna	·7	·5	·4	·3	1·7	7·8	11·4	10·7	7·8	2·9	·2	·1

Represent these graphically on a single diagram.

12. The mean monthly temperature (in degrees) of Benares and Bombay for twelve months is given in the following table :

Months	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Benares	60·0	65·3	76·6	86·8	91·3	89·4	84·1	83·1	83·0	77·9	67·8	60·2
Bombay	74·5	74·8	78·0	82·1	84·6	82·4	79·5	79·4	79·4	80·7	79·3	76·4

Represent these graphically on a single diagram.

13. The following table gives the population (in millions) of two countries P and Q for the years specified :

Years	1881	1891	1901	1911	1921	1931
P	4·4	4·6	5·0	5·6	6·4	7·4
Q	6·5	5·9	5·4	5·0	4·7	4·5

Represent these graphically on the same diagram and find when the population was the same in each country and the year in which this happened.

14. The following table gives, in inches, the average height of a boy, and of a girl, at various ages :

Age	0	2	4	6	8	10	12	14	16
Boy	18	31	$37\frac{1}{2}$	42	46	50	53	$56\frac{1}{2}$	$62\frac{1}{2}$
Girl	18	$30\frac{1}{4}$	$36\frac{1}{2}$	$40\frac{3}{4}$	$44\frac{1}{2}$	49	$53\frac{1}{2}$	$57\frac{3}{4}$	61

Exhibit these graphically on the same diagram, and find the ages at which the average height of a boy is the same as that of a girl.

15. The annual income (in rupees) of a man is given in the following table for the years specified :

Years	1922	1923	1924	1925	1926	1927
Income	1700	1800	2100	2600	3300	4200

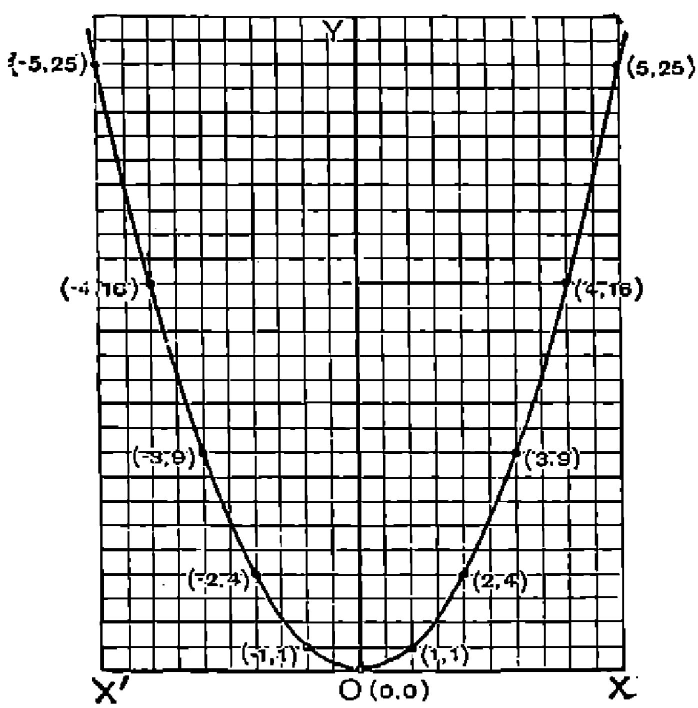
Represent these graphically. If his income goes on increasing at the same rate, find in what year his income will be Rs. 6600.

CHAPTER XXXIV

GRAPHS OF QUADRATIC EQUATIONS

176. We have seen in Chapter XVII that graphs of simple equations can be obtained by plotting a number of points whose co-ordinates satisfy the equations and then joining them. In this chapter we shall consider the graphs of quadratic equations and shall confine ourselves to some special cases. The method is precisely the same *i.e.*, we plot a number of points whose co-ordinates satisfy the equation and then join them in order. In particular cases, however, the work may be shortened, as shown in the following examples.

EXAMPLE 1. *Draw the graph of* $y=x^2$.



Giving different integral values to x, we have, when

$x =$	5	4	3	2	1	0	-1	-2	-3	-4	-5
$y =$	25	16	9	4	1	0	1	4	9	16	25

Taking 2 divisions along OX and 1 division along OY as unit, we plot the points (5, 25), (4, 16), (3, 9),... Joining these points, we get the required graph, which is a continuous curve.

The students should observe that :

(i) For every value of x, positive or negative, y is always positive, *i.e.*, the curve lies entirely above the axis of x.

(ii) For values of x which are equal in magnitude but opposite in sign, the value of y remains the same, *i.e.*, the curve is *symmetrical* about the axis of y.

(iii) As x increases, y also increases but more rapidly than x. If x becomes infinitely great, y also becomes infinitely great, *i.e.*, the curve is unlimited in size and extends indefinitely in the two directions.

This curve is called a **Parabola**.

EXAMPLE 2. *Draw the graph of* xy=1.

The equation can be written in the form $y=\frac{1}{x}$.

Giving positive values to x and y, we have, when

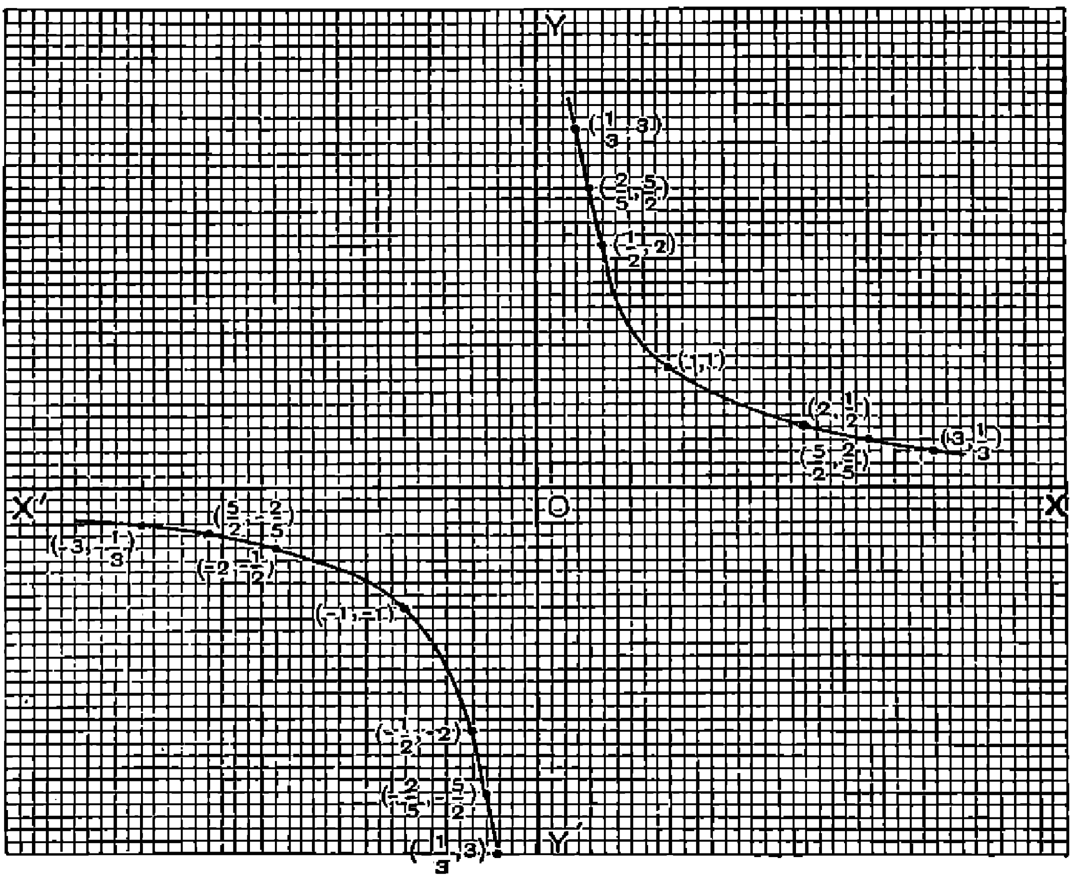
$x =$	0	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	1	2	$\frac{5}{2}$	3	...	∞
$y =$	∞	3	$\frac{5}{2}$	2	1	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	...	0

Again, giving negative values to x and y, we have, when

$x =$	-0	$-\frac{1}{3}$	$-\frac{2}{5}$	$-\frac{1}{2}$	-1	-2	$-\frac{5}{2}$	-3	...	$-\infty$
$y =$	$-\infty$	-3	$-\frac{5}{2}$	-2	-1	$-\frac{1}{2}$	$-\frac{2}{5}$	$-\frac{1}{3}$	...	-0

NOTE. We know that when the denominator of a fraction becomes smaller and smaller, the value of the fraction becomes greater and greater. When the denominator becomes infinitely small or zero, the value of the fraction becomes infinitely great. This infinitely great value is called *infinity*, and is denoted by the symbol ∞.

Taking 1 centimetre as unit, first plot the points $(\frac{1}{3}, 3)$, $(\frac{2}{5}, \frac{5}{2})$,... and then the points $(-\frac{1}{3}, -3)$, $(-\frac{2}{5}, -\frac{5}{2})$,... Joining them in order we get the required graph which has two distinct branches, one lying in the first quadrant and the other in the third, as shown in the diagram.



We also notice that as x increases, y decreases, and, when x is infinitely great, y is infinitely small, *i.e.*, the graph continually approaches nearer and nearer to the x-axis and that it does not actually meet the x-axis, however small x may be, until $x=\infty$. Similarly when y increases, x decreases, and, when y is infinitely great, x is infinitely small, *i.e.*, the graph also approaches nearer and nearer to the y-axis and that it does not actually meet the y-axis however small y may be, until $y=\infty$.

This curve is called a **hyperbola**, and the axes of co-ordinates which continually approach the branches of the curve are called its **asymptotes.**

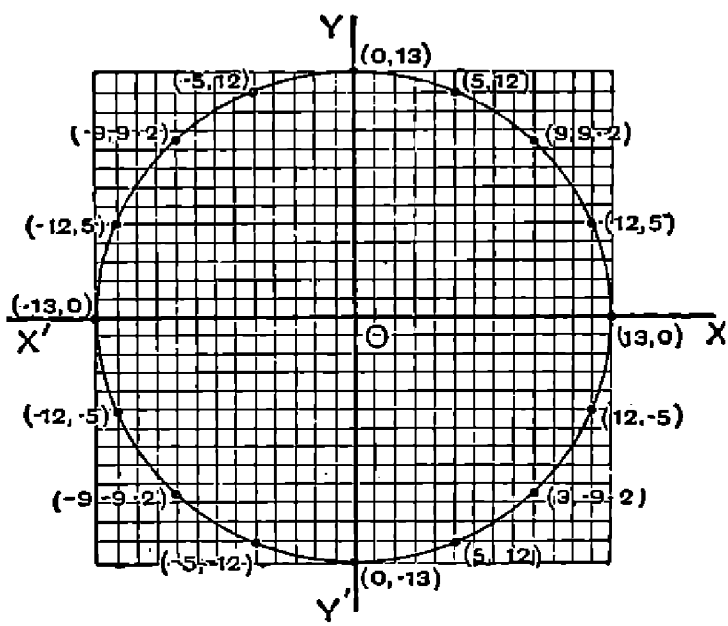
EXAMPLE 3. *Draw the graph of* $x^2+y^2=169$.

The equation can be written as

$$y^2=169-x^2,$$
$$y=\pm\sqrt{169-x^2}.$$

Giving different integral values to x, both positive and negative, we have, when

$x=$	-13	-12	-9	-5	0	5	9	12	13
$y=$	0	± 5	$\pm 9{\cdot}2$	± 12	± 13	± 12	$\pm 9{\cdot}2$	± 5	0



From the above we notice, that for every value of x, y has two values, equal in magnitude but opposite in sign *i.e.*, the curve is *symmetrical* about the axis of x. Similarly, writing the equation in the form $x=\pm\sqrt{169-y^2}$, we can say that the curve is also *symmetrical* about the axis of y.

Plotting the points $(-13, 0)$, $(-12, 5)$, $(-12, -5)$, $(-9, 9{\cdot}2)$, $(-9, -9{\cdot}2)$,...and joining them we see that the curve is a **circle** whose centre is at the origin and whose radius is equal to 13 units.

NOTE. Students should note that the greatest possible values that can be given to x or y are ± 13, *i.e.*, the graph is enclosed between the lines $x=13$, $x=-13$, $y=13$ and $y=-13$.

177. EXAMPLE. 1. *Solve the equation* $x^2-3x-10=0$ *graphically.*

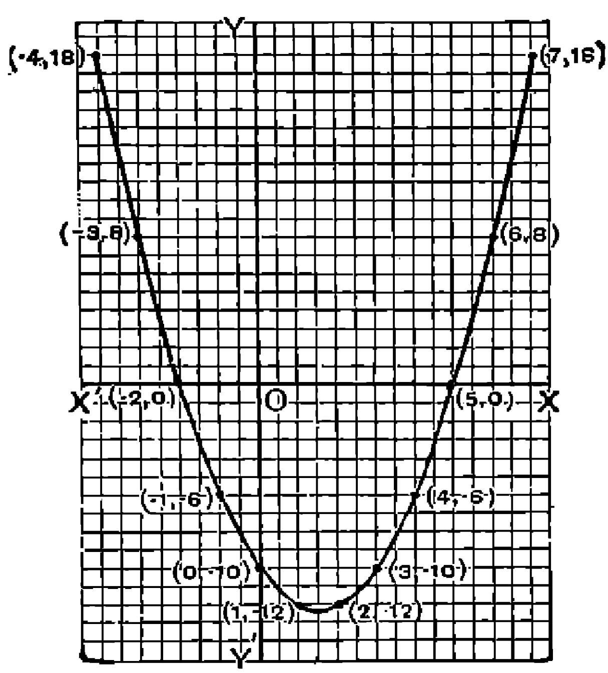
Putting the expression $x^2-3x-10$ equal to y, we first draw the graph of the equation $y=x^2-3x-10$ and then find those values of x which will make the ordinate y zero. The value of the ordinate is zero at every point on the axis of x, therefore the abscissæ of the points where the curve cuts the axis of x will be the roots of the equation.

To draw the graph of $y=x^2-3x-10$, we give different values to x and find corresponding values of y thus :

When

$x=$	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$y=$	18	8	0	-6	-10	-12	-12	-10	-6	0	8	18

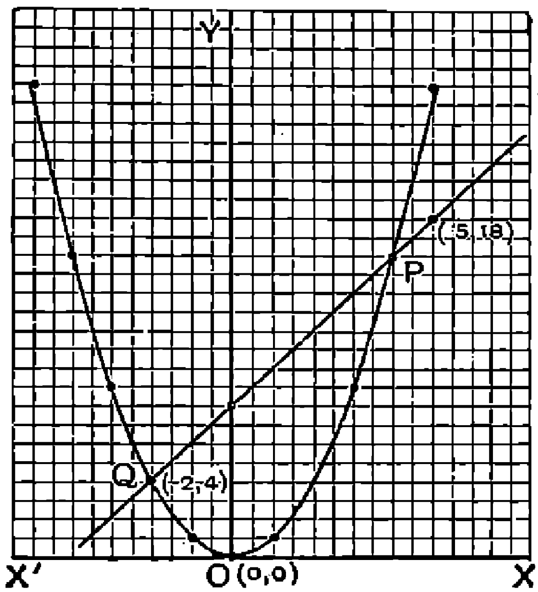
Now plotting the points $(-4, 18)$, $(-3, 8)$, $(-2, 0)$,...and joining them, we get the graph. Since the graph cuts the x-axis in points



whose co-ordinates are $(5, 0)$ and $(-2, 0)$, therefore the roots of the equation are 5 and -2.

EXAMPLE 2. *Solve graphically the equations* $y=x^2$ *and* $y=2x+8$.

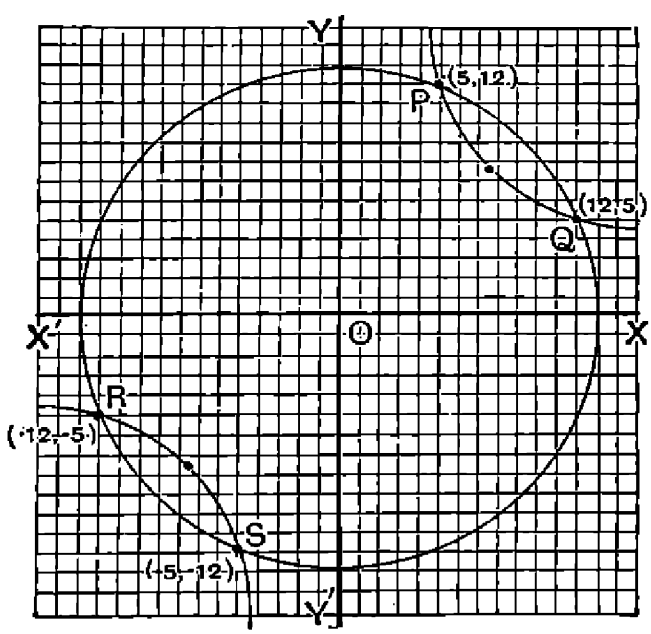
We have already drawn the graph of $y=x^2$. (See page 371). On the same diagram and to the same scale draw the graph of the straight



line $y=2x+8$, convenient points on the graph being (0, 8) and (5, 18). The two graphs intersect at the points P and Q whose co-ordinates are (4, 16) and (-2, 4).

Hence the solution is $x=4$, $y=16$; and $x=-2$, $y=4$.

EXAMPLE 3. *Solve graphically the equations* $x^2+y^2=169$ *and* $xy=60$.



We have already drawn the graph of $x^2+y^2=169$. (See page 374). On the same diagram and to the same scale, draw the graph of $xy=60$.

The two graphs intersect at the points P, Q, R and S, whose co-ordinates are (5, 12), (12, 5), (−12, −5) and (−5, −12).

Hence the solution is $x=5,\ y=12$; $x=12,\ y=5$; $x=-12,\ y=-5$; and $x=-5,\ y=-12$.

EXAMPLES CIV

1. Draw the graphs of

(i) $y=4x^2$. (ii) $4y=x^2$. (iii) $y=16x^2$. (iv) $y=25x^2$.

2. Draw the graph of $y=ax^2$, when

(i) $a=1$. (ii) $a=3$. (iii) $a=10$. (iv) $a=-1$. (v) $a=-5$.

3. Draw the graphs of

(i) $x^2=y$. (ii) $x^2=10y$. (iii) $3x^2=y$. (iv) $x^2=2y$.

4. On the same diagram and to the same scale draw the graphs of $y^2=4ax$ and $y^2=-4ax$. What is the relation between the two ?

5. For values of x from −5 to +5 and with 1 inch or 1 centimetre as unit for both the axes, draw the graphs of

(i) $y=1+x^2$. (ii) $x=3y-y^2$. (iii) $y=x^2-4$. (iv) $y=\frac{1}{5}x^2$.

6. Draw the graphs of

(i) $xy=4$. (ii) $xy=16$. (iii) $xy=40$. (iv) $xy=-1$.

7. On the same diagram and to the same scale draw the graphs of $xy=1$ and $xy=-1$. What is the relation between the two ?

8. Draw the graphs of

(i) $x^2+y^2=1$. (ii) $x^2+y^2=4$. (iii) $x^2+y^2=25$. (iv) $x^2+y^2=100$.

9. On the same diagram and to the same scale draw the graphs of $y^2=4ax$, when

(i) $a=1$. (ii) $a=2$. (iii) $a=4$. (iv) $a=\frac{1}{2}$. (v) $a=\frac{1}{4}$.

10. On the same diagram and to the same scale draw the graphs of $x^2+y^2=a^2$, when

(i) $a=\cdot 8$. (ii) $a=1$. (iii) $a=1\cdot 2$. (iv) $a=1\cdot 5$.

11. With 1 inch as unit for both the axes, draw the graphs of

(i) $9x^2+4y^2=36$. (ii) $36x^2+100y^2=225$.

12. Solve the following equations graphically :

(i) $x^2-1=0$. (ii) $x^2-5x+6=0$. (iii) $x^2+7x+12=0$.
(iv) $x^2+2x-15=0$. (v) $6x^2-5x+1=0$. (vi) $2x^2-5x-3=0$.

13. Draw the graph of $y=x^2-x$, and thence find the roots of the following equations :

(i) $x^2-x-6=0$. (ii) $x^2-x-20=0$. (iii) $x^2-x-\frac{3}{4}=0$.
(iv) $x^2-x=0$. (v) $9x^2-9x-4=0$.

14. On the same diagram and with the same scale draw the graphs of $y=3x^2$ and $y=6x$, and find the co-ordinates of the points of intersection of the two graphs.

15. On the same diagram and with the same scale draw the graphs of $x^2+y^2=25$ and $4x=3y$, and find the co-ordinates of the points of intersection of the two graphs.

16. On the same diagram and with the same scale draw the graphs of $x^2+y^2=100$ and $xy=48$, and find the co-ordinates of the points of intersection of the two graphs.

17. Solve the following pairs of equations graphically :

(i) $x+y=5$, $xy=6$. (ii) $x^2+y^2=16$, $x+y=4$. (iii) $y=x^2-1$, $y-x=5$. (iv) $2x=y^2$, $2x+3y=4$.

(v) $y=x^2$, $xy=1$. (vi) $x^2+y^2=10$, $xy=2$. (vii) $y^2=8x$, $y=x^2$.

CHAPTER XXXV

RATIO AND PROPORTION

Ratio

178. When two quantities are compared, their comparision may be obtained either by finding their difference or by finding how many times one is contained in the other. The second comparison shows more clearly the relative magnitude of the two quantities compared. The relation which one quantity bears to another with respect to *magnitude*, is called the **ratio** of the one to the other.

Thus, to find the ratio of Rs. 2 to Rs. 5, we see 'how many times Rs. 2 is contained in Rs. 5' or 'what fraction Rs. 2 is of Rs. 5'. This is expressed arithmetically by the fraction $\frac{2}{5}$. Thus the ratio of Rs. 2 to Rs. 5 is $\frac{2}{5}$. Similarly, the ratio of a to b is $\frac{a}{b}$. This is also expressed as $a \div b$ or $a : b$.

NOTE. It is necessary that the two quantities to be compared must be *of the same kind*, or can be expressed in the same denomination. Concrete quantities of different kinds can have no ratio to one another; for we cannot compare the magnitude of 3 yards with 10 days, but we can compare 3 yards with 10 feet, if both are expressed in terms of yards or feet. Thus the ratio of 3 yards to 10 feet is the same as the ratio of 9 feet to 10 feet, *i.e.* $\frac{9}{10}$.

179. In the ratio $a : b$, the quantity a is called the **first term** or **antecedent** of the ratio, and b the **second term** or **consequent** of the ratio.

180. Since a ratio may be taken as a fraction, it follows that whatever has been proved for fractions is also true for ratios.

Thus, as we have seen in Article 135, a fraction is unaltered if both its numerator and denominator are multiplied by the same number, similarly *a ratio is unaltered in value if each of its terms are multiplied by the same quantity.*

Thus, since $\frac{b}{c}=\frac{ab}{ac}$, hence the ratio $b : c$ and $ab : ac$ are equal.

Similarly the ratios $3 : 4$, $9 : 12$ and $3m : 4m$ are all equal.

NOTE. We shall not commit any mathematical error if we work with fractions instead of ratios, though strictly speaking the *idea* of ratio is not quite the same as the *idea* of fraction.

181. To compare two or more ratios, we express the equivalent fractions with a common denominator and then compare the numerators.

Thus, to compare the ratios $5 : 4$, $17 : 15$ and $7 : 6$, we express the equivalent fractions $\frac{5}{4}$, $\frac{17}{15}$ and $\frac{7}{6}$ with a common denominator 60 (which is generally the L.C.M. of the denominators) as $\frac{75}{60}$, $\frac{68}{60}$ and $\frac{70}{60}$. Now the ratios can be arranged in descending order of magnitude as $5 : 4$, $7 : 6$ and $17 : 15$.

EXAMPLE. *Which is the greater ratio* $3 : 5$ *or* $11 : 19$?

The fractions representing these ratios are $\frac{3}{5}$ and $\frac{11}{19}$. These can be expressed with a common denominator as $\frac{57}{95}$ and $\frac{55}{95}$.

Hence $\frac{3}{5}$ is the greater ratio.

182. The **inverse** of a given ratio is the ratio which has for its antecedent the consequent of the given ratio, and for its consequent the antecedent of the given ratio, *i.e.* it can be obtained by inverting the given ratio.

Thus the inverse of the ratio $\frac{2}{3}$ is $\frac{3}{2}$ and that of $\frac{a}{b}$ is $\frac{b}{a}$.

183. When two or more ratios are multiplied together they are said to be **compounded**.

Thus the ratio compounded of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ac}{bd}$, and that of $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ is $\frac{ace}{bdf}$.

184. When a ratio is compounded with itself twice, the resulting ratio is called the **duplicate** of the given ratio.

Thus $\frac{a^2}{b^2}$ is the duplicate of the ratio $\frac{a}{b}$.

When a ratio is compounded with itself thrice the resulting ratio is called the **triplicate** of the given ratio.

Thus $\frac{a^3}{b^3}$ is the triplicate of the ratio $\frac{a}{b}$.

The ratio $\frac{\sqrt{a}}{\sqrt{b}}$ is called the **sub-duplicate** of the ratio $\frac{a}{b}$.

NOTE. If a and b denote concrete quantities then a^2, b^2 ; a^3, b^3 ; $\sqrt{a}$, $\sqrt{b}$ have no meaning, so that the duplicate ratio, the triplicate ratio or the sub-duplicate ratio cannot be defined as $a^2 : b^2$, $a^3 : b^3$ or $\sqrt{a} : \sqrt{b}$.

185. A homogeneous equation in two variables can give us the ratio of the two variables.

Thus, the equation $2x^2 + xy - 6y^2 = 0$, can be written as

$$(x + 2y)(2x - 3y) = 0,$$

$$\therefore \quad x = -2y,\ 2x = 3y\ ;$$

$$\frac{x}{y} = -2 \text{ or } \frac{3}{2}.$$

186. EXAMPLE 1. *Find the value of* x, *if the duplicate ratio of* $\frac{3x+1}{x-1}$ *is equal to* 25.

The duplicate ratio of $\frac{3x+1}{x-1}$ is $\frac{(3x+1)^2}{(x-1)^2}$ or $\frac{9x^2+6x+1}{x^2-2x+1}$.

$$\frac{9x^2+6x+1}{x^2-2x+1} = 25,$$

$$\therefore \quad 9x^2+6x+1 = 25x^2-50x+25,$$

$$16x^2-56x+24 = 0,$$

$$\therefore \quad 2x^2-7x+3 = 0,$$

$$\therefore \quad (x-3)(2x-1) = 0,$$

$$\therefore \quad x = 3 \text{ or } \tfrac{1}{2}.$$

EXAMPLE 2. *If* $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, *prove that each of the ratios* is equal to $\frac{a+c+e}{b+d+f}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$,

then $a=bk$, $c=dk$, $e=fk$.

$$\frac{a+c+e}{b+d+f}=\frac{bk+dk+fk}{b+d+f}=\frac{k(b+d+f)}{b+d+f}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

EXAMPLES CV

Which is the greater ratio

1. 2:3 or 3:4 ?

2. 3:5 or 10:17 ?

3. Arrange the ratios 2:5, 4:7 and 13:21 in descending order of magnitude.

4. Express the ratio of Rs. 3. 12a. to Rs. 10. 10a. in its simplest form.

5. Express the ratio of 2 md. 5 sr. to 3 md. 7 sr. 8 chk. in its simplest form.

Find the ratio compounded of

6. 2:3, 6:7, 7:8.

7. $a:b$, $b:c$, $c:d$, $d:e$.

8. $(a+b):(x+y)$, $(a-b):(x-y)$, $(x^2-y^2):(a^2-b^2)$.

9. $(x^2-2xy+y^2):(x^2+2xy+y^2)$, $(x^3+3x^2y+3xy^2+y^3):(x^3-3x^2y+3xy^2-y^3)$.

10. Find the duplicate ratio of 2:3.

11. Find the triplicate ratio of 4:5.

12. Find the triplicate ratio of $x:y$.

13. Find the sub-duplicate ratio of 9:25.

14. Find the sub-duplicate ratio of 16:9.

15. Find the ratio compounded of the duplicate ratio of 10:3 and the triplicate ratio of 2:5.

16. Find the ratio compounded of the duplicate ratio of 2:3 and the sub-duplicate ratio of 36:49.

17. Find the ratio compounded of the triplicate ratio of $x:y$ and the sub-duplicate ratio of $4y^4 : 9x^4$.

18. If $a:b=10:3$, find the value of $(a-3b):(2a-5b)$.

19. If $(2a+7b):(9a-b)=5:3$, find the value of $a:b$.

20. If $\frac{7x+2y}{5x-3y}=\frac{29}{3}$, find the value of $\frac{x}{y}$.

21. If $\frac{3x-y}{4x+5y}=\frac{1}{2}$, find the value of $\frac{x}{y}$.

22. If $9x^2-6xy+y^2=0$, find the value of $x:y$.

23. If $4x^2-20xy+25y^2=0$, find the value of $\frac{x}{y}$.

24. If $x^2-5xy+6y^2=0$, find the value of $\frac{x}{y}$.

25. What number must be added to each term of the ratio $\frac{11}{21}$ to make it equal to $\frac{3}{5}$?

26. What number must be taken from each term of the ratio $\frac{21}{37}$ to make it equal to $\frac{1}{2}$?

27. What number must be taken from each term of the ratio $\frac{23}{30}$ to make it equal to $\frac{2}{3}$?

28. Divide Rs. 4. 4 a. into two parts having the ratio of 1:3.

29. Divide Rs. 13. 7a. into two parts having the ratio of 2:3.

30. Divide 20 into two parts in the ratio of $a:b$.

31. Divide 5 into two parts in the ratio of $1:k$.

32. Divide a straight line a inches long into two parts having the ratio of 2:3.

33. Divide a straight line a inches long into two parts having the ratio of $p:q$.

34. The perimeter of a rectangle is 18 ft. If one side is double the other, find the length of the sides.

35. The sides of a triangle are in the ratio 5:6:7. If its perimeter is 108 yards, find the length of each side.

36. The angles of a triangle are in the ratio 1:2:3. If their sum is 180°, find the value of each angle.

37. If the ratio 2:3 is equal to the sub-duplicate ratio of $x-1 : x+4$, find x.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each of the ratios is equal to

38. $\frac{a-c+e}{b-d+f}$. **39.** $\frac{2a+3c+4e}{2b+3d+4f}$. **40.** $\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$.

41. $\sqrt[3]{\frac{a^3+c^3+e^3}{b^3+d^3+f^3}}$ **42.** $\sqrt[3]{\frac{2ac^2+3ce^2+5ea^2}{2bd^2+3df^2+5fb^2}}$.

Proportion

187. Four quantities are said to be **in proportion** when the ratio of the first to the second is equal to the ratio of the third to the fourth.

For instance, 2, 3, 4, 6 are in proportion, for $2:3=4:6$. Similarly a, b, c, d will be in proportion if $a:b=c:d$ *i.e.*, if $\frac{a}{b}=\frac{c}{d}$.

This is sometimes expressed by the notation $a:b::c:d$, and read "*a is to b as c is to d.*"

When four quantities are in proportion, the first and the fourth are called the **extremes**, the second and the third are called the **means**, and the fourth is called the **fourth proportional.**

188. If four quantities a, b, c, d are in proportion,

$$\frac{a}{b}=\frac{c}{d}.$$

Multiplying both sides by the L.C.M. of the denominators, we have

$$\frac{a}{b}\times bd=\frac{c}{d}\times bd,$$

$$\therefore \quad ad=bc,$$

i.e., the product of the means is equal to the product of extremes.

Conversely, *if $ad=bc$, then a, b, c, d will be in proportion.*

From the above we see that if any of the three terms in proportion are given, the fourth may be found.

Thus, to find the fourth proportional to 2, 5, 6, denote it by an unknown quantity x,

$$\text{then} \quad \frac{2}{5}=\frac{6}{x},$$

$$\therefore \quad 2x=30,$$

$$\therefore \quad x=15.$$

Hence the fourth proportional is 15.

189. Quantities are said to be **in continued proportion** if the ratios of the first to the second, the second to the third, the third to the fourth, etc., are equal.

Thus $a, b, c, d, e, \ldots$ will be in continued proportion, if $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\ldots$

If three quantities a, b, c, are in continued proportion, then b is called the **mean proportional** between a and c, and c is called the **third proportional** to a and b.

When a, b, c are in continued proportion, we have

$$\frac{a}{b}=\frac{b}{c},$$

$$\therefore \quad b^2=ac,$$

$$\text{or} \quad b=\sqrt{ac},$$

i.e., the mean proportional between two quantities is the square root of their product.

Conversely, *if* $ac=b^2$, *then* a, b, c *will be in proportion.*

190. If four quantities are in proportion, we can form many other propositions from them. Some of the important ones are given below.

(i) *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{b}{a}=\frac{d}{c}$. [*Invertendo*]

If $\frac{a}{b}=\frac{c}{d}$,

$$1 \div \frac{a}{b}=1 \div \frac{c}{d},$$

$$\frac{b}{a}=\frac{d}{c}.$$

(ii) *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a}{c}=\frac{b}{d}$. [*Alternando*]

If $\frac{a}{b}=\frac{c}{d}$,

$$\frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{b}{c},$$

$$\frac{a}{c}=\frac{b}{d}$$

(iii) *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a+b}{b}=\frac{c+d}{d}$. [*Componendo*]

If $\frac{a}{b}=\frac{c}{d}$,

$\frac{a}{b}+1=\frac{c}{d}+1$,

$\frac{a+b}{b}=\frac{c+d}{d}$.

(iv) *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a-b}{b}=\frac{c-d}{d}$. [*Dividendo*]

If $\frac{a}{b}=\frac{c}{d}$,

$\frac{a}{b}-1=\frac{c}{d}-1$,

$\frac{a-b}{b}=\frac{c-d}{d}$.

(v) *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. [*Componendo and Dividendo*]

If $\frac{a}{b}=\frac{c}{d}$,

$\therefore$ from (iii) we have $\frac{a+b}{b}=\frac{c+d}{d}$,

and from (iv) we have $\frac{a-b}{b}=\frac{c-d}{d}$.

Hence by division we have $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

191. EXAMPLE 1. *If* $\frac{a}{b}=\frac{b}{c}$, *shew that* $\frac{a}{c}=\frac{a^2}{b^2}$.

Since $\frac{a}{b}=\frac{b}{c}$,

$\frac{a}{b}\times\frac{a}{b}=\frac{b}{c}\times\frac{a}{b}$,

$\therefore \quad \frac{a^2}{b^2}=\frac{a}{c}$.

EXAMPLE 2. *If* $\frac{a}{b}=\frac{c}{d}$, *shew that* $\frac{ma+nb}{mc+nd}=\frac{ma-nb}{mc-nd}$.

Since $\frac{a}{b}=\frac{c}{d}$,

$$\frac{m}{n}\times\frac{a}{b}=\frac{m}{n}\times\frac{c}{d},$$

$$\frac{ma}{nb}=\frac{mc}{nd},$$

$$\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc-nd}, \quad \text{[Componendo and Dividendo]}$$

$$\frac{ma+nb}{mc+nd}=\frac{ma-nb}{mc-nd}. \quad \text{[Alternando]}$$

EXAMPLE 3. *If* $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, *shew that*

$$\sqrt{a^3c^3+c^3e^3+e^3a^3} : \sqrt{b^3d^3+d^3f^3+f^3b^3}=ace : bdf.$$

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$,

so that $a=bk$, $c=dk$, $e=fk$.

$$\therefore \frac{\sqrt{a^3c^3+c^3e^3+e^3a^3}}{\sqrt{b^3d^3+d^3f^3+f^3b^3}}=\frac{\sqrt{(bk)^3(dk)^3+(dk)^3(fk)^3+(fk)^3(bk)^3}}{\sqrt{b^3d^3+d^3f^3+f^3b^3}}$$

$$=\frac{\sqrt{b^3d^3k^6+d^3f^3k^6+f^3b^3k^6}}{\sqrt{b^3d^3+d^3f^3+f^3b^3}}=\frac{k^3\sqrt{b^3d^3+d^3f^3+f^3b^3}}{\sqrt{b^3d^3+d^3f^3+f^3b^3}}=k^3.$$

And $\frac{ace}{bdf}=\frac{bk.dk.fk.}{bdf}=\frac{k^3.bdf}{bdf}=k^3$.

Hence $\sqrt{a^3c^3+c^3e^3+e^3a^3}:\sqrt{b^3d^3+d^3f^3+f^3b^3}=ace : bdf$.

EXAMPLE 4. *If* a, b, c, d *are in continued proportion, shew that* $a^3+b^3+c^3 : b^3+c^3+d^3=a : d$.

Let $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k$,

so that $c=dk$,
$b=ck=dk^2$,
and $a=bk=dk^3$.

$$\frac{a^3+b^3+c^3}{b^3+c^3+d^3}=\frac{(dk^3)^3+(dk^2)^3+(dk)^3}{(dk^2)^3+(dk)^3+d^3}=\frac{d^3k^9+d^3k^6+d^3k^3}{d^3k^6+d^3k^3+d^3}$$

$$=\frac{d^3k^3(k^6+k^3+1)}{d^3(k^6+k^3+1)}=k^3.$$

And $\frac{a}{d}=\frac{dk^3}{d}=k^3$.

Hence $a^3+b^3+c^3 : b^3+c^3+d^3=a : d$.

EXAMPLES CVI

Find a fourth proportional to

1. $1, 2, 5$. **2.** $3, 1\frac{3}{4}, 12$. **3.** a, b, c.

4. $2a^2b^2, 3bc, 4a^3b$. **5.** $a^2-b^2, (a-b)^2, (a+b)^2$.

Find a third proportional to

6. $3, 6$. **7.** $1, a$. **8.** a^2, ab. **9.** $(a+b)^2, a^2-b^2$.

Find a mean proportional to

10. $2, 8$. **11.** $16x, 9x^3y^2$. **12.** $27a^3, 3ab^4$.

13. $4(a+b)^2, 49(a-b)^2$.

If $\frac{a}{b}=\frac{c}{d}$, shew that

14. $\frac{a+b}{b}=\frac{c+d}{d}$. **15.** $\frac{a+c}{c}=\frac{b+d}{d}$. **16.** $a^2d^2=b^2c^2$.

17. $\frac{2a+3b}{b}=\frac{2c+3d}{d}$. **18.** $\frac{ma+nb}{b}=\frac{mc+nd}{d}$.

19. $\frac{ma+nb}{mc+nd}=\frac{pa+qb}{pc+qd}$. **20.** $\frac{ac}{bd}=\frac{c^2}{d^2}$. **21.** $\frac{ab}{cd}=\frac{a^2}{c^2}$.

22. $\frac{a^2}{c^2}=\frac{a^2-b^2}{c^2-d^2}$. **23.** $\frac{a^2+c^2}{b^2+d^2}=\frac{ac}{bd}$.

24. $\frac{ma^2+nc^2}{mb^2+nd^2}=\frac{a^2}{b^2}$. **25.** $\frac{2a+5c}{5a+2c}=\frac{2b+5d}{5b+2d}$.

26. $\frac{(a+c)^3}{(b+d)^3}=\frac{a(a-c)^2}{b(b-d)^2}$. **27.** $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$.

28. $\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}=\frac{a}{b}$. **29.** $\frac{\sqrt{a+3c}}{\sqrt{b+3d}}=\frac{\sqrt{3a+c}}{\sqrt{3b+d}}$.

30. $\frac{a+3b+5c+15d}{a+3b-5c-15d}=\frac{a-3b+5c-15d}{a-3b-5c+15d}$.

31. $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$.

If a, b, c are in continued proportion, shew that

32. $\frac{a^2-b^2}{b^2-c^2}=\frac{a}{c}$. **33.** $\frac{2a+3b}{2b+3c}=\frac{5a-3b}{5b-3c}$.

34. $\frac{a^2+b^2}{b^2+c^2}=\frac{(a+b)^2}{(b+c)^2}$. **35.** $\frac{a(a+b)}{b(b+c)}=\frac{b(b-a)}{c(c-b)}$.

36. $\frac{a^2-b^2}{a^2-c^2}=\frac{b^2}{b^2+c^2}$.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

37. $\frac{2a-3c+5e}{2b-3d+5f}=\frac{c}{d}$.

38. $\frac{3a^2-5ace+7e^2f}{3b^2-5bdf+7f^3}=\frac{ae}{bf}$.

39. $\frac{a^4+3c^2e^2+e^4}{b^4+3d^2f^2+f^4}=\frac{a^2c^2}{b^2d^2}$.

40. $\frac{\sqrt{a^6+c^6+e^6}}{\sqrt{b^6+d^6+f^6}}=\frac{ace}{bdf}$.

41. $\frac{a^2+c^2+e^2}{ab+cd+ef}=\frac{ab+cd+ef}{b^2+d^2+f^2}$.

If a, b, c, d are in continued proportion, shew that

42. $\frac{a}{d}=\frac{a^3}{b^3}$.

43. $\frac{a+b}{b+c}=\frac{b+c}{c+d}$.

44. $(b+c)(b+d)=(c+d)(c+a)$.

45. $\frac{a}{d}=\frac{la^3+mb^3+nc^3}{lb^3+mc^3+nd^3}$.

192. EXAMPLE 1. *If* $\frac{x+y+z+w}{x+y-z-w}=\frac{x-y+z-w}{x-y-z+w}$, *shew that* $\frac{x}{y}=\frac{z}{w}$.

Since $\frac{x+y+z+w}{x+y-z-w}=\frac{x-y+z-w}{x-y-z+w}$,

$$\frac{2x+2y}{2z+2w}=\frac{2x-2y}{2z-2w}, \quad \text{[Componendo and Dividendo]}$$

$$\frac{2x+2y}{2x-2y}=\frac{2z+2w}{2z-2w}, \quad \text{[Alternando]}$$

$$\frac{4x}{4y}=\frac{4z}{4w}, \quad \text{[Componendo and Dividendo]}$$

$$\frac{x}{y}=\frac{z}{w}.$$

EXAMPLE 2. *If* $\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}$, *shew that* $\frac{a}{2y+2z-x}=\frac{b}{2z+2x-y}=\frac{c}{2x+2y-z}$.

Let $\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}=k$,

so that $x=k(2b+2c-a)$, $y=k(2c+2a-b)$, $z=k(2a+2b-c)$.

$\therefore \quad 2y+2z-x=k(4c+4a-2b+4a+4b-2c-2b-2c+a)=9ak$.

Similarly $2z+2x-y=9bk$,

and $\quad 2x+2y-z=9ck$.

$$\frac{a}{2y+2z-x}=\frac{b}{2z+2x-y}=\frac{c}{2x+2y-z}=\frac{1}{9k}.$$

Hence the three ratios are equal.

EXAMPLES CVII

1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each ratio is equal to

$$\frac{c+e}{d+f}=\frac{2a-3e}{2b-3f}=\frac{a+c-e}{b+d-f}=\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}.$$

2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each ratio

$$=\frac{pa+qc+re}{pb+qd+rf}=\frac{pa-qc+re}{pb-qd+rf}=\frac{pa-qc-re}{pb-qd-rf}.$$

3. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that each ratio

$$=\frac{x+y+z}{a+b+c}=\frac{lx+my+nz}{la+mb+nc}=\frac{ax+by+cz}{a^2+b^2+c^2}.$$

4. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that $\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each ratio is equal to $\sqrt{\frac{ac+ce+ea}{bd+df+fb}}$.

6. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that each ratio is equal to

$$\sqrt{\frac{2yz+3zx+4xy}{2bc+3ca+4ab}}.$$

7. If $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, prove that $\frac{x+y}{l+m}=\frac{y+z}{m+n}=\frac{z+x}{n+l}$.

8. If $\frac{a}{x}=\frac{b}{y}$, prove that $\frac{al+xm}{bl+ym}=\frac{ap+xq}{bp+yq}$.

9. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{ma-nc}{mb-nd}=\frac{ma+nc}{mb+nd}$.

10. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, prove that $x+y+z=0$.

11. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, prove that $(b-c)x+(c-a)y+(a-b)z=0$.

12. If $\frac{x}{2a+b}=\frac{y}{a+b+c}=\frac{z}{b+2c}$, prove that $x-2y+z=0$.

13. If $\frac{x}{2p-q-r}=\frac{y}{2q-r-p}=\frac{z}{2r-p-q}$, prove that $x+y+z=0$.

14. If $\frac{bz-cy}{a}=\frac{cx-az}{b}=\frac{ay-bx}{c}$, prove that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

15. If $\frac{y+z}{a}=\frac{z+x}{b}=\frac{x+y}{c}$, prove that $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$.

16. If $\frac{x}{a+b-c}=\frac{y}{b+c-a}=\frac{z}{c+a-b}$, prove that each ratio is equal to $\frac{x+y+z}{a+b+c}$.

17. If $\frac{x^2}{a^2-b^2}=\frac{y^2}{b^2-c^2}=\frac{z^2}{c^2-a^2}$, prove that

(i) $x^2+y^2+z^2=0$. (ii) $c^2x^2+a^2y^2+b^2z^2=0$.

193. EXAMPLE. *Two numbers are in the ratio* 5 : 7 ; *if* 4 *be subtracted from each, the differences are in the ratio* 2 : 3. *Find the numbers.*

Let the numbers be represented by $5x$ and $7x$.

$$\frac{5x-4}{7x-4}=\frac{2}{3},$$

$$15x-12=14x-8,$$

$$\therefore \quad x=4.$$

Hence the numbers are 5×4 and 7×4 *i.e.*, 20 and 28.

EXAMPLES CVIII

1. Two numbers are in the ratio 3:4 ; if 4 be subtracted from the smaller number and 8 be added to the greater, the new numbers are in the ratio 4:7. Find the numbers.

2. Two numbers are in the ratio 3:4 ; if 6 and 5 be added to the two numbers respectively, the sums are in the ratio 4:5. Find the numbers.

3. Two numbers are in the ratio $1\frac{1}{2}$: $2\frac{2}{3}$; if 15 be added to each, the sums are in the ratio $1\frac{2}{3}$: $2\frac{1}{2}$. Find the numbers.

4. The numbers of scholars in two schools are in the ratio 3:5. If there had been 50 more scholars in the first school and 80 less in the second, the numbers would have been in the ratio 5:6. Find the numbers of scholars on roll in the two schools.

5. In a certain examination the number of those who passed was four times the number of those who failed. If there had been 8 candidates less and if 4 more had failed, the numbers would have been in the ratio 2:1. Find the number of candidates.

6. On a certain map a road 3500 yards long is represented by 1·75 inches. Determine the scale of the map. What area on the map would represent 5 square miles ?

7. A man has to pay a debt of Rs. 10500 to four of his creditors A, B, C and D. The debts due to A and B are in the ratio 2:3 ; those due to B and C in the ratio 4:5 ; and those due to C and D in the ratio 6:7. What sum will each get ?

8. A sum of money is divided into two parts in the ratio $a:b$. Two persons divide between themselves the first part in the ratio 2:3, and the second part in the ratio 5:4. If they receive equal amounts, find the ratio of a to b.

9. A quantity of milk is increased in the ratio 4:5 by adding water to it, and after selling 5 seers, the rest by being mixed with 1 seer of water is increased in the ratio 2:1. How many seers of milk were there at first ?

10. Two vessels contain mixtures of milk and water, the first in the ratio of 1:3 and the second in the ratio of 3:5. What quantities must be taken from each vessel to form a mixture which shall contain 5 seers of milk and 9 seers of water ?

CHAPTER XXXVI

INDICES

194. We have already seen in Article 28, that *when powers of any quantity are multiplied together the index of the product is the sum of the indices of the factors.*

Thus $a^3 \times a^4 = a^{3+4} = a^7$,

and $a^m \times a^n = a^{m+n}$. ....(1)

We have also seen in Article 39, that *when one power of a quantity is divided by another power of the same quantity, the index of the quotient is obtained by substracting the index of the divisor from that of the dividend.*

Thus $a^6 \div a^2 = a^{6-2} = a^4$.

If, however, the index of the divisor is greater than the index of the dividend, *i.e.*, if there are more factors of the same quantity in the denominator than in the numerator, the factors of the numerator will cancel with those of the denominator, and *the index of the product of the remaining factors in the denominator can be obtained by substracting the index of the dividend from that of the divisor.*

$$\text{Thus} \quad a^2 \div a^6 = \frac{a^2}{a^6} = \frac{a \times a}{a \times a \times a \times a \times a \times a}$$

$$= \frac{1}{a \times a \times a \times a} = \frac{1}{a^4}.$$

Similarly if $m > n$, we have

$$a^m \div a^n = a^{m-n}, \qquad \text{..............................(2)}$$

and if $m < n$, we have

$$a^m \div a^n = \frac{1}{a^{n-m}}. \qquad \text{..........................(3)}$$

195. In Article 28, a^n is defined as the product of n factors each equal to a, when n is a positive integer. Using this definition and assuming the indices m and n to be positive integers, we have established results (1), (2) and (3) of the previous article. We shall now extend the meaning of a^n so as to include fractional and negative values of n and in doing so we shall assume that *the* **Index Law** (1) *of the previous article is true for all values of the indices, whether positive or negative, fractional or integral.*

NOTE. The principle on which we have extended the meaning of the index is based on the assumption, that a law in Algebra, which is true to certain restrictions, is generally true provided that the removal of the restrictions is not inconsistent with the truth of the law.

196. Since $a^m \times a^n = a^{m+n}$ has been assumed to be true for all values of m and n, we have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

But $\sqrt{a} \times \sqrt{a} = a,$

$$a^{\frac{1}{2}} = \sqrt{a}.$$

In the same way, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$

and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a,$

$$\therefore \quad a^{\frac{1}{3}} = \sqrt[3]{a}.$$

Similarly $a^{\frac{1}{n}} = \sqrt[n]{a}.$

EXAMPLE. *Simplify* $8^{\frac{1}{3}}$.

$$8^{\frac{1}{3}} = \sqrt[3]{8} = \sqrt[3]{2.2.2} = 2.$$

197. Since $a^0 = a^{m-m} = a^m \div a^m = \dfrac{a^m}{a^m} = 1.$

Hence a^0 is always equal to one *whatever the value of a may be.*

NOTE. 0^0 is not necessarily equal to unity. It is an indeterminate quantity.

EXAMPLE. *Simplify* $2^{\frac{2}{3}} \times 2^{-\frac{2}{3}}$.

$$2^{\frac{2}{3}} \times 2^{-\frac{2}{3}} = 2^{\frac{2}{3}-\frac{2}{3}} = 2^0 = 1.$$

198. Since $a^m \times a^n = a^{m+n}$ has been assumed to be true for all values of m and n, replacing m by $-n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0 = 1,$$

$$a^{-n} = \frac{1}{a^n};$$

$$\text{and } a^n = \frac{1}{a^{-n}}.$$

Hence we see that *any quantity can be transferred from the numerator to the denominator, or from the denominator to the numerator of a fraction by changing the sign of its index.*

EXAMPLE 1. *Simplify* 2^{-5}.

$$2^{-5} = \frac{1}{2^5} = \frac{1}{32}.$$

EXAMPLE 2. *Simplify* $27^{-\frac{1}{3}}$.

$$27^{-\frac{1}{3}} = \frac{1}{27^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{\sqrt[3]{3.3.3}} = \frac{1}{3}.$$

199. Since $a^m \times a^n = a^{m+n}$ has been assumed to be true for all values of m and n, replacing both m and n by $\frac{p}{q}$, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}}.$

Proceeding in the same way for q factors, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \text{.........to } q \text{ factors}$$

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\text{.........to } q \text{ terms}}$$

$$= a^{\frac{qp}{q}} = a^p.$$

i. e., $\left(a^{\frac{p}{q}}\right)^q = a^p.$

Therefore, by taking the q^{th} root,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

That is, $a^{\frac{p}{q}}$ *is the* q^{th} *root of* a^p.

Example 1. *Simplify* $27^{\frac{2}{3}}$.

$$27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{27 \times 27} = \sqrt[3]{3.3.3.3.3.3} = 3.3 = 9.$$

Example 2. *Simplify* $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$.

$$\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \frac{8^{-\frac{2}{3}}}{27^{-\frac{2}{3}}} = \frac{27^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \frac{\sqrt[3]{27^2}}{\sqrt[3]{8^2}} = \frac{9}{4} = 2\frac{1}{4}.$$

Example 3. *Simplify* $\frac{\sqrt[3]{x} \cdot \sqrt[6]{x}}{\sqrt[4]{x^3} \cdot \sqrt[9]{x^6}}$.

$$\text{The expression} = \frac{x^{\frac{1}{3}} \cdot x^{\frac{1}{6}}}{x^{\frac{3}{4}} \cdot x^{\frac{6}{9}}} = \frac{x^{\frac{1}{3}+\frac{1}{6}}}{x^{\frac{3}{4}+\frac{2}{3}}} = \frac{x^{\frac{3}{6}}}{x^{\frac{17}{12}}} = x^{\frac{1}{2}-\frac{17}{12}} = x^{-\frac{11}{12}} = \frac{1}{x^{\frac{11}{12}}}.$$

Example 4. *Find the value of* $4^{\frac{1}{2}} - 9^{\frac{1}{2}} + 16^{\frac{1}{2}} - 25^{\frac{1}{2}} + 36^{\frac{1}{2}}$.

$$\text{The expression} = \sqrt{4} - \sqrt{9} + \sqrt{16} - \sqrt{25} + \sqrt{36} = 2 - 3 + 4 - 5 + 6 = 4.$$

EXAMPLES CIX

(*Examples* 1 *to* 22 *may be taken orally*)

Find the value of

1. $4^{\frac{1}{2}}$. **2.** $25^{\frac{1}{2}}$. **3.** $8^{\frac{1}{3}}$. **4.** $32^{\frac{3}{5}}$. **5.** 36^{-2}.

6. 5^{-1}. **7.** 3^{0}. **8.** $49^{-\frac{1}{2}}$. **9.** $8^{-\frac{1}{3}}$. **10.** $8^{\frac{2}{3}}$.

11. $16^{\frac{3}{4}}$. **12.** $4^{\frac{3}{2}}$. **13.** $27^{\frac{2}{3}}$. **14.** $16^{-\frac{3}{4}}$. **15.** $100^{\frac{3}{2}}$.

16. $(\frac{1}{9})^{\frac{1}{2}}$. **17.** $(\frac{1}{4})^{\frac{3}{2}}$. **18.** $(\frac{2}{3})^{0}$. **19.** $(\frac{8}{27})^{\frac{1}{3}}$. **20.** $(\frac{27}{64})^{-\frac{1}{3}}$.

21. $\frac{1}{3^{-1}}$. **22.** $\frac{1}{2^{-3}}$.

Express with positive indices

23. x^{-2}. 24. $2x^{-3}$. 25. $3x^{-\frac{1}{2}}$. 26. $2a^{-\frac{2}{3}}$.

27. $2\div a^{-1}$. 28. $5\div b^{-3}$. 29. $\dfrac{1}{3a^{-2}}$. 30. $\dfrac{1}{5x^{-\frac{1}{2}}}$.

31. $\dfrac{4}{3x^{-\frac{1}{5}}}$. 32. $\dfrac{1}{a^{-\frac{1}{n}}}$. 33. $\dfrac{3x^{-2}y^{3}}{4a^{-3}b^{2}}$. 34. $\dfrac{x^{-5}y^{-7}}{a^{-3}b^{-4}}$.

35. $\dfrac{2a^{\frac{1}{2}}}{5b^{-\frac{1}{2}}c^{-\frac{1}{2}}}$.

Express with radical signs and positive indices

36. $x^{\frac{1}{2}}$. 37. $x^{-\frac{1}{2}}$. 38. $2x^{-\frac{1}{5}}$. 39. $3a^{\frac{1}{n}}$.

40. $\dfrac{1}{3a^{\frac{1}{3}}}$. 41. $\dfrac{2}{x^{-\frac{2}{3}}}$. 42. $\dfrac{1}{x^{-\frac{1}{n}}}$. 43. $a^{\frac{2}{n}}$.

44. $a^{\frac{n}{2}}$.

Simplify

45. $a^{-\frac{1}{2}}\times a^{-\frac{1}{3}}$. 46. $a^{-\frac{1}{3}}\times 2a^{-\frac{1}{4}}$. 47. $2a^{-2}\div 3a^{2}$.

48. $4x^{-1}\div 2x^{-\frac{1}{2}}$. 49. $\sqrt{a}\times\sqrt[3]{a}$.

50. $\dfrac{3a^{-2}}{4a^{-\frac{3}{2}}}$. 51. $\dfrac{a^{-\frac{1}{3}}}{5a}$. 52. $\dfrac{\sqrt[3]{x^{-1}}}{\sqrt[3]{x^{2}}}$.

53. $\sqrt[3]{a^{2}}\times\sqrt{a^{3}}$. 54. $\sqrt[3]{a^{-n}}\div\sqrt[3]{a^{-2n}}$. 55. $2^{2}.2^{3}.2^{-4}$.

56. $3^{5}.3^{-7}.3^{9}$. 57. $\sqrt{5}.\sqrt[3]{5}.\sqrt[5]{5}$. 58. $\sqrt{a}.\sqrt[3]{a}.\sqrt[4]{a}$.

59. $7^{0}.7^{-1}.7^{2}.7^{-3}.7^{4}.7^{-5}$. 60. $a^{\frac{1}{2}}.a^{\frac{1}{3}}.a^{\frac{1}{4}}.a^{\frac{1}{5}}$.

61. $a^{-\frac{1}{2}}.a^{-\frac{1}{3}}.a^{-\frac{1}{4}}.a^{-\frac{1}{5}}$. 62. $2\div\sqrt{2}$.

63. $9\div\sqrt{9}$. 64. $8\div\sqrt[3]{8}$. 65. $\sqrt{x^{3}}.\sqrt[6]{x^{2}}.\sqrt[12]{x^{3}}$.

66. $x\sqrt{yz}.y\sqrt{zx}.z\sqrt{xy}$. 67. $5^{-2}\div 5^{-1}$.

68. $2^{-4}\div 2^{5}$. 69. $\dfrac{\sqrt{x}.\sqrt[4]{x}}{\sqrt[3]{x}}$. 70. $\dfrac{\sqrt[3]{x^{2}}.\sqrt[6]{x^{5}}}{\sqrt[12]{x^{7}}}$.

71. $\sqrt[4]{\dfrac{x^{2}y^{4}}{a^{4}b^{6}}}.\sqrt{\dfrac{a^{2}b^{3}}{xy^{2}}}$. 72. $\dfrac{\sqrt{a^{2}bc}.\sqrt{ab^{2}c}.\sqrt{abc^{2}}}{\sqrt{abc}}$.

73. $\dfrac{\sqrt[3]{x^{3}y^{2}z}.\sqrt[3]{x^{2}y^{3}z}}{\sqrt[3]{x^{2}y^{2}z^{2}}}$.

Find the value of

74. $\sqrt{1}+\sqrt{4}+\sqrt{9}+\sqrt{16}+\sqrt{25}$.

75. $\sqrt[3]{1}-\sqrt[3]{8}-\sqrt[3]{27}+\sqrt[3]{64}$.

76. $0^{\frac{1}{4}}+16^{\frac{1}{4}}+81^{\frac{1}{4}}-625^{\frac{1}{4}}$.

77. $4^{\frac{3}{2}}-16^{\frac{1}{4}}+(\frac{1}{27})^{-\frac{2}{3}}-(\frac{1}{4})^{-\frac{5}{2}}$.

78. $3^0+4^{-1}-\frac{1}{5^{-1}}+\left(\frac{1}{49}\right)^{\frac{1}{2}}+2^{-2}$

79. $2^{\frac{3}{2}}\,.\,5^{-\frac{1}{2}}\,.\,6^{\frac{1}{2}}\,.\,10^{\frac{1}{2}}\,.\,12^{\frac{1}{2}}$.

80. $2^{-\frac{1}{3}}\,.\,4^{\frac{1}{2}}\,.\,16^{\frac{3}{4}}\,.\,2^{-\frac{8}{3}}$.

200. *To prove that* $(a^m)^n=a^{mn}$ *for all values of* m *and* n.

CASE I. Let n be a positive integer.

Then, whatever m may be,

$$(a^m)^n = a^m \times a^m \times a^m \times \ldots\ldots\ldots \text{to } n \text{ factors}$$
$$= a^{m+m+m+\cdots\cdots\cdots \text{to m terms}}$$
$$= a^{mn}.$$

CASE II. Let n be a positive fraction, say $\frac{p}{q}$, where p and q are positive integers.

Then, whatever m may be,

$$(a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} \qquad \text{[Article 199]}$$
$$= \sqrt[q]{a^{mp}} \qquad \text{[Case I]}$$
$$= a^{\frac{mp}{q}} \qquad \text{[Article 199]}$$
$$= a^{mn}.$$

CASE III. Let n be negative, say $-r$, where r is a positive quantity, fractional or integral.

Then, whatever m may be,

$$(a^m)^n = (a^m)^{-r} = \frac{1}{(a^m)^r} \qquad \text{[Article 198]}$$
$$= \frac{1}{a^{mr}} \qquad \text{[Case I or II]}$$
$$= a^{-mr} \qquad \text{[Article 198]}$$
$$= a^{m(-r)} = a^{mn}.$$

Hence $(a^m)^n = a^{mn}$ is universally true.

EXAMPLES. (i) $(a^{\frac{5}{3}})^{\frac{6}{5}} = a^{\frac{5}{3}\times\frac{6}{5}} = a^2$.

(ii) $\{(a^{-2})^4\}^{-6} = (a^{-8})^{-6} = a^{48}$.

(iii) $\left(x^{\frac{1}{a+b}}\right)^{a^2-b^2} = x^{\frac{1}{a+b}\times(a^2-b^2)} = x^{a-b}$.

201. *To prove that* $(ab)^n = a^n b^n$ *for all values of* n.

CASE I. Let n be a positive integer.

Then $(ab)^n = ab.\ ab.\ ab......$to n factors

$= (a.\ a.\ a.......$to n factors)

$\times$ $(b.\ b.\ b.......$to n factors)

$= a^n b^n$.

CASE II. Let n be a positive fraction, say $\frac{p}{q}$, where p and q are positive integers.

The q^{th} power of $(ab)^n$

$= \{(ab)^n\}^q = \{(ab)^{\frac{p}{q}}\}^q = (ab)^p$ [Article 199]

$= a^p b^p$. [Case I]

Also the q^{th} power of $a^n b^n$

$= (a^n b^n)^q = (a^{\frac{p}{q}} b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q \times (b^{\frac{p}{q}})^q$ [Case I]

$= a^p b^p$.

$\therefore \quad (ab)^n = a^n b^n$.

CASE III. Let n be negative, say $-r$, where r is a positive quantity, fractional or integral.

Then $(ab)^n = (ab)^{-r} = \dfrac{1}{(ab)^r}$ [Article 198]

$= \dfrac{1}{a^r b^r}$ [Case I or II]

$= \dfrac{1}{a^r} \times \dfrac{1}{b^r} = a^{-r} b^{-r}$ [Article 198]

$= a^n b^n$.

Hence $(ab)^n = a^n b^n$ is universally true.

EXAMPLES. (i) $\left(a^{\frac{3}{2}}b^{\frac{5}{4}}\right)^8 = a^{\frac{3}{2}\times 8}b^{\frac{5}{4}\times 8} = a^{12}b^{10}$.

(ii) $(32a^{-5}b^{10})^{\frac{1}{5}} = (2^5 . a^{-5}b^{10})^{\frac{1}{5}} = 2^{5\times\frac{1}{5}} . a^{-5\times\frac{1}{5}} . b^{10\times\frac{1}{5}}$

$= 2a^{-1}b^2 = \frac{2b^2}{a}$.

EXAMPLE. *Simplify* $\left\{\sqrt[4]{\left(x^{-\frac{2}{3}}y^{\frac{1}{3}}z^{\frac{2}{5}}\right)^3}\right\}^{-\frac{2}{3}}$.

The expression $= \left\{\left(x^{-\frac{2}{3}}y^{\frac{1}{3}}z^{\frac{2}{5}}\right)^{\frac{3}{4}}\right\}^{-\frac{2}{3}} = \left(x^{-\frac{2}{3}}y^{\frac{1}{3}}z^{\frac{2}{5}}\right)^{-\frac{1}{2}}$

$= x^{\frac{1}{3}}y^{-\frac{1}{6}}z^{-\frac{1}{5}} = \frac{x^{\frac{1}{3}}}{y^{\frac{1}{6}}z^{\frac{1}{5}}}$.

202. *To prove that* $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ *for all values of* n.

From the previous article, we have

$$a^n = \left(\frac{a}{b}\times b\right)^n = \left(\frac{a}{b}\right)^n \times b^n,$$

i.e. $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$.

EXAMPLE. *Simplify* $\left(\frac{x^{-3}}{y^{\frac{2}{3}}z^{-1}}\right)^{-\frac{3}{2}} \times \left(\frac{y^{-1}z^{\frac{10}{3}}}{x^{-\frac{15}{4}}}\right)^{\frac{2}{5}}$.

The expression $= \frac{(x^{-3})^{-\frac{3}{2}}}{\left(y^{\frac{2}{3}}z^{-1}\right)^{-\frac{3}{2}}} \times \frac{\left(y^{-1}z^{\frac{10}{3}}\right)^{\frac{2}{5}}}{\left(x^{-\frac{15}{4}}\right)^{\frac{2}{5}}} = \frac{x^{\frac{9}{2}}}{y^{-1}z^{\frac{3}{2}}} \times \frac{y^{-\frac{2}{5}}z^{\frac{4}{3}}}{x^{-\frac{3}{2}}}$

$= x^{\frac{9}{2}+\frac{3}{2}}y^{-\frac{2}{5}+1}z^{\frac{4}{3}-\frac{3}{2}} = x^6y^{\frac{3}{5}}z^{-\frac{1}{6}} = \frac{x^6y^{\frac{3}{5}}}{z^{\frac{1}{6}}}$.

EXAMPLES CX

(*Examples* 1 *to* 24 *may be taken orally*)

Write down the square of

1. 10^2. **2.** 4^5. **3.** 6^{-3} **4.** x^{-1}.

5. x^5. **6.** x^n. **7.** x^{2n}. **8.** x^2.

Write down the cube of

9. 2^2. **10.** 2^3. **11.** 5^{-3}. **12.** 10^{-2}.

13. x^2. **14.** x^n. **15.** x^{-n} **16.** $x^{-\frac{n}{3}}$.

Write down the square root of

17. 6^2. **18.** 8^4. **19.** a^{-2}. **20.** a^{-4}.

21. $a^{-4}b^{10}$. **22.** a^2b^{10}. **23.** x^{2n}. **24.** x^{3n}.

Simplify and express with positive indices

25. $(a^4)^{\frac{1}{2}}$. **26.** $(a^6)^{\frac{1}{3}}$. **27.** $(a^3)^{-\frac{2}{3}}$. **28.** $(a^2b)^{\frac{3}{4}}$.

29. $(4a^{-2})^{\frac{3}{2}}$. **30.** $(16a^{12})^{\frac{3}{4}}$. **31.** $(8a^9)^{-\frac{2}{3}}$. **32.** $(9x^8)^{\frac{3}{2}}$.

33. $(9x^{-8})^{-\frac{3}{2}}$. **34.** $(x^{\frac{1}{2}})^{\frac{2}{3}}$. **35.** $(x^{\frac{1}{2}}y^{\frac{1}{3}})^{\frac{6}{5}}$.

36. $(x^{-\frac{2}{3}}y^{\frac{3}{5}})^{-\frac{15}{16}}$. **37.** $(a^2x)^{\frac{1}{3}}(ax^2)^{\frac{1}{2}}$. **38.** $(a^{-2}b)^{-2}(ab)^{-3}$.

39. $\dfrac{(a^2x)^{\frac{1}{3}}}{(ax^2)^{\frac{1}{2}}}$. **40.** $\left(\dfrac{a^{-2}b^{-1}}{a^2b^3}\right)^{-2}$. **41.** $\left(\dfrac{a^2b^2}{16a^4}\right)^2$.

42. $\left(\dfrac{8x^{-3}}{27y^{-3}}\right)^{-\frac{1}{3}}$. **43.** $\left(\dfrac{\sqrt{x}}{\sqrt{y}}\right)^{\frac{1}{2}}$. **44.** $\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{y}}\right)^3$.

45. $(\sqrt{x^2y^3})^6$. **46.** $(x^ny^{-m})^3 \times (x^3y^2)^{-n}$.

47. $\left(\dfrac{16a^2}{x^{-2}}\right)^{-\frac{1}{4}}$. **48.** $\left(\dfrac{x}{\sqrt[n]{x}}\right)^n$. **49.** $\left(\dfrac{a^{m+n}}{a^n}\right)^m$.

50. $\left(\dfrac{x^2y^2}{a^3b^2}\right)^{\frac{1}{4}}\left(\dfrac{a^3y^2}{xy}\right)^{\frac{1}{2}}$. **51.** $(a^{-2}b)^{\frac{1}{2}} \times (ab^{-3})^{\frac{1}{3}}$.

52. $\left(\dfrac{\sqrt{a}}{\sqrt{b}}\right)^{\frac{1}{2}}\left(\dfrac{\sqrt{b}}{\sqrt{c}}\right)^{\frac{1}{2}}\left(\dfrac{\sqrt{c}}{\sqrt{a}}\right)^{\frac{1}{2}}$. **53.** $(x^3)^n.\ (x^{-2})^{3n}.\ (x^3)^{3n}$.

54. $(x^m)^n \times (x^n)^m \times (x^{-m})^{-n}$. **55.** $\sqrt[6]{x.\ \sqrt[3]{x^{-1}}}$. **56.** $\left(\dfrac{\sqrt[n]{x^n}}{\sqrt[m]{x}}\right)^{\frac{1}{1-m}}$.

57. $\sqrt{(x+y)^3} \times \sqrt[3]{(x+y)^2}$. **58.** $(xy)^{a-b}(yz)^{b-c}(zx)^{c-a}$.

59. $\left\{(a-b)^{-2}\right\}^n \div \left\{(a+b)^n\right\}^2$. **60.** $\sqrt{(a+b)^3} \times (a^2-b^2)^{-\frac{1}{2}}$.

61. $\left(\dfrac{x^{\frac{2}{3}}y^{-\frac{1}{2}}}{yx^{-\frac{2}{3}}} \div \sqrt{\dfrac{xy^{-2}}{yx^{-1}}}\right)^6$. **62.** $\left(\dfrac{a^{-\frac{2}{3}}b^{\frac{1}{2}}}{b^{-1}a}\right)^2 \times \sqrt[3]{\dfrac{b^{-3}}{a^{-1}}}$.

203. Examples on multiplication and division of compound expressions involving fractional and negative indices are worked out in the ordinary way as if the indices are positive integers. If root symbols occur, they should first be changed into index forms.

NOTE. In Article 161, we pointed out that the terms of the expression $\frac{9x^2}{16}+\frac{4}{81x^2}-\frac{3x}{4}+\frac{2}{9x}-\frac{1}{12}$ were arranged in descending powers of x thus:

$$\frac{9x^2}{16}-\frac{3x}{4}-\frac{1}{12}+\frac{2}{9x}+\frac{4}{81x^2}.$$

A reason for this arrangement may be seen if we write the terms of the expression thus:

$$\frac{9}{16}x^2-\frac{3}{4}x-\frac{1}{12}x^0+\frac{2}{9}x^{-1}+\frac{4}{81}x^{-2}.$$

EXAMPLE 1. *Multiply* $x^{-\frac{1}{3}}+2x-3x^{\frac{2}{3}}$ *by* $x^{\frac{1}{3}}-1$.

Arranging the two expressions in descending powers of x, the product

$$=(2x-3x^{\frac{2}{3}}+x^{-\frac{1}{3}})\times(x^{\frac{1}{3}}-1)$$
$$=2x\times x^{\frac{1}{3}}-3x^{\frac{2}{3}}\times x^{\frac{1}{3}}+x^{-\frac{1}{3}}\times x^{\frac{1}{3}}-2x+3x^{\frac{2}{3}}-x^{-\frac{1}{3}}$$
$$=2x^{\frac{4}{3}}-3x+x^0-2x+3x^{\frac{2}{3}}-x^{-\frac{1}{3}}$$
$$=2x^{\frac{4}{3}}-5x+3x^{\frac{2}{3}}+x^0-x^{-\frac{1}{3}}$$
$$=2x^{\frac{4}{3}}-5x+3x^{\frac{2}{3}}+1-x^{-\frac{1}{3}}.$$

EXAMPLE 2. *Divide* $x-9x^{\frac{1}{3}}-12-x^{\frac{2}{3}}$ *by* $3x^{\frac{1}{3}}+x^{\frac{2}{3}}+3$.

First arranging both the expressions in descending powers of x and then dividing, we have

$$\begin{array}{l} x^{\frac{2}{3}}+3x^{\frac{1}{3}}+3)\,x-x^{\frac{2}{3}}-9x^{\frac{1}{3}}-12\,(\,x^{\frac{1}{3}}-4 \\ \qquad\qquad\quad \underline{x+3x^{\frac{2}{3}}+3x^{\frac{1}{3}}\qquad\quad} \\ \qquad\qquad\qquad -4x^{\frac{2}{3}}-12x^{\frac{1}{3}}-12 \\ \qquad\qquad\qquad \underline{-4x^{\frac{2}{3}}-12x^{\frac{1}{3}}-12} \end{array}$$

Hence the quotient is $x^{\frac{1}{3}}-4$.

EXAMPLES CXI

Multiply

1. $\sqrt{7}-\sqrt{5}$ by $\sqrt{7}$.
2. $3\sqrt{2}+1$ by $\sqrt{2}$.
3. $\sqrt{x}-\sqrt{y}$ by $\sqrt{x}$.
4. $\sqrt{5}-\sqrt{3}$ by $\sqrt{5}+\sqrt{3}$.
5. $\sqrt{x}-\sqrt{y}$ by $\sqrt{x}+\sqrt{y}$.
6. $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}+\sqrt{2}$.
7. $\sqrt{x}+\sqrt{y}$ by $\sqrt{x}+\sqrt{y}$.
8. $\sqrt{x}+2\sqrt{y}$ by $\sqrt{x}+3\sqrt{y}$.
9. $\sqrt{a+b}+\sqrt{a-b}$ by $\sqrt{a+b}-\sqrt{a-b}$.
10. $\sqrt{\frac{a+b}{2}}-\sqrt{\frac{a-b}{2}}$ by $\sqrt{\frac{a+b}{2}}+\sqrt{\frac{a-b}{2}}$.
11. $x^2+x^{-2}+1$ by $x^2+x^{-2}-1$.
12. $7x^{\frac{2}{3}}-2x^{\frac{1}{2}}+1$ by $3x^{\frac{1}{3}}+1$.
13. $3a^{\frac{1}{3}}-3a^{-\frac{1}{3}}+2a^{-1}$ by $5a^{\frac{2}{3}}+4$.
14. $2x-4x^{\frac{2}{3}}-5$ by $3x^{\frac{2}{3}}+4x^{\frac{1}{3}}-2$.
15. $a^{\frac{1}{2}}+3a^{\frac{1}{4}}+2$ by $a^{\frac{1}{2}}+3a^{\frac{1}{4}}+2$.

Divide

16. $x^{\frac{2}{3}}-7x^{\frac{1}{3}}+12$ by $x^{\frac{1}{3}}-3$.
17. $9x^{\frac{2}{3}}+6x^{\frac{1}{3}}y^{\frac{1}{3}}-35y^{\frac{2}{3}}$ by $3x^{\frac{1}{3}}+7y^{\frac{1}{3}}$.
18. $2x-3x^{\frac{2}{3}}-6x^{\frac{1}{3}}-1$ by $2x^{\frac{2}{3}}-5x^{\frac{1}{3}}-1$.
19. $x+x^{\frac{3}{4}}+7x^{\frac{1}{2}}-6x^{\frac{1}{4}}+8$ by $x^{\frac{1}{2}}+2x^{\frac{1}{4}}+8$.
20. $x-5x^{\frac{4}{5}}+9x^{\frac{3}{5}}-6x^{\frac{2}{5}}-x^{\frac{1}{5}}+2$ by $x^{\frac{2}{5}}-3x^{\frac{1}{5}}+2$.
21. $a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

CHAPTER XXXVII

SURDS

Simple Surds

204. The root of a number or quantity which cannot be exactly found is called a **surd** or **irrational quantity**.

Thus, $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, $\sqrt[5]{4}$,...are surds, for the square roots of 2 and 5, cube roots of 3 and 4, fifth root of 4,... cannot be exactly found, though we can obtain their values to as many decimal places as we please.

If the root of a quantity can be exactly found, it is called a **rational quantity.**

Thus, $\sqrt{4}$, $\sqrt{9}$, $\sqrt[3]{8}$, $\sqrt[5]{32}$,...are rational quantities.

205. Since $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{3}$, $\sqrt[5]{4}$,...can be written as $2^{\frac{1}{2}}$, $5^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{5}}$,... hence we see that surds can be expressed as quantities with fractional indices and are therefore subject to the principles which govern other algebraical symbols.

206. The **degree** or **order** of a surd is denoted by its root symbol. Thus $\sqrt{2}$, $\sqrt{5}$ are surds of the *second order*, $\sqrt[3]{3}$ is a surd of the *third order* and $\sqrt[5]{4}$ is a surd of the *fifth order*.

207. Since $\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6}$,

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4},$$

$$\text{and } \sqrt[6]{7} = 7^{\frac{1}{6}} = 7^{\frac{2}{12}} = \sqrt[12]{7^2},$$

hence we see that *surds of different orders can be transformed into surds of the same order.*

EXAMPLE. *Express* $\sqrt{a^3}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a}$ *as surds of the same lowest order.*

The L.C.M. of 2, 3, 4 is 12. Therefore expressing the given surds as surds of the twelfth order, we have

$$\sqrt{a^3}=a^{\frac{3}{2}}=a^{\frac{18}{12}}=\sqrt[12]{a^{18}},$$
$$\sqrt[3]{a^2}=a^{\frac{2}{3}}=a^{\frac{8}{12}}=\sqrt[12]{a^8},$$
$$\sqrt[4]{a}=a^{\frac{1}{4}}=a^{\frac{3}{12}}=\sqrt[12]{a^3}.$$

Hence $\sqrt[12]{a^{18}}$, $\sqrt[12]{a^8}$, $\sqrt[12]{a^3}$ are the required surds of the same lowest order.

208. Since $x=\sqrt{x^2}=\sqrt[3]{x^3}=\ldots=\sqrt[n]{x^n}$.

Hence we see that *any quantity which is not a surd can be written in the form of a surd of any order.*

209. Sometimes surds are simplified by the help of factors. Thus

(i) $\sqrt{192}=\sqrt{8.8.3}=8\sqrt{3}$.

(ii) $\sqrt[3]{500}=\sqrt[3]{5.5.5.4}=5\sqrt[3]{4}$.

(iii) $\sqrt[5]{32a^5b^{12}}=\sqrt[5]{2^5.a^5.b^{10}.b^2}=2ab^2.\sqrt[5]{b^2}$.

NOTE. Students should remember that $\sqrt{ab}=\sqrt{a}.\sqrt{b}$, but $\sqrt{a+b}$ is not equal to $\sqrt{a}+\sqrt{b}$.

210. Since $3\sqrt{2}=\sqrt{9}.\sqrt{2}=\sqrt{18}$,

$$2\sqrt[3]{3}=\sqrt[3]{8}.\sqrt[3]{3}=\sqrt[3]{24},$$
$$a.\sqrt[n]{b}=\sqrt[n]{a^n}.\sqrt[n]{b}=\sqrt[n]{a^nb},$$

hence we see that *the product of a rational quantity and a surd can be expressed in the form of a surd.*

EXAMPLES CXII

Express in terms of the smallest possible surds

1. $\sqrt{98}$.
2. $\sqrt{150}$.
3. $\sqrt{288}$.
4. $\sqrt[3]{256}$.
5. $\sqrt[3]{1029}$.
6. $\sqrt[3]{-448}$.
7. $\sqrt{49a^3}$.
8. $\sqrt{12a^2b^5}$.
9. $\sqrt{75a^3x^7}$.
10. $\sqrt[3]{32a^4}$.
11. $\sqrt[3]{3\frac{3}{8}}$.

12. $\sqrt[4]{117r^3s}$. **13.** $\sqrt[3]{-108x^6y^4}$. **14.** $\sqrt[n]{x^ny^{2n}}$.

15. $\sqrt[n]{x^{2n}y^{n+1}}$. **16.** $\sqrt{(a+b)^3}$. **17.** $\sqrt{a^3-2a^2b+ab^2}$.

Express as entire surds

18. $5\sqrt{2}$. **19.** $3\sqrt{7}$. **20.** $8\sqrt{3}$. **21.** $2\sqrt[3]{5}$.

22. $10\sqrt[3]{5}$. **23.** $a\sqrt{2a}$. **24.** $5x\sqrt{3x}$. **25.** $2x^2.\sqrt[3]{x^2}$.

26. $ab^2\sqrt{3ab}$. **27.** $3a^2b.\sqrt[3]{2a^2b}$. **28.** $a\sqrt[n]{a}$. **29.** $a^2.\sqrt[n]{a^2}$.

Express as surds of the twelfth order

30. $\sqrt{x}$. **31.** $\sqrt{x^3}$. **32.** $\sqrt{2x^5}$. **33.** $\sqrt[3]{x^4}$.

34. $\sqrt[3]{3x^2}$. **35.** $\sqrt[3]{5xy^5}$. **36.** $\sqrt[4]{3x^2}$. **37.** $\sqrt[4]{5x^3y^7}$.

38. $\sqrt[4]{2x^2y^3}$. **39.** $\sqrt[6]{2x^5}$. **40.** $\sqrt[4]{a^2b^5c^7}$.

Express as surds of the n[th] order

41. $\sqrt{x}$. **42.** $\sqrt[3]{a}$. **43.** $\sqrt[4]{2x^2}$. **44.** $\sqrt[7]{x^2y^3}$.

45. $\sqrt[5]{2x^3y^4}$.

Express as surds of the same lowest order

46. $\sqrt{a}$, $\sqrt[3]{a^2}$. **47.** $\sqrt[3]{a}$, $\sqrt[4]{a^2}$. **48.** $\sqrt{a}$, $\sqrt[5]{a^2}$.

49. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$. **50.** $\sqrt[3]{a^2}$, $\sqrt[4]{a^3}$, $\sqrt[5]{a^4}$.

51. $\sqrt{ax^3}$, $\sqrt[3]{a^2x}$, $\sqrt[4]{a^3x^2}$. **52.** $\sqrt{xy}$, $\sqrt[3]{x^2y^2}$, $\sqrt[6]{x^5y^7}$.

211. Surds are said to be **like** or **similar** when they have, or can be reduced to have, the same irrational factor, other factors being rational.

Thus $\sqrt{2}$, $3\sqrt{2}$, $5\sqrt{2}$ and $\frac{1}{2}\sqrt{2}$ are like surds. $\sqrt{3}$, $\sqrt{27}$ and $\sqrt{147}$ are also like surds, for they can be reduced to $\sqrt{3}$, $3\sqrt{3}$ and $7\sqrt{3}$.

Surds which are not like or similar are said to be **unlike** or **dissimilar.**

Thus, $\sqrt{2}$ and $\sqrt{5}$ are unlike surds, and $\sqrt{3}$ and $\sqrt[3]{2}$ are also unlike surds.

EXAMPLE 1. *Simplify* $\sqrt{20}-\sqrt{45}+\sqrt{125}$.

Since $\sqrt{20}=\sqrt{2.2.5}=2\sqrt{5}$,

$\sqrt{45}=\sqrt{3.3.5}=3\sqrt{5}$,

and $\sqrt{125}=\sqrt{5.5.5}=5\sqrt{5}$.

Therefore the expression $=2\sqrt{5}-3\sqrt{5}+5\sqrt{5}=4\sqrt{5}$.

Hence we see that *to find the sum or difference of the surds of the same order, we first reduce them to their simplest form and then find their sum or difference by affixing to their irrational part the sum or difference of their coefficients.*

NOTE. The sum or difference of unlike surds cannot be simplified.

EXAMPLE 2. *Multiply* $4\sqrt{2}$ *by* $5\sqrt{3}$.

The product $=4\sqrt{2}\times 5\sqrt{3}=4\times 5\times\sqrt{2}\times\sqrt{3}=20\sqrt{6}$.

EXAMPLE 3. *Simplify* $2\sqrt[3]{5}\times 5.\sqrt[3]{6}\times 4.\sqrt[3]{7}$.

The expression $=2\times 5\times 4\times\sqrt[3]{5}\times\sqrt[3]{6}\times\sqrt[3]{7}=40.\sqrt[3]{210}$.

From Examples 2 and 3, we see that *to find the product of the surds of the same order, we multiply separately their rational factors and irrational factors.*

EXAMPLE 4. *Multiply* $4\sqrt{5}$ *by* $5.\sqrt[3]{2}$.

Expressing as surds of the same lowest order, we have

$$4\sqrt{5}=4.5^{\frac{1}{2}}=4.5^{\frac{3}{6}}=4.\sqrt[6]{5^3},$$

$$\text{and}\quad 5.\sqrt[3]{2}=5.2^{\frac{1}{3}}=5.2^{\frac{2}{6}}=5.\sqrt[6]{2^2}.$$

$\therefore$ the product $=4.\sqrt[6]{5^3}\times 5.\sqrt[6]{2^2}=20.\sqrt[6]{5^3 2^2}=20.\sqrt[6]{500}$.

Hence we see that *to multiply surds of different orders, we first reduce them to the surds of the same order.*

212. Suppose it is required to find the value of $\frac{20}{\sqrt{2}}$.

It can be written as $\frac{20}{1{\cdot}41421}$ and then by division its value can be found.

A much shorter method of finding the value is to multiply both the numerator and the denominator by $\sqrt{2}$, so as to make the denominator a rational quantity. Thus

$$\frac{20}{\sqrt{2}}=\frac{20}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{20\sqrt{2}}{2}=10\sqrt{2}=10\times 1{\cdot}41421\ldots$$

$$=14{\cdot}1421\ldots$$

This process of removing surds from the denominator of a fraction is called **rationalising the denominator**.

EXAMPLE. *Rationalise the denominator of* $\frac{\sqrt{x}}{\sqrt{y}}$.

The fraction $=\frac{\sqrt{x}}{\sqrt{y}}\times\frac{\sqrt{y}}{\sqrt{y}}=\frac{\sqrt{xy}}{y}$.

Compound Surds

213. A compound expression involving two or more simple surds is called a **compound surd**.

Thus $\sqrt{2}-3\sqrt{5}$, $2\sqrt{a}+5\sqrt{b}$, $\sqrt[3]{x}-\sqrt[5]{y}$ are compound surds.

The product of two compound surds is found by multiplying each term of the first by each term of the second as in ordinary multiplication of compound expressions.

214. When two binomial surds of the second order differ only in the sign which connects their terms, they are said to be **conjugate.**

Thus $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ are conjugate surds ; $2\sqrt{a}-3\sqrt{b}$ and $2\sqrt{a}+3\sqrt{b}$ are also conjugate surds.

Let us now consider the product of $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$.

$$\text{The product}=(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=(\sqrt{3})^2-(\sqrt{2})^2$$
$$=3-2=1.$$

Similarly the product of $2\sqrt{a}-3\sqrt{b}$ and $2\sqrt{a}+3\sqrt{b}$ is $(2\sqrt{a})^2-(3\sqrt{b})^2=4a-9b$.

Thus we see that *the product of two conjugate surds is rational.*

215. In simplifying a fraction involving a compound surd in the denominator, it is advisable to rationalise the denominator by multiplying both the numerator and the denominator by the conjugate surd.

EXAMPLE 1. *Simplify* $\dfrac{5\sqrt{3}-3\sqrt{2}}{5\sqrt{3}+3\sqrt{2}}$.

Multiplying both the numerator and the denominator by the conjugate of the denominator, we have the fraction

$$=\frac{5\sqrt{3}-3\sqrt{2}}{5\sqrt{3}+3\sqrt{2}}\times\frac{5\sqrt{3}-3\sqrt{2}}{5\sqrt{3}-3\sqrt{2}}=\frac{(5\sqrt{3})^2+(3\sqrt{2})^2-2.5\sqrt{3}.3\sqrt{2}}{(5\sqrt{3})^2-(3\sqrt{2})^2}$$
$$=\frac{75+18-30\sqrt{6}}{75-18}=\frac{93-30\sqrt{6}}{57}=\frac{31-10\sqrt{6}}{19}.$$

EXAMPLE 2. *Rationalise the denominator of* $\dfrac{\sqrt{1+x^2}+2\sqrt{1-x^2}}{2\sqrt{1+x^2}-\sqrt{1-x^2}}$.

The conjugate of the denominator is $2\sqrt{1+x^2}+\sqrt{1-x^2}$.

Multiplying both the numerator and the denominator by this quantity, we have the fraction

$$=\frac{\sqrt{1+x^2}+2\sqrt{1-x^2}}{2\sqrt{1+x^2}-\sqrt{1-x^2}}\times\frac{2\sqrt{1+x^2}+\sqrt{1-x^2}}{2\sqrt{1+x^2}+\sqrt{1-x^2}}$$

$$=\frac{2(\sqrt{1+x^2})^2+4\sqrt{1+x^2}.\sqrt{1-x^2}+\sqrt{1+x^2}.\sqrt{1-x^2}+2(\sqrt{1-x^2})^2}{(2\sqrt{1+x^2})^2-(\sqrt{1-x^2})^2}$$

$$=\frac{2(1+x^2)+5\sqrt{1-x^4}+2(1-x^2)}{4(1+x^2)-(1-x^2)}=\frac{4+5\sqrt{1-x^4}}{3+5x^2}.$$

EXAMPLES CXIII

Simplify

1. $\sqrt{8}+\sqrt{50}$.
2. $\sqrt{50}-\sqrt{18}$.
3. $2\sqrt{a}+3\sqrt{a}+5\sqrt{a}$.
4. $\sqrt{12}+\sqrt{27}+\sqrt{48}$.
5. $a\sqrt{x}+2a\sqrt{x}+3a\sqrt{x}$.
6. $\sqrt{50}-\sqrt{18}+\sqrt{8}$.
7. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{250}$.
8. $\sqrt{a^3b}+\sqrt{ab^3}$.
9. $\sqrt[3]{a^4b}-\sqrt[3]{ab^4}$.
10. $\sqrt{175}-\sqrt{63}+\sqrt{28}-4\sqrt{7}$.
11. $4\sqrt{18}+2\sqrt{32}-3\sqrt{8}-5\sqrt{2}$.
12. $5.\sqrt[3]{54}+3\sqrt[3]{16}-\sqrt[3]{2}-2\sqrt[3]{128}$.
13. $\sqrt{20x^3}-\sqrt{45x^3}+\sqrt{80x^3}-\sqrt{125x^3}$.
14. $5b\sqrt{a^3b}+2a\sqrt{ab^3}+\sqrt{a^3b^3}-ab\sqrt{9ab}$.
15. $3\sqrt{\frac{2}{5}}-2\sqrt{\frac{1}{10}}$.
16. $\sqrt{\frac{a}{3}}+\sqrt{3a}+\sqrt{27a}$.
17. $\sqrt{(x+y)^2z}+\sqrt{(x-y)^2.z}+\sqrt{x^2z}-\sqrt{y^2z}$.
18. $\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x^3}}+\sqrt{\frac{1}{x^5}}$.
19. $3\sqrt{5}.\,4\sqrt{45}$.
20. $2\sqrt{27}.\,6\sqrt{6}.\,5\sqrt{2}$.
21. $5\,.\sqrt{32}\,.\sqrt{48}\,.\,3\sqrt{54}$.
22. $5.\sqrt[3]{12}.\,4\sqrt[3]{18}$.
23. $\sqrt[3]{168}.\sqrt[3]{147}$.
24. $\sqrt{a}.\sqrt[3]{a}$.
25. $\sqrt[3]{a}.\sqrt[4]{a^3}$.
26. $\sqrt[3]{x}.\sqrt[4]{y}$.
27. $4\,.\sqrt[4]{3}\,.\,3\sqrt[3]{4}$.
28. $2\sqrt{5}\,.\,5\sqrt[3]{2}$.
29. $\sqrt[m]{a}\,.\sqrt[n]{a}$.
30. $\sqrt{a+b}.\sqrt[4]{a-b}$.

Rationalise the denominators of

31. $\frac{5}{\sqrt{2}}$.
32. $\frac{5}{\sqrt{7}}$.
33. $\frac{1}{2+\sqrt{3}}$.
34. $\frac{4}{\sqrt{5}-1}$.

35. $\frac{3}{1+\sqrt{2}}$. 36. $\frac{1}{1-\sqrt{x}}$. 37. $\frac{1}{\sqrt{3}-\sqrt{2}}$. 38. $\frac{4+3\sqrt{2}}{3-2\sqrt{2}}$.

39. $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$. 40. $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$. 41. $\frac{1}{a-\sqrt{a^2-b^2}}$.

42. $\frac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$. 43. $\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$. 44. $\frac{x^2}{\sqrt{x^2+y^2}+y}$.

45. $\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

Square Root of Compound Surds

216. To prove that *the square root of a rational quantity cannot be partly rational and partly irrational.*

If possible let $\sqrt{n}=a+\sqrt{m}$.

Squaring, $$n=a^2+m+2a\sqrt{m},$$
$$\sqrt{m}=\frac{n-a^2-m}{2a}.$$

That is, a surd or an irrational quantity is equal to a rational quantity, which is absurd. Hence the proposition.

217. *If* $x+\sqrt{y}=a+\sqrt{b}$, *where* x *and* a *are rational and* $\sqrt{y}$ *and* $\sqrt{b}$ *are irrational, then* $x=a$ *and* $y=b$.

If x is not equal to a, let $x=a+c$,

then $a+c+\sqrt{y}=a+\sqrt{b}$,

$\therefore \quad c+\sqrt{y}=\sqrt{b}$,

which is impossible (see Article 216), unless $c=0$, in which case $x=a$ and therefore $y=b$.

NOTE. If $x+\sqrt{y}=a+\sqrt{b}$, it follows that $x-\sqrt{y}$ will be equal to $a-\sqrt{b}$.

218. *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, *then will* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Since $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.

Squaring, $a+\sqrt{b}=x+y+2\sqrt{xy}$,

$\therefore \quad a=x+y$ and $\sqrt{b}=2\sqrt{xy}$. [Article 217]

$\therefore \quad a-\sqrt{b}=x+y-2\sqrt{xy}$,

$\therefore \quad \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

219. *Find the square root of* $a+\sqrt{b}$.

Let $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.

Then, squaring, we have

$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

$\therefore \quad x+y=a$ ....(1)

and $2\sqrt{xy}=\sqrt{b}$.

$$\therefore \quad (x-y)^2=(x+y)^2-4xy$$
$$=a^2-b.$$

$\therefore \quad x-y=\sqrt{a^2-b}$ ....(2)

Adding (1) and (2), we have

$$2x=a+\sqrt{a^2-b},$$

$$\therefore \quad x=\tfrac{1}{2}(a+\sqrt{a^2-b}).$$

Subtracting (2) from (1), we have

$$2y=a-\sqrt{a^2-b},$$

$$\therefore \quad y=\tfrac{1}{2}(a-\sqrt{a^2-b}).$$

Hence $\sqrt{a+\sqrt{b}}=\sqrt{\tfrac{1}{2}(a+\sqrt{a^2-b})}+\sqrt{\tfrac{1}{2}(a-\sqrt{a^2-b})}$.

NOTE. From the values we have obtained for x and y, it is evident that this process of finding the square root of $a+\sqrt{b}$ is of no practical use unless a^2-b is a perfect square.

EXAMPLE 1. *Find the square root of* $5+2\sqrt{6}$.

Let $\sqrt{5+2\sqrt{6}}=\sqrt{x}+\sqrt{y}$.

Then $5+2\sqrt{6}=x+y+2\sqrt{xy}$,

$\therefore \quad x+y=5$ ....(1)

and $2\sqrt{xy}=2\sqrt{6}$.

$$\therefore \quad (x-y)^2=(x+y)^2-4xy$$
$$=25-24$$
$$=1.$$

$x-y=\pm 1$ ....(2)

Now from (1) and (2), we have $x=3$ or 2, and $y=2$ or 3. Hence $\sqrt{5+2\sqrt{6}}=\sqrt{3}+\sqrt{2}$.

NOTE. In the same way we can shew that $\sqrt{5-2\sqrt{6}}=\sqrt{3}-\sqrt{2}$.

EXAMPLE 2. *Find the square root of* $5\sqrt{2}+2\sqrt{12}$.

Since $5\sqrt{2}+2\sqrt{12}=5\sqrt{2}+2\sqrt{2}.\sqrt{6}=\sqrt{2}(5+2\sqrt{6})$,

$\therefore \quad \sqrt{5\sqrt{2}+2\sqrt{12}}=\sqrt{\sqrt{2}(5+2\sqrt{6})}=\sqrt[4]{2}.\sqrt{5+2\sqrt{6}}$.

From Example 1, we know that $\sqrt{5+2\sqrt{6}}=\sqrt{3}+\sqrt{2}$,

$$\sqrt{5\sqrt{2}+2\sqrt{12}}=\sqrt[4]{2}(\sqrt{3}+\sqrt{2}).$$

EXAMPLES CXIV

Find the square root of

1. $4+2\sqrt{3}$. **2.** $3+2\sqrt{2}$. **3.** $8-2\sqrt{15}$.

4. $12+2\sqrt{35}$. **5.** $5-2\sqrt{6}$. **6.** $11+6\sqrt{2}$.

7. $16+5\sqrt{7}$. **8.** $2\sqrt{8}-2\sqrt{6}$. **9.** $3\sqrt{5}-2\sqrt{10}$.

10. $\sqrt{80}+\sqrt{60}$. **11.** $x-2\sqrt{x-1}$. **12.** $2a+\sqrt{a^2-x^2}$.

Equations involving Surds

220. Here we shall consider only those irrational equations in which unknown quantities appear under the square root sign. In such cases by the process of squaring both the sides, we can get rid of the radical signs and ultimately reduce the equation to the form of a rational equation.

Since in the process of squaring, roots may be introduced which do not satisfy the original equation, it is therefore necessary to test whether each solution obtained does, or does not, satisfy the given equation.

EXAMPLE 1. *Solve* $2\sqrt{x}-\sqrt{3x-11}=2$.

Transposing $2\sqrt{x}-2=\sqrt{3x-11}$.

Squaring both sides, $4x-8\sqrt{x}+4=3x-11$,

$$\therefore \quad 8\sqrt{x}=x+15.$$

Squaring again, $64x = x^2 + 30x + 225$,

$$\therefore \quad x^2 - 34x + 225 = 0,$$

$$\therefore \quad (x-25)(x-9) = 0,$$

$$\therefore \quad x = 25,\ 9.$$

Since both the values satisfy the equation, hence the solution is $x = 25,\ 9$.

EXAMPLE 2. *Solve* $\dfrac{2\sqrt{x}-3}{3\sqrt{x}} = \dfrac{4\sqrt{x}-1}{7(\sqrt{x}+4)}$.

Multiplying across, we have

$$(2\sqrt{x}-3)\times 7(\sqrt{x}+4) = 3\sqrt{x}(4\sqrt{x}-1),$$

$$\therefore \quad 14x + 35\sqrt{x} - 84 = 12x - 3\sqrt{x},$$

$$\therefore \quad 2x + 38\sqrt{x} - 84 = 0,$$

$$\therefore \quad x + 19\sqrt{x} - 42 = 0,$$

$$\therefore \quad (\sqrt{x}-2)(\sqrt{x}+21) = 0,$$

$$\therefore \quad \sqrt{x} = 2,\ -21\ ;$$

$$\therefore \quad x = 4,\ 441.$$

EXAMPLE 3. *Solve* $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} = \dfrac{3}{2}$.

By Componendo and Dividendo, the equation can be written as

$$\frac{2\sqrt{1+x}}{2\sqrt{1-x}} = \frac{3}{1},$$

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} = 5,$$

$$\frac{1+x}{1-x} = 25,$$

$$1 + x = 25 - 25x,$$

$$26x = 24,$$

$$x = \frac{24}{26} = \frac{12}{13}.$$

EXAMPLES CXV

Solve the following equations :

1. $\sqrt{x-6} = 2$.
2. $3 + \sqrt{x-3} = 7$.
3. $15 + \sqrt{x-7} = 19$.
4. $x + 2 + \sqrt{x+3} = 5$.
5. $x + \sqrt{x^2+9} = 9$.
6. $\sqrt{5x+16} + \sqrt{x} = 8$.

7. $\sqrt{5x-1}-\sqrt{5x-2}=1.$

8. $\sqrt{5-x}+\sqrt{3+x}=\sqrt{5}+\sqrt{3}.$

9. $\sqrt{x}+\sqrt{x-1}=\dfrac{2}{\sqrt{x}}.$

10. $\sqrt{x}+\sqrt{4+x}=\dfrac{2}{\sqrt{x}}.$

11. $2(x+2)=1+\sqrt{4x^2+9x+14}.$

12. $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}.$

13. $\dfrac{3x-1}{\sqrt{3x}+1}=4+\dfrac{\sqrt{3x}-1}{2}.$

14. $\sqrt{(x-a)^2+2ab+b^2}=x-a+b.$

15. $\sqrt{a^2+x^2}+\sqrt{a^2-x^2}=4.$

16. $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1+x}}=\dfrac{1}{3}.$

17. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{1}{2}.$

18. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{2}{x}.$

19. $\dfrac{\sqrt{px}+\sqrt{q}}{\sqrt{px}-\sqrt{q}}=\dfrac{\sqrt{p}+\sqrt{q}}{\sqrt{q}}.$

20. $\dfrac{\sqrt{a-\sqrt{a-\sqrt{a^2-ax}}}}{\sqrt{a+\sqrt{a-\sqrt{a^2-ax}}}}=\dfrac{1}{4}.$

MISCELLANEOUS EXAMPLES IV

(*Most of these examples have been selected from the question papers of the Indian Universities*).

A

1. Resolve into factors :

 (i) $6x^2+5x-4.$ (ii) $a^2-2ab+b^2-a+b.$

2. If $a : b=c : d$, shew that

 (i) $a : b=a\pm c : b\pm d.$ (ii) $a^2\pm c^2 : b^2\pm d^2=(a\pm c)^2 : (b\pm d)^2.$

3. Find the square root of $4x^4-12x^3a-11x^2a^2+30xa^3+25a^4.$

4. Simplify

 (i) $\left(\dfrac{a+b}{a-b}-\dfrac{b-a}{a+b}\right)\div\left(\dfrac{a}{b}+\dfrac{b}{a}\right).$

 (ii) $\dfrac{1}{a-b}-\dfrac{2a+b}{a^2-b^2}+\dfrac{a^3+ab^2}{a^4-b^4}.$

5. Solve

 (i) $ax+by=bx-ay=a^2+b^2.$

 (ii) $b(2x-b)+c^2=c(2x-c)+b^2.$

6. Two persons started at the same time from A. One rode on horseback at the rate of 10 miles an hour and arrived at B 40 minutes later than the other who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between A and B.

B

1. Resolve into factors :

(i) $8a^3+27b^3$. (ii) $(ax+by)^2-(bx-ay)^2$.

2. Solve

(i) $a(x-c)-c(x-a)=b(a-c)$.

(ii) $x-cy=cx-y=c$.

3. Find the square root of

$$\frac{a^2}{b^2}-3\frac{a}{b}+\frac{17}{4}-\frac{3b}{a}+\frac{b^2}{a^2}.$$

4. Simplify

(i) $\left(\frac{a}{a-b}-\frac{a}{a+b}\right)\div\frac{2b^2}{a^2-b^2}$.

(ii) $\frac{2}{x+3}-\frac{x}{x^2+5x+6}+\frac{1}{x^2+7x+12}$.

5. A man starts from a place and travels uniformly at the rate of 10 miles an hour, but rests $\frac{1}{2}$ hour at the end of every hour. After $1\frac{1}{4}$ hour, another man starts from the same place and overtakes the first man in $18\frac{1}{2}$ miles. Find graphically the rate at which the second man travels.

6. A man bought a certain number of eggs at 4 for 5 annas. Two of the eggs were broken, and on selling the rest at 3 for 4 annas, he lost 2 annas. How many eggs did he buy ?

C

1. Resolve into factors :

(i) $2x^3-3x+1$. (ii) $x^2-cx-cd-d^2$.

2. (i) Find the H.C.F. of $x^4-9a^2x^2+10a^3x$ and $ax^3-a^2x^2-4a^4$.

(ii) Find the L.C.M. of $\sqrt{x}-\sqrt{a}$, $\sqrt{3ax}$, x^2+ax, x^2-a^2 and $3ax^2+3a^2x$.

3. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$.

4. Simplify

(i) $1-\frac{1}{1-x}+\frac{3}{(1-x)^2}-\frac{1}{(1-x)^3}$.

(ii) $\left\{\frac{x}{a}+\frac{2x^2}{a(b-x)}\right\}\left\{\frac{a}{x}-\frac{2ax}{x(b+x)}\right\}$.

5. Solve

(i) $ax-by=c,\ bx+ay=c.$

(ii) $\dfrac{x+a}{x+b}=\dfrac{x+3a}{x+a+b}.$

6. A man rows 44 miles down and 30 miles up a stream in 10 hours and 55 miles down and 40 miles up in 13 hours. Find the rate of his rowing and the rate of the flow of the stream.

D

1. Resolve into factors :

(i) $x^2-x-20.$ (ii) $x^8-16a^8.$ (iii) $a^8+a^4x^4+x^8.$

2. Solve

(i) $ax+by-c=0,\ a^2x+b^2y-c^2=0.$

(ii) $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

3. Find the H.C.F. of $x^3+6x^2+11x+6$ and $x^3+2x^2-8x-16.$

4. Simplify

(i) $\dfrac{6x+5}{2x+3}-\dfrac{3x-4}{x+1}-\dfrac{7}{(x-1)(2x+3)}.$

(ii) $\dfrac{x^2+7x+10}{x^2-10x+24}\times\dfrac{x^2-7x+12}{x^2-2x-35}\div\dfrac{x^2-x-6}{x^2-13x+42}.$

5. A starts at 2 p.m. to drive along a road from a point P at the uniform rate of 6 miles an hour ; B starts at 1.45 p.m. to walk along the same road at 4 miles an hour from a point Q, which is 4 miles in advance of P ; find graphically when and where A will be 4 miles in front of B.

6. A boy buys a certain number of oranges at 3 for 2 pence, and one-third of that number at 2 for 1 pence ; at what price must he sell them to get 20 per cent. profit ? If his profit be 4*s.* 2*d.*, find the number bought.

E

1. Find the H. C. F. of $2x^4-2x^3+3x^2-x+1$ and $3x^3-5x^2+5x-2.$

2. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, shew that each ratio is equal to $\dfrac{a+c+e}{b+d+f}.$

If $\dfrac{a+b-c}{a+b}=\dfrac{b+c-a}{b+c}=\dfrac{c+a-b}{c+a}$, and $a+b+c$ is not equal to zero, shew that $a=b=c.$

3. Solve

(i) $\frac{6}{x}+\frac{4}{y}=3,\ \frac{9}{x}-\frac{1}{y}=2\frac{3}{4}.$ (ii) $\sqrt{5x-1}=1+\sqrt{5x-2}.$

4. Prove that

$$\frac{a(b+c)}{(a-b)(a-c)}+\frac{b(c+a)}{(b-c)(b-a)}+\frac{c(a+b)}{(c-a)(c-b)}+1=0.$$

5. Shew that $(x-1)(x-2)(x-3)(x-4)+1$ is a perfect square.

6. A challenged B to ride a bicycle race of 1040 yards. He first gave B 120 yards start, but lost by 5 seconds; he then gave B 5 seconds start and won by 120 feet. How long would each take to ride the distance?

F

1. Resolve into factors:

(i) $81a^4+64b^4.$ (ii) $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$
(iii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$

2. Solve

(i) $a(2x-a)+b(2x-b)=2ab.$

(ii) $\frac{2x+3}{x-1}-\frac{2-3x}{1+x}=\frac{25}{4}.$

3. Find the square root of

$$\frac{4a^2-12ab-6bc+4ac+9b^2+c^2}{4a^2+9c^2-12ac}.$$

4. Simplify

(i) $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y).$

(ii) $\dfrac{\frac{c}{a-c}-\frac{a}{a+c}}{\frac{a+c}{a}+\frac{c-a}{c}}\times(a^2-c^2).$

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

(i) $\frac{ma+nb}{mc+nd}=\frac{b^2c}{d^2a}.$ (ii) $a^2+c^2+e^2 : b^2+d^2+f^2=ce : df.$

6. The dimensions of a rectangular court are such that if its length were increased by 3 yards and breadth diminished by 3 yards, its area would be diminished by 18 square yards, and if both length and breadth were increased by 3 yards, its area would be increased by 60 square yards; find the dimensions.

G

1. Resolve into factors:

(i) $(a+b)^2+3(a^2-b^2).$ (ii) $(a^2-bc)^3+8b^3c^3.$
(iii) $2(ab+cd)-a^2-b^2+c^2+d^2.$

2. Solve

(i) $\frac{x+b}{a-b}=\frac{x-b}{a+b}$.

(ii) $x+y-3z=-a,\ z+x-3y=-b,\ y+z-3x=-c$.

3. If $x=b+c$, $y=c-a$ and $z=a-b$, prove that

$x^2+y^2+z^2-2xy-2zx+2yz=4b^2$.

4. Prove that

$bc(b-c)+ca(c-a)+ab(a-b)+(b-c)(c-a)(a-b)=0$.

5. The distance from a place P to another place Q is $3\frac{1}{2}$ miles; two persons A and B start together from P to go to Q, the former by carriage which travels at the rate of 6 miles an hour, the latter walking at the rate of 3 miles an hour. If A remains at Q for 15 minutes, and then returns by carriage to P, find graphically where he will meet B.

6. A tradesman sells two articles for Rs. 46 making 10 per cent. profit on one and 20 per cent. on the other. If he had sold each article at 15 per cent. profit, the result would have been the same. At what price does he sell each article?

H

1. Resolve into factors:

(i) x^3+x^2y-x-y. (ii) $1+x^2(y-z^2)-x^3yz$.

(iii) $x^6-6x^4+11x^2-6$.

2. If $a=p(p+q)-q(p-q)$ and $b=p(p-q)-q(q-p)$, prove that $a^2-b^2=4p^2q^2$.

3. Solve

(i) $\frac{x}{x+a-b}+\frac{x}{x+b-c}=2$.

(ii) $3x+4y-11=0,\ 5y-6z+8=0,\ 7z-8x-13=0$.

4. Simplify

$$\frac{2}{a+x}-\frac{1}{a-x}+\frac{3x}{a^2-x^2}+\frac{ax}{a^3+x^3}.$$

5. If $x=a+b-2c$, $y=b+c-2a$, $z=c+a-2b$, find the value of $x^3+y^3+z^3-3xyz$.

6. Divide 834 into two parts such that 30 per cent. of the one exceeds 40 per cent. of the other by 6.

I

1. Resolve into factors:

(i) $(x+1)(x+3)(x+5)(x+7)+15$.

(ii) $x^4-(a^2+2)x^2y^2+y^4$.

2. Simplify

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}.$$

3. Find the square root of

$(a^2+b^2+c^2)(x^2+y^2+z^2)-(bz-cy)^2-(cx-az)^2-(ay-bx)^2.$

4. If $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}=1$, and $a+b+c$ is not equal to zero, prove that $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$.

5. Solve

(i) $\frac{x}{x-1}+\frac{x}{x+2}=2.$ (ii) $\frac{1}{x}+\frac{2}{y}=10,\ \frac{4}{y}+\frac{3}{z}=18,\ \frac{2}{z}+\frac{3}{x}=16.$

6. A man bought some eggs at the rate of 2 for 1 anna and half the number at the rate of 3 for 1 anna. He sold them at the rate of 5 for 2 annas and lost 4 annas. How many eggs did he buy ?

J

1. Find the square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$.

2. Find the H.C.F. of x^3-3x^2+3x-1, x^3-x^2-x+1 and x^4-2x^3+2x-1.

3. Solve

(i) $(x-a)(x-b)=(x-c)(x-d)$.

(ii) $ax+by+c=0,\ a'x+b'y+c'=0$.

4. Simplify

(i) $$\frac{\frac{1}{a-b}-\frac{1}{a-c}}{\frac{1}{(a-b)^2}-\frac{1}{(a-c)^2}}.$$

(ii) $$\left\{\frac{x}{1-\frac{1}{x}}-x-\frac{1}{1-x}\right\}\div\left\{\frac{x}{1+\frac{1}{x}}+x-\frac{1}{1+x}\right\}.$$

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that each ratio is equal to

(i) $\frac{ka+lc+me}{kb+ld+mf}$. (ii) $\left(\frac{ace}{bdf}\right)^{\frac{1}{3}}$.

6. A man walks the distance between P and Q in a certain time. If he had walked 1 mile an hour faster, he would have reached Q 30 minutes earlier, and if he had walked 1 mile an hour slower, he would have reached Q 45 minutes late. Find his speed and the distance between P and Q.

K

1. If $\left(a+\frac{1}{a}\right)^2=5$, shew that $a^3+\frac{1}{a^3}=2\sqrt{5}$.

2. Simplify

$$\frac{bc(x-a)}{(a-b)(a-c)}+\frac{ca(x-b)}{(b-a)(b-c)}+\frac{ab(x-c)}{(c-a)(c-b)}.$$

3. Solve

(i) $x+y+z=0,$
$bcx+cay+abz=0,$
$ax+by+cz+(b-c)(c-a)(a-b)=0.$

(ii) $16\left(\frac{a-x}{a+x}\right)^3=\frac{a+x}{a-x}.$

4. If $\frac{x}{(b-c)(b+c-2a)}=\frac{y}{(c-a)(c+a-2b)}=\frac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$.

5. Draw the graphs of $x^2+y^2=25$ and $3x+4y=25$, and shew that the second is a tangent to the first. Find the co-ordinates of the point of contact.

6. In a quarter-mile race, A gives B a start of 22 yards and beats him by 2 seconds; and in a 300 yards race, he gives him a start of 2 seconds, and beats him by $10\frac{1}{3}$ yards. Find the rate of each.

L

1. Simplify

$$\left\{\frac{\sqrt{x+a}}{\sqrt{x-a}}-\frac{\sqrt{x-a}}{\sqrt{x+a}}\right\}\times\frac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}.$$

2. Solve

(i) $\frac{bc(ax-1)}{b+c}+\frac{ca(bx-1)}{c+a}+\frac{ab(cx-1)}{a+b}=a+b+c.$

(ii) $\frac{b}{x}+\frac{a+c}{y}=m,\quad \frac{a-c}{x}+\frac{b}{y}=n.$

3. Shew that

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}=\frac{(a-b)(b-c)(c-a)}{(1+ab)(1+bc)(1+ca)}.$$

4. If $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$, find the value of $\dfrac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$.

5. Find the square root of

$$\left(\frac{x^4}{y^4}+\frac{y^4}{x^4}\right)-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5.$$

6. A number consists of three digits. The middle digit is equal to the sum of the other two digits; the square of the middle digit is less than five times the product of the other two digits by 1; and five times the digit in the hundreds place is equal to twice the digit in the units place. Find the number.

M

1. Find the H.C.F. and the L.C.M. of $2x^2+(6p-10q)x-30pq$ and $3x^2-3(3p+5q)x+45pq$.

2. Solve

(i) $\dfrac{1}{(a-b)(x-a)}-\dfrac{1}{(c-d)(x-c)}=\dfrac{1}{(a-b)(x-b)}-\dfrac{1}{(c-d)(x-d)}$.

(ii) $x+y+z=a+b+c$,

$\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=3$,

$ax+by+cz=a^2+b^2+c^2$.

3. Simplify

$$\frac{x^2-a^2}{x^2+(d-a)x-da}+\frac{x^2-b^2}{x^2+(d-b)x-db}+\frac{x^2-c^2}{x^2+(d-c)x-dc}.$$

4. If $a+b+c=s$, prove that
$(as+bc)(bs+ca)(cs+ab)=(a+b)^2(b+c)^2(c+a)^2$.

5. Simplify

$$\left(x^{1+\frac{b}{a}}\right)^{\frac{a}{a+b}}\div\sqrt[a]{\frac{x^{2a}}{(x^{-1})^{-a}}}.$$

6. Two vessels contain mixtures of wine and water; in one there is twice as much wine as water, and in the other three times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds 15 gallons, in order that its contents may be half wine and half water.

N

1. Resolve into factors :

(i) $(a+b)^2+(a+c)^2-(b+d)^2-(c+d)^2$.

(ii) $(a+b-3c)^2-a-b+3c$.

2. Simplify $\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}$.

3. Solve

(i) $c(b+x)-ac=d(b+x)-ad$.

(ii) $ax+by=1=bx-\frac{b}{a}+ay-\frac{a}{b}$.

4. If $x=a(b-c)$, $y=b(c-a)$ and $z=c(a-b)$, prove that

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}-\frac{3xyz}{abc}=0.$$

5. Shew that $(y-z)^3+(x-y)^3+3(x-y)(y-z)(x-z)=(x-z)^3$.

6. The denominator of a fraction exceeds the numerator by 4, and if 5 be taken from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5. Find the original fraction.

O

1. Resolve into factors :

(i) $a^3-b^3-a(a^2-b^2)+b(a-b)^2$.

(ii) $x^3+(p-q-r)x^2+(qr-pq-pr)x+pqr$.

2. Simplify

$$\frac{(2x-3y)^2-x^2}{4x^2-(3y+x)^2}+\frac{4x^2-(3y-x)^2}{9(x^2-y^2)}+\frac{9y^2-x^2}{(2x+3y)^2-x^2}.$$

3. Solve

(i) $\frac{a-x^2}{bx}-\frac{b-x}{c}=\frac{c-x}{b}-\frac{b-x^2}{cx}$.

(ii) $x^2+y^2=a^2$, $xy=b^2$.

4. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, find the value of $(b-c)x+(c-a)y+(a-b)z$.

5. A man first gets $\frac{x}{y}$ of Rs. 100 and then $\frac{y}{x}$ of Rs. 100. If he spends Rs. 200, how much has he spent from his pocket ?

6. Two men A and B are employed on a piece of work which has to be finished in 14 days. In 3 days they do $\frac{1}{3}$ of the work, and then A's place is taken by C. B and C work for one day and do $\frac{1}{20}$ of the whole work, and then B's place is taken by A. A and C finish the work a day before the appointed time. Find the time in which the work could have been done (i) by each working separately ; and (ii) by all working together.

EXAMINATION PAPERS

Board of High School and Intermediate Education, United Provinces.

High School Examination

1931

5. (a) Find the L.C.M. of $4x^3+16x^2-3x-45$ and $10x^3+63x^2+119x+60$.

(b) Find the H.C.F. of $27a^5-45a^4-16$ and $18a^5-45a^4-5a-14$.

6. (a) Solve the equation

$$\frac{x-3{\cdot}5}{2x+5}=\frac{3x-1{\cdot}5}{2x-1}.$$

(b) Simplify

$$\frac{(a+b)^2-ab}{(b-c)(c-a)}+\frac{(b+c)^2-bc}{(c-a)(a-b)}+\frac{(c+a)^2-ca}{(a-b)(b-c)}.$$

7. (a) What number must be added to

$$x^4+4x^3+10x^2+12x+3$$

to make it a perfect square ?

(b) A says to B, "I am twice as old as you were when I was as old as you are." The sum of their present ages is 63. Find their ages.

8. 0° Centigrade = 32° Fahrenheit. 100°C. = 212°F. Draw a graph, showing the relation between the Centigrade and Fahrenheit scales, and find the Centigrade reading corresponding to 73° F.

Or,

Construct the graphs of the equations

(i) $4x+6y=24$ and (ii) $2x+3y=6$.

Find graphically the co-ordinates of the points of intersection of $4x-3y=6$ with (i) and (ii).

1932

4. (a) Solve

$$7x+\frac{3}{x}=35\frac{3}{5}.$$

(b) Simplify

$$\left(x+\frac{a-x}{1+ax}\right)\times\frac{x}{a}\div\left\{1-\frac{x(a-x)}{1+ax}\right\}.$$

5. Resolve into factors :

(i) $x^6 - 64b^3$. (ii) $x^2 - 1 - 2a - a^2$. (iii) $(a+2b)a^3 - (b-2a)b^3$.

6. (a) A man distributed Rs. 100 equally among his friends ; if there had been five more friends each would have received one rupee less. How many friends had he ?

(b) If $x + \frac{1}{x} = 5$, find the value of $x^3 + \frac{1}{x^3}$.

7. Using the same axes and unit, draw the graphs of (i) $y + x = 0$, (ii) $5y = 3x$, (iii) $y = 3x + 12$. Find the co-ordinates of the vertices of the triangle formed by the straight lines (i), (ii) and (iii).

1933

5. Resolve into factors :

(i) $x^2 + 3x - y^2 - 3y$. (ii) $x^3(y-z) + y^3(z-x) + z^3(x-y)$.
(iii) $125x^5y^2 - 27x^2y^5$.

6. (a) Simplify

$$\frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)}.$$

(b) Solve $\frac{3}{x+3} + \frac{4}{x+5} = \frac{7}{x+11}$.

7. (a) Solve $ax + by + c = 0$, $Ax + By + C = 0$.

(b) Divide the number 77 into three parts such that the sum of the first and second multiplied by 3, the sum of the second and third diminished by 3, and the sum of the first and third increased by 3 may be all equal.

8. Plot the points $(4, -3)$ and $(0, 3)$ and write down the equation of the straight line joining them. Draw the graphs of the equations $2x - 3y = 12$ and $x = 1$. Find the co-ordinates of the vertices of the triangle formed by these two lines and the axis of x.

1934

5. Factorise

(i) $a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc$. (ii) $30x^2 + 97xy - 28y^2$.
(iii) $(bx+ay)^2 + (ax-by)^2$.

6. (a) Simplify

$$\frac{a^2 - bc}{(a-b)(a-c)} + \frac{b^2 - ca}{(b-c)(b-a)} + \frac{c^2 - ab}{(c-a)(c-b)}.$$

(b) Solve

$$3y + 4x = 3xy, \quad \frac{9}{x} + \frac{8}{y} = 7.$$

7. At an election there were two candidates A and B. $\frac{2}{5}$ of the electors voted for A who was elected by a majority of 200 over B, while $\frac{1}{3}$ of the electors did not vote at all. How many electors were there altogether ?

8. *A* walks at the rate of 4 miles an hour and rests for 18 minutes at the end of every hour. Two hours later *B* runs at the rate of 6 miles an hour. Find graphically when and where they will meet. Find the equation of the inclined portion of the graph between the first and second haltage.

Or,

Draw the graphs of the following straight lines :

(i) $4x-y=10$. (ii) $2x-y=4$. (iii) $x=3$. (iv) $y=2$.

Solve graphically the first two simultaneous equations.

Draw a straight line which cuts off intercepts of 3 and 4 on the axes of x and y respectively and find its equation. (The same units are taken in all cases.)

1935

5. Factorise

(i) $(2x-3y)^3+(3y-z)^3+(z-2x)^3$.

(ii) $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

(iii) $x^2-\left(a+\frac{1}{a}\right)x+1$.

6. Solve

(i) $\frac{2}{2x-7}-\frac{4}{9x-15}=\frac{5}{18-27}-\frac{3}{3x-5}$.

(ii) $\frac{4}{x}+\frac{7}{y}=29$, $\frac{3}{y}+\frac{1}{x}=11$.

7. Tin appears to lose one-seventh of its weight when weighed in water. Lead appears to lose one-twelfth of its weight under the same conditions. An alloy of tin and lead, which weighs 270 lbs. in air, appears to weigh only 240 lbs. when weighed in water. How much of each metal does it contain ?

8. Explain clearly what you understand by the graph of an equation.

Draw the graph of the equation $\frac{x}{3}+\frac{y}{4}=1$, and measure its intercept between the two axes.

1936

5. Factorize

(i) x^4+4. (ii) $ax^2+(a^2+1)x+a$.

(iii) $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

6. Solve

(i) $\frac{8}{2x-1}+\frac{9}{3x-2}=\frac{7}{x+1}$.

(ii) $x+6y=5z$,
$7x+z=6y$,
$5x+6y-4z=24$.

7. A certain number between 10 and 100 is eight times the sum of its digits and if 45 be subtracted from it, the digits will be reversed. Find the number.

8. A can do a piece of work in 6 hours and B in 8 hours. Find by means of a graph the time they would take in finishing it working together.

Board of High School Education, Central Provinces.

High School Certificate Examination

1931

5. Find the factors of

(i) $x^2(b-c)+b^2(c-x)+c^2(x-b)$. (ii) $x^2-xy-2y^2+x+y$.
(iii) x^3-7x-6.

6. A walks a distance of 24 miles at the rate of 4 miles per hour, and B, starting one hour later, walks the distance in 4 hours. Draw graphs of their motions and from the diagram determine when and where B will overtake A.

7. Solve the following equations :

(i) $\frac{3}{x}-\frac{2}{y}=5,\ 4x+5y=23xy$. (ii) $\frac{2x}{x-1}+\frac{3x-1}{x+2}=\frac{5x-11}{x-2}$.

8. *Either*,

(*a*) If $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=1$, prove that

$$(x+y)(y+z)(z+x)=0.$$

Or,

(*b*) In a mile race between a bicycle and a tricycle, the rates of which are as 5 : 4, the tricycle had a minute's start but was beaten by 176 yards. Find the rates of each in yards per minute.

1932

5. Simplify

(i) $1+\frac{a}{b}-\frac{b}{a+b}-\frac{a^2}{ab-b^2}+\frac{2a^2}{a^2-b^2}$.

(ii) $\left(1-\frac{2xy}{x^2+y^2}\right)\div\left(\frac{x^3-y^3}{x-y}-3xy\right)$.

6. Find the factors of x^4+4 ; and apply the Remainder Theorem to show that

$$x^4-4x^3+2x^2+x+6$$

is exactly divisible by x^2-5x+6. Find the quotient.

7. Solve the equations :

(a) $x+y=3,\ \frac{2}{x}+\frac{1}{y}=2.$ (b) $x^2+xy=28,\ xy+y^2=21.$

8. If $a : b :: c : d$, show that

$$4(a+b)(c+d)=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2.$$

9. Solve *either* (a) *or* (b).

(a) A and B compared their incomes and found that A's income was to that of B's as $5 : 7$, and that one-half of A's income exceeded the difference of their incomes by Rs. 35. Find the income of each.

(b) The cost of printing the first 100 copies of a book is Rs. 25, and for each additional 100 copies the cost is Rs. 5. Draw a *graph* to show the cost of printing any number of copies up to 500 ; and find from the graph the cost of printing 363 copies and the number of copies printed for Rs. 31.

1933

5. Resolve into simple factors :

(i) $(x^2+5x)(x^2+5x-2)-24.$ (ii) $2x^4-5x^3+6x^2-5x+2.$

(iii) $(x-y)^3+(y-z)^3+(z-x)^3.$

6. Solve *any two* of the following equations :

(i) $\frac{2}{x+4}+\frac{3}{x+16}=\frac{5}{x+8}.$

(ii) $\frac{1}{x+\sqrt{2-x^2}}+\frac{1}{x-\sqrt{2-x^2}}=\frac{2}{x}.$

(iii) $\frac{2}{x+1}+\frac{3}{y+2}=2,\ \frac{3}{x+1}+\frac{10}{y+2}=4.$

7. The salary of a person is increased each year by a fixed amount. After 5 years of service his salary is raised to Rs. 125 and after 15 years to Rs. 165. Draw a graph from which his salary may be read off for any year and determine (a) his initial salary, and (b) his salary at the end of 20 years.

8. *Either*,

If $x-\frac{1}{x}=n$, prove that

$$x^3+x^2-x+\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}=(n+1)(n^2+2).$$

Or,

If $a : b :: c : d$, prove that

$$\left(\frac{1}{a}+\frac{1}{d}\right)-\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{(a-b)(a-c)}{abc}.$$

9. Two bags A and B contain together 100 balls ; half the balls in A are transferred to B and then half the balls that B now contains are transferred to A which has now the same number of balls that B had at first. Find the number of balls in A and B at first.

1934

5. *Either*, (a) Apply the Remainder Theorem to find the remainder when $4x^4-12x^3+25x^2+13x-16$ is divided by $x+3$.

Factorise $2x^4-11x^3+20x^2-17x+6$.

Or,

(b) Factorise

(i) $a^3+b^3+c^3-3abc$. (ii) $b^2(a+c)-c^2(a+b)$.

6. *Either*, (a) Simplify $\dfrac{1}{4-\dfrac{3}{2+\dfrac{x}{1-x}}}$

What will be the value of x if the above expression is equal to 2 ?

Or,

(b) If $x=\dfrac{4ab}{a+b}$, find the value of

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}.$$

7. Solve the following equations :

(i) $\dfrac{3x+40}{3x+1}+\dfrac{4x-24}{2x+1}=\dfrac{3x+32}{x+2}$.

(ii) $\dfrac{x}{3}-\dfrac{2}{y}=1,\quad \dfrac{x}{4}+\dfrac{3}{y}=3$.

8. A sum of money has been distributed among A, B and C, such that A gets half of the whole amount, A and B together get Rs. 76, and A and C get Rs. 62. What does each get ?

9. *Either*, (a) If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \quad\text{and}\quad \frac{2a^2+c^2}{2b^2+d^2}=\frac{2c^2+e^2}{2d^2+f^2}.$$

Or,

(b) A square room is carpeted with a square carpet, leaving a border all round, two feet wide uncarpeted. If 'x' feet is the length of the side of the room. and 'y' square feet the area of the uncarpeted border, draw a graph showing the relation between 'y' and 'x'. Write down the value of 'x' when $y=120$.

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4. Simplify

$$\frac{1}{x-3}+\frac{1}{x+3}+\frac{x+1}{x^2-3x+9}-\frac{2x^2+x+12}{x^3+27}.$$

5. Solve the following equations :

(i) $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{x+a+b}+\frac{1}{x}$.

(ii) $x+20=4y-10,\ \frac{4(x-1)}{4x-5}=\frac{3(y-3)}{3y-8}$.

8. A train left A for B, and at the same time another train left B for A. The trains crossed each other after four hours. If the train coming from B to A travelled 16 miles per hour faster than the first, and the distance between A and B is 216 miles, find out the speed of the trains.

Show the motions of the trains on a graph, and find, from your graph, their distances from A five hours after the start.

9. Factorise

(i) $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
(ii) $x(x+1)(x+2)(x+3)+1$.

10. (i) If $a+b : b+c :: c+d : d+a$, prove that

either $a=c$,

or $a+b+c+d=0$.

(ii) If $x+\frac{1}{x}=3$, find the value of $x^3+\frac{1}{x^3}$.

11. If C is the H.C.F. of two expressions A and B, prove that C is also the H.C.F. of $A+B$ and $A-B$.

Find the H.C.F. of $2x^4-7x^3+8x^2-10x+3$ and $3x^4-10x^3+8x^2-7x+2$.

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1. (a) Factorise

$$(a^2+2a)^2-(a^2+2a)-6.$$

(b) Simplify

$$\frac{1}{a-x}-\frac{1}{a+x}+\frac{2x}{a^2-x^2}-\frac{4x^3}{a^4-x^4}.$$

2. Solve the equations :

(a) $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+4}{x+5}-\frac{x+5}{x+6}$.

(b) $\frac{2}{x+1}+\frac{3}{y+2}=1\frac{3}{4},\ \frac{3}{x+1}+\frac{10}{y+2}=4$.

4. A man leaves a town A at 9 a.m. and walks at 4 miles an hour towards town B which is 20 miles from A. At 10 a.m. his friend leaves B on a bicycle and comes to meet him, cycling at 8 miles an hour. At 10-30 a.m. the walker rests for 15 minutes and then proceeds to walk as before. Draw a graph to show when and where they meet, and verify your result by calculation.

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6. (a) Factorize *any two* of the following :

(i) x^6+27. (ii) $(x-y)^2-(1-xy)^2$.

(iii) $15(x^2-1)-72x$.

(b) Prove that

$$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a).$$

7. (a) Find the continued product of
$a+b+c$, $b+c-a$, $c+a-b$ and $a+b-c$.

(b) If $x+5y$ exactly divides $x^3+3x^2y+5my^3$, find the value of m.

8. Solve *any two* of the following equations :

(i) $y(x+7)=x(y+1)$, $1-2y=20-3x$.

(ii) $\dfrac{7}{3x-1}-\dfrac{4}{x+1}=\dfrac{1}{4}$. (iii) $\dfrac{3}{x-2}+\dfrac{5}{x-6}=\dfrac{8}{x+3}$.

9. A man walks a certain distance at a certain rate. Had he walked $\frac{1}{2}$ mile an hour faster, he would have taken 1 hour less time ; but if he had gone 1 mile an hour slower, he would have taken 3 hours longer. Find the distance.

10. *Either*,

(a) Exhibit graphically the following data showing the scale of charges as quoted by the Manager of a hotel :

No. of guests	150	200	250	300	350	400
Charge per head	Rs. 6.	Rs. 4. 14a.	Rs. 4. 2a.	Rs. 3. 10a.	Rs. 3.4a.	Rs. 3.

and estimate the charge per head for 175, 225 and 375 guests.

Or,

(b) Using the same unit for both the axes, draw the graph of the line $4x+y=0$, and plot the points $(3, -4)$ and $(-2, 6)$. Find the co-ordinates of the points in which the line joining these points meets

(i) the above line, and (ii) the axes of co-ordinates.

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6. (a) Factorize (i) $4a^2b^2-(a^2+b^2-c^2)^2$. (ii) $x^3-x^2y-xy^2-2y^3$.

(b) Simplify $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

7. (a) If $a : x=b : y$, prove that
$(x^2+y^2)(a^2+b^2)=(ax+by)^2$.

(b) What number must be subtracted from each of the numbers 9, 5, 18 and 8, so that the remainders may be in proportion ?

8. Solve *any two* of the following equations, and verify your results :

(i) $\dfrac{12x+17}{3x+4}+\dfrac{15x-7}{5x-2}=7$.

(ii) $\left.\begin{array}{l}(x-6)(y+4)=xy-x-1\\ y(5-x)=2x-xy-2y\end{array}\right\}$.

(iii) $\dfrac{x}{x+1}+\dfrac{x+1}{x}=2\dfrac{1}{6}$.

9. A man travels a certain distance, and finds that if he had gone one more mile per hour he would have saved an hour and a half ; but that, if he had gone slower than he did by half a mile an hour he would have taken one hour longer. Find the distance and his rate.

10. (a) A boy was measured on his 12th, 14th, 16th, 18th, 20th, 22nd and 24th birthdays, when his height was found to be $4\frac{1}{3}$ft., 4ft. $10\frac{3}{4}$ in., $5\frac{1}{4}$ft., 5ft. $6\frac{1}{4}$ in., 5ft. $8\frac{3}{4}$ in., 5ft. $10\frac{1}{4}$ in. and 5ft. $11\frac{2}{3}$ in. respectively. Exhibit his growth graphically, and estimate his height at $13\frac{1}{2}$ years of age. At what age was he just 5 feet high ?

Or,

(b) Find graphically the co-ordinates of the vertices of the triangle formed by the straight lines

(i) $2y+x+8=0$, (ii) $x=0$, (iii) $8y=3(x+1)$.

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6. (a) Factorise any two of the following :

(i) $a^2-b^2+8bc-16c^2$. (ii) $(a^2+2a)^2-(a^2+2a)-2$.
(iii) $x^3-8y^3+27z^3+18xyz$.

(b) If $2s=a+b+c$, shew that
$a(b-c)(s-a)^2+b(c-a)(s-b)^2+c(a-b)(s-c)^2=0$.

7. (a) Solve

$$\frac{x+a}{x-a}-\frac{x-b}{x+b}=\frac{2(a+b)}{x}.$$

Or,

$$\frac{4}{x}+\frac{10}{y}=2 \text{ and } \frac{3}{x}+\frac{2}{y}=\frac{19}{20}.$$

(b) State the relations between the roots and the coefficients of the general quadratic equation.

Find the values of x which satisfy the equation

$$\frac{2(3x-1)}{4x-3}=\frac{5x}{x+2}-2.$$

8. (a) Simplify

$$\frac{1}{x+\sqrt{x^2-1}}+\frac{1}{x-\sqrt{x^2-1}}.$$

(b) Find the square root of

$$\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right).$$

9. (a) Simplify

$$\left\{(a^m)^{m-\frac{1}{m}}\right\}^{\frac{1}{m+1}}.$$

(b) A man bought oranges at 4a. a dozen. He found 50 of them spoiled, and selling the remainder at 9 for 4a. made a profit of Rs. 6. 4a. How many oranges did he buy ?

10. The average weight in pounds of a boy and of a girl at various ages in years are shown in the following table :

Age	6	8	10	12	14	16
Boy's weight	42·5	52·5	64	75	87	110
Girl's weight	41	50	60	73	92·5	108

Exhibit this graphically on the same diagram. Find the average weights of a boy and of a girl at the age of 9. Also find how old they are when their weights are the same.

Or,

Draw the triangle whose sides are represented by the equations $3x+y=13$, $3y-x=9$ and $x+7y=11$; find the co-ordinates of their vertices.

ANSWERS

PART I

EXAMPLES I. Page 4

1. $a \times b$. **2.** $x \times y$. **3.** $5 \times a$. **4.** $l \div b$.
5. $x+y+z$. **6.** abc. **7.** $8+7=15$. **8.** $a+a=2a$.
9. $9-7=2$. **10.** $3x-2x=x$. **11.** $10+8=13+5$.
12. $a+b=x+y$. **13.** $10-6=8-4$. **14.** $x-y=a-b$.
15. Yes. **16.** Yes. **17.** No. **18.** Yes.
34. $10a+b$. **35.** $10x+y$. **36.** $10y+x$; $10x+y$.
38. $100a+10b+c$. **39.** $100z+10y+x$; $100x+10y+z$.
41. (i) 3, (ii) 1, (iii) 100, (iv) $\frac{1}{2}$. **42.** (i) 5, (ii) a, (iii) abc, (iv) $10p$.
43. (i) 6, (ii) 1, (iii) 20, (iv) $\frac{3}{5}$. **46.** $60x$ min., $3600x$ sec.
47. $1760a$ yd., $5280a$ ft., $63360a$ in.
48. $10y$ dm., $100y$ cm., $1000y$ mm.
49. $9x$ sq. ft., 1296 sq. in. **50.** 12 miles. **51.** $3x$ miles.
52. $4y$ miles. **53.** xy miles. **54.** Rs. $\frac{x}{6}$. **55.** lb sq. yd.
56. $\frac{9a}{l}$ ft. **57.** $a+b+c$; 17. **58.** $x-y$; 15.
59. $a-b$; 35. **60.** $x+y+z+w$; 22.

EXAMPLES II. Page 8

23. $7a+12x$. **24.** $3a+7b$. **25.** $4a+9b+12c$.
26. $7a+10b+8c$. **27.** $7xy+9yz$. **28.** $12ab+15bc+18ca$.
29. $x+y$. **30.** $a+b+c$. **31.** $ab+bc+ca$.
32. $x+y$. **33.** $x+y+z$. **34.** $x+y+5$.
35. $10x$. **36.** $4x$. **37.** $205x$.
38. $192+12x+y$. **39.** $240x+12y+z$. **40.** $3x+y+3$.

EXAMPLES III. Page 11

1.	$3a+3b$.	**2.**	$3x+8y$.	**3.**	$4x+7$.
4.	$7a+11b$.	**5.**	$6x+6y$.	**6.**	$3x+4y+5z$.
7.	$3x+8y+13z$.	**8.**	$7ab+7xy$.	**9.**	$6a+9bc$.
10.	$10a+16b$.	**11.**	$6a+6b+6c$.	**12.**	$9d+15f+14$.
13.	$14a+24b+33c$.	**14.**	$23a+24b+31c$.	**15.**	$6a+9b+8c$.
16.	$7l+17m+20n$.	**17.**	$2a+2b+2c$.	**18.**	$2ab+2bc+2ca$.
19.	$a+b+c$.	**20.**	$2a+2b+2c$.	**21.**	$21ab+25c$.
22.	$3xy+3yz+3zx$.	**23.**	$11pq+15qr+19rp$.		
24.	$12a+4b+9c+8$.	**25.**	$6+11x+3y+11xy$.		

EXAMPLES V. Page 14

1.	$a+2b$.	**2.**	$3a$.	**3.**	$a+b$.
4.	$3b$.	**5.**	$2x$.	**6.**	0.
7.	$\frac{1}{2}x+\frac{2}{3}y$.	**8.**	$\frac{1}{3}x+\frac{2}{5}y$.	**9.**	$x+y+z$.
10.	$8a+3b+c$.	**11.**	$a+\frac{3}{2}b+\frac{4}{5}c$.	**12.**	$\frac{1}{3}x+\frac{1}{4}y+\frac{3}{7}z$.
13.	$ab+bc$.	**14.**	$xy+7yz+5zx$.	**15.**	$3ab+c+6$.
16.	$2mn+np$.	**17.**	$c+2d+3$.	**18.**	$2x+4y$.
19.	$a+b+c$.	**20.**	y.	**21.**	$2l+m$.
22.	$ab+bc+3ca$.	**23.**	$\frac{1}{2}y+\frac{1}{4}z$.	**24.**	$3xy+\frac{1}{2}zx$.

EXAMPLES VII. Page 31

1.	18.	**2.**	7.	**3.**	8.	**4.**	$x+y-z$.
5.	$x-y+z$.	**6.**	20.	**7.**	$a-b-c+d$.	**8.**	$3x+y$.
9.	$-x+5y$.	**10.**	$4m-8$.	**11.**	$-2m+2$.	**12.**	$-a-b$.
13.	$2x$.	**14.**	$-3a+2b$.	**15.**	$-3x+2y$.	**16.**	$2a$.

17.	$2b$.	**18.**	$2ab+2bc+2ca$.	**19.**	0.
20.	$-2n-m-p$.	**21.**	$-6c-d$.	**22.**	$-2a-b$.
23.	$2x-2y+2z$.	**24.**	$4x+7y-2z$.	**25.**	$-x+10y+3z$.
26.	$7b-6c-x-2$.	**27.**	$m-n-11p$.	**28.**	18.
29.	23.	**30.**	$a+b+c-d$.	**31.**	$a+b-c+d$.
32.	$x+2y-3z$.	**33.**	0.	**34.**	$x-y+z-a$.
35.	10.	**36.**	$8+3x-12y-3yz$.	**37.**	0.
38.	$12x$.	**39.**	$5a$.	**40.**	$2l-4m+4n$.
41.	1.	**42.**	432.	**43.**	16.
44.	$-x+y-2$.	**45.**	$x+y-z-u+w$.		

46. $ba-bc+bd-be-bf$. **47.** $2a$. **48.** $y-9z$.
49. $4a-12b+24c-24d-32$. **50.** $8m+48n+48p+8q-324$.
51. a. **52.** $5x+z$. **53.** $5m+n$.
54. $a-b+c-d+e-f$. **55.** $-4a+6b$. **56.** $2m$.
57. $-a$. **58.** 0.

EXAMPLES VIII. Page 32

1. 7, −5, 11, −15, 15. **2.** 7, −12, 16, −26, 30.
4. −6, 3, −4, 14, −11. **5.** −6, 9, −7, 18, −25.

EXAMPLES IX. Page 34

1. $20a$. **2.** $31x$. **3.** $-15a$. **4.** $-25m$.
5. $-10a$. **6.** x. **7.** $-xy$. **8.** $-2ab$.
9. $\frac{3}{2}x$. **10.** $2a$. **11.** $2a$. **12.** $6m+5$.
13. $k-l$. **14.** $8x-8y-7z$. **15.** $6a$. **16.** $2a$.
17. $8x$. **18.** $-3ab$. **19.** $16mn$. **20.** ab.
21. $x+1$. **22.** $7b$. **23.** $-4xy+8ab$. **24.** $-10xy$.
25. $\frac{2}{3}x$. **26.** $\frac{4}{9}x$. **27.** $2x$. **28.** $5a$.
29. $2y$. **30.** $5a+2b$. **31.** $2a+b$. **32.** $5a-2b$.
33. $a+\frac{1}{6}b$. **34.** $a-b$. **35.** $a+b$. **36.** $5a-b$.
37. $x+5y$. **38.** $11x-16$. **39.** $9-4x$. **40.** $x+y$.
41. $5a-b$. **42.** $-\frac{1}{2}x+\frac{1}{4}y$. **43.** $2ax$. **44.** 0.
45. $2bx$. **46.** $5a$. **47.** $-4ax-6a$. **48.** $6a+6b$.
49. $5x+y$. **50.** y. **51.** $5a+7b$. **52.** $-2m-10n$.
53. $a+b+c$. **54.** $9a-3b-6c$. **55.** $4a+6b+2c$.
56. $7x+5y$. **57.** $9a+3b-5c$. **58.** $2n$. **59.** $4q-2r$.
60. $-4x+2y+16z$. **61.** $4l-m$. **62.** $-3c+6e$. **63.** 0.
64. $13ax+6by-10cz$. **65.** $pq+qr+rp$. **66.** $6a+2c$.
67. $6a+9b-10c$. **68.** $-\frac{1}{3}x+y+\frac{1}{3}z$. **69.** $\frac{2}{9}a+\frac{4}{9}b-\frac{4}{9}c$.
70. $6ab-bc$. **71.** $-3ab$. **72.** $3a-2b+8c-d$.
73. $2x+2y-6z+4w$. **74.** $4a-b-5c-5$. **75.** $4x+2y$.
76. $\frac{23}{12}x-\frac{7}{12}y-z+4$.

EXAMPLES X. Page 37

1. a. **2.** $a-3b$. **3.** $3a-b$. **4.** $2b$.
5. $-2y$. **6.** $10c+13d$. **7.** $a-b+3c$. **8.** $4l-9m+2n$.
9. $-p+7q+3r$. **10.** $2x+2y+2z$. **11.** $4a+4b-c$.

12. $-6a-4c$. **13.** $2x-2z$. **14.** $a+c$.
15. $2a-2b-3c$. **16.** $6a+4b-6c$. **17.** $-2xy-2zx$.
18. $2bc+2da$. **19.** $a-6b+19c$. **20.** $-10x+y-9z$.
21. $-4a-11b+10c$. **22.** $6a-15b$. **23.** $-8a-6b+11c$.
24. $a+3b+4c$. **25.** $-9x+11y-2z$. **26.** $a+b+4c$.
27. $3x-3z$. **28.** $-15d+e$. **29.** $3x-3y-3z$.
30. $3b-2c$. **31.** $-5a-b+c$. **32.** $x+2y-8z$; 36 ; 12.
33. $2y-z$; first class, 14 ; second class, 67 ; third class, 193.
34. $x-6y+6z$; 240, 220, 320.

EXAMPLES XI. Page 38

1. 14. **2.** 15. **3.** −1. **4.** 27. **5.** −30. **6.** 2.
7. 10. **8.** −17. **9.** 51. **10.** 2. **11.** 8. **12.** 16.
13. −39. **14.** $2\frac{1}{2}$. **15.** $-\frac{2}{3}$. **16.** 192. **17.** 41. **18.** −1.
19. −68. **20.** 5.

EXAMPLES XIII. Page 46

16. a^{15}. **17.** $-a^{18}$. **18.** $-1+x^3$. **19.** $a^4b^4c^4$.
20. $-x^{11}y^{10}z^9$. **21.** 24. **22.** abc. **23.** $-x^3$.
24. $-x^3$. **25.** x^{15}. **26.** a^7. **27.** $-8a^6$.
28. $-x^3y^3$. **29.** $a^2b^2c^2x^3$. **30.** $l^3m^3n^3$. **31.** $-12a^2b^6c^4$.
32. $x^2y^2z^2$. **33.** $-x^3y^3z^3$. **34.** $a^4b^6c^8$. **35.** $a^2b^3c^2x^2y^3z^2$.
36. $\frac{1}{4}l^3m^4n^{10}p^5$.

EXAMPLES XIV. Page 46

1. (i) 1, −1, 4, 8, 4, −4, −8, 4, −8. (ii) 0, 5, 6, −3, −8, 7, −15.
2. 0, 1, −11, 5, −3, −1. **3.** 49, 25, 51, 26. **4.** −24, 0, 146, 18.
5. 32, 132, 16, 128. **6.** −5, 19, 35, 97.
7. 4, 8, −216, 1, 1216, −4. **9.** 6, 2, 0, 0, 12, 20, 30.
10. 16, 9, 1, 1, 36, 64. **11.** −12, 3, 54, 9, 66, 159.
12. 1, 1, 1, 7, −5, 121, −119.

EXAMPLES XV. Page 48

19. $x^2y^2z+xy^2z^2+x^2yz^2$. **20.** $-a^2b^2c+ab^2c^2-a^2bc^2$.
21. $-10a^2b^2c+15ab^2c^2-20a^2bc^2$. **22.** $x^5+2x^4y+x^3y^2$.
23. $a^3bc+ab^3c+abc^3$. **24.** $-3x+6x^2+9x^3-3x^4$.
25. $2x^4-6x^3+6x^2+2x$. **26.** $-3x^7+6x^5-9x^3$.

27. $2l^4mn+2lm^4n+2lmn^4$. 28. $30x^6y^2z+6x^4y^6z^4-54x^3y^7z^3$.

29. $-\frac{1}{10}a^2b^2c+\frac{3}{5}ab^3c+\frac{2}{3}ab^2c^2$.

30. $-10p^4q^4r^4+12p^2q^5r^5-\frac{35}{2}p^3q^3r^6$.

EXAMPLES XVI. Page 50

1. $27a^2+25a$. 2. $11a^2-5b^2$. 3. 0. 4. $16a-13a^2$.
5. $2x+12x^2$. 6. $12x^2+15x+18$. 7. $6a^2+3a+8$.
8. a^3-a. 9. x^2+2y^2. 10. $2x^2$. 11. $2a^2-2b^2$.
12. $4a^2+7a-6$. 13. $-3a^3-3a^2+8a-7$. 14. $4x^3+5x^2-5x-3$.
15. $11+2x^2-2x^4$. 16. $2a^2+7a-2$. 17. $5x^2-3$.
18. $3x^2+9$. 19. $a^2-2ab-4b^2$ 20. $-a^2+ab-b^2$.
21. $-3ab-6b^2$. 22. $4a^3-6a^2+8a-10$. 23. $6a^2-2a-5$.
24. a^3-3a^2+5a+4. 25. $-7x^4+9x^3-11x^2+13x-6$.
26. $-5x^3+9x^2y-9xy^2-7y^3$. 27. $2x^3+2x^2y+3xy^2$.
28. x. 29. $3a^2$. 30. $12xy$.
31. 0. 32. $2x^3y-2xy^3$. 33. x^2-y^2.
34. x^2-y^2. 35. $2x^2+x$. 36. x^2.
37. $3-3a^2$. 38. $6x^2-3x-37$. 39. $2x^2-2y^2-2z^2$.
40. $-1+a+a^3$.

EXAMPLES XVII. Page 55

1. x^2+3x+2. 2. x^2-5x+6. 3. x^2-x-6.
4. x^2+x-6. 5. $x^2+11x+24$. 6. x^2-x-42.
7. x^2+4x+3. 8. $x^2+11x+30$. 9. x^2-1.
10. $1-4x-21x^2$. 11. x^2-9. 12. $49-x^2$.
13. $x^2+2xy+y^2$. 14. $x^2+5ax+6a^2$. 15. x^2-4y^2.
16. $2x^2-5xy+2y^2$. 17. $a^2-ab-6b^2$. 18. $-20a^2+9ab-b^2$.
19. $6x^2-x-12$. 20. $16x^2-25$. 21. $p^2x^2-q^2$.
22. $c^2a^2-b^2$. 23. a^4-b^4. 24. $p^4+p^2q^2-12q^4$.
25. $a^4-a^3b^3-ab+b^4$. 26. $pqy^2+qy-py-1$.
27. $abx^2+bx-ax-1$. 28. $p^4+p^2q-3p^2r-3qr$.
29. $x^3+x^2a+xa^2+a^3$. 30. $x^3-x^2a-xa^2+a^3$.
31. $ab-b^2+2bc-ac-c^2$. 32. x^3-3x^2+3x-1.
33. $4x^3-8x^2+5x-1$. 34. x^3+a^3.
35. $x^3+9x^2+27x+27$. 36. $2x^3-11x^2+5x+25$.
37. $-c^3-4c^2d-5cd^2-2d^3$. 38. x^6-y^6. 39. a^4-b^2+4b-4.
40. $a^4-6a^3+11a^2-6a+1$. 41. $a^2+b^2+c^2+2ab+2bc+2ca$.
42. $x^2-y^2-z^2+2yz$. 43. $4x^2-y^2+6yz-9z^2$.

EXAMPLES XVIII. Page 57

1. Rs. $80x$. **2.** Rs. $5a$. **3.** Rs. $10x$. **4.** Rs. xy.
5. $2x^2$ miles. **6.** Rs. ay. **7.** x^2 annas. **8.** Rs. $5x^2$.
9. Rs. x^3. **10.** Rs. $6x^4$. **11.** Rs. $(5x+4y)$.
12. $50x+60y$. **13.** Rs. $(x^2+y^2+z^2)$. **14.** Rs. $14x$.
15. $2x^4$ annas. **16.** $23x^2$ miles. **17.** x^5 sq. ft.
18. (a^2-b^2) sq. ft. **19.** Rs. $a(3x+y)$.

EXAMPLES XIX. Page 60

34. $5x$. **35.** $2ab$. **36.** x. **37.** $\frac{8}{9}x$. **38.** 1.
39. xy^5. **40.** $-\frac{1}{6}$. **41.** $\frac{1}{36}$. **42.** $-1\frac{1}{2}$. **43.** $4\frac{1}{4}$.
44. $\frac{5}{13}$. **45.** $-\frac{1}{5}$. **46.** $1\frac{7}{32}$. **47.** $-1\frac{2}{3}$. **48.** $10\frac{1}{4}$.
49. -1. **50.** 6. **51.** $-4\frac{2}{5}$. **52.** $35\frac{9}{10}$.

EXAMPLES XX. Page 62

15. $1-5a^2+3a^4$. **16.** $-b^2+1-a^2$. **17.** $-a-b-ab$.
18. $1-2xy+3x^2y^2$. **19.** $-l^3m^2n^2+lmn-m$. **20.** $xyz-yz-z$.
21. $-a-x-ax$. **22.** $-1-5ab+3b$. **23.** $-xy^3+2x^2y^2+7x^3y$.
24. $-7q^2r^4-5p^2q^4+2p^4r^2$. **25.** $\frac{13}{15}x+\frac{11}{15}$.
26. $\frac{1}{8}x+\frac{5}{8}$. **27.** $-\frac{19}{21}x+1\frac{17}{21}$. **28.** $\frac{5}{3}x-25$.
29. $-\frac{3}{5}a+\frac{9}{10}$. **30.** $\frac{1}{2}x^2-\frac{33}{28}x+\frac{2}{7}$. **31.** $-\frac{2}{3}x^2+\frac{29}{12}x-\frac{4}{3}$.
32. $\frac{7}{20}x-\frac{7}{60}$. **33.** $-\frac{7}{12}x+6\frac{5}{12}$. **34.** $\frac{1}{6}x-\frac{1}{12}$. **35.** $-\frac{8}{9}x+\frac{8}{9}$.

EXAMPLES XXI. Page 66

1. $x+2$. **2.** $x+3$. **3.** $x-4$. **4.** $x-1$.
5. $x-7$. **6.** $3x+2$. **7.** $2x-1$. **8.** $4x+1$.
9. $5x+3y$. **10.** $3x+y$. **11.** $3x+5a$. **12.** $1+3x$.
13. $5-3a$. **14.** $5+x$. **15.** x^2-2x+3. **16.** x^2-4.
17. $2x^2+1$. **18.** $3-4a+a^2$. **19.** $4-x^2$. **20.** x^2+6x-3.
21. $x^2+11x-13$. **22.** $7x^2+x+8$. **23.** $7x^2+6x+1$.
24. $9x^2+6xy+4y^2$. **25.** $9-x^3$. **26.** $4a^2-4ab+b^2$.
27. $a^2+2ab+b^2$. **28.** $x^2-2xy+y^2$. **29.** $7x^2-xy-9y^2$.
30. a^2-a+1. **31.** x^2+x+1. **32.** $px+2$.
33. $ax+c$. **34.** $9x+bc$. **35.** $9k^2+9k+5$.
36. $4a^2-12ab+9b^2$. **37.** $a+b+c$. **38.** $a+b-c$.
39. x^2+x+1. **40.** a^4-a^2+1. **41.** $x+1$, rem. $2x+7$.

42. $x-5$. 43. $4x-5$. 44. $x+3y$.
45. $3x-2y$. 46. $a^5+a^4+a^3+a^2+a+1$.
47. $3k-2$. 48. $3p-4$.
49. a^2+3a+2, rem. $2a+1$. 50. m^2+m+1.
51. x^2+5x-3, rem. $4x+5$. 52. $4x^4-2x^2y^2+y^4$.
53. $27x^3+9x^2+3x+1$. 54. $x^6+x^3y^3+y^6$. 55. $5a^3+2a^2-4a+3$.
56. x^2+14x. 57. $3+3x+x^2$. 58. $5x-4a$.
59. $3a^2-4a+5$. 60. x^2-3b^2.

EXAMPLES XXII. Page 68

1. $\frac{x}{5}$. 2. Rs. $\frac{20}{x}$. 3. $\frac{100}{a}$. 4. $\frac{x}{y}$.
5. Rs. $\frac{a}{x}$. 6. Rs. $\frac{a}{x}$. 7. Rs. $\frac{b}{40a}$. 8. Rs. $\frac{y}{x}$.
9. Rs. $\frac{100y}{x}$. 10. x. 11. Rs. x^5. 12. Rs. $\frac{x-y}{3}$.
13. $\frac{a-b}{35}$. 14. Rs. $\frac{x-y}{z}$. 15. $\frac{x+y}{a}$. 16. Rs. $\frac{x+y+z}{a}$.
17. £$(x-2)$. 18. $5x^2-1$. 19. Rs. $(2+y)$. 20. $\frac{1}{2}x^2+x+\frac{3}{2}$.

MISCELLANEOUS EXAMPLES I. Page 69

A

1. $x+y+z$. 2. 15 ; $p+q$. 3. 9 ; $p-q$.
4. $2a$, $4a$, $4a+5$, $4a-5$. 5. 0, 0, 0. 6. (i) 6, 0, 2, -4.
(ii) 11, 5, -1, -7. (iii) 1, 2, 5, $-\frac{1}{3}$, -1, -1. (iv) 6, $\frac{1}{3}$, $\frac{2}{3}$, 6, $1\frac{1}{2}$.

B

1. $a+b+c$, $a+c+b$, $b+a+c$, $b+c+a$, $c+a+b$, $c+b+a$.
2. Rama goes x miles west ; Rama owes x rupees ;
Rama has x rupees. 3. (i) 16 ; (ii) 8 ; (iii) 48 ; (iv) 3.
4. (i) -6 ; (ii) 6 ; (iii) $-\frac{2}{3}$; (iv) $\frac{2}{3}$; (v) $-\frac{2}{3}$. 5. $x+y-z$.
6. $x+y$; $(2a+3y)$ annas.

C

3. 15. 4. $a+b$. 5. $-(a+b)$. 6. $4x$ feet.

D

1. (i) $5x$, (ii) x^5 ; 10, 32. 2. 15. 3. a^5, $3x^7$.
4. xx, $xxxxx$; 7. 5. 1. 6. 8.

E

1. $-b$. **2.** 1. **3.** a. **4.** (i) $\frac{x}{y}$; (ii) 1.

5. (i) $y+xy$. (ii) x^2y-x. (iii) $a+b-c$. (iv) $x^2+2xy-3y^2$.

6. lb sq. in.

F

2. (i) -1 ; (ii) 0 ; (iii) -8. **3.** The journey from Allahabad to Jabbulpore ; the journey from Jabbulpore to Allahabad.

4. $(a+b+c)$ inches. **5.** $(2l+2b)$ feet.

6. (i) 11 ; (ii) 13 ; (iii) 15 ; (iv) 21.

G

2. $a^2-ab-2b^2$. **3.** $(a+b)(a-b)$. **4.** At A.

5. 30 miles, 90 miles ; $\frac{x}{h}$ miles, $\frac{xy}{h}$ miles. **6.** 720 ; xyz.

H

1. 81, 12. **3.** (i) 12 $(x-y)$. (ii) $\frac{1}{16}$ $(x-y)$. **4.** Go 5 miles west ; go 10 miles east ; go 10 miles west. **5.** (i) 16 ; (ii) 15.

6. x.

I

1. (i) $8a+11b+7c$, 51. (ii) $13a+22b+20c$, 117.
(iii) $11a-4b-2c$, -3. (iv) $24a+12b-19c$, -9.

2. (i) $a+3b+5c$, 35. (ii) $7a+b+c$, 35.
(iii) $-b-2c$, -12. (iv) $a+11b-8c$, -14.

3. $\frac{4}{3}a-b+\frac{13}{6}c$, $2a+b+\frac{19}{6}c$. **4.** $x^6+2a^3x^3-4a^4x^2+a^6$.

5. $x^2+2xy+2y^2$.

J

2. $2a^2-2b^2+4$. **3.** $1+2x-x^2+5x^3-x^5-2x^6$.

4. $2x^2-4x+3$. **5.** $2a-3b+6c+3ac$, $2a-b-3c+3ac$; $2b-9c$.

6. $\frac{m}{h}$; $\frac{60h}{m}$.

EXAMPLES XXIII. Page 76

26. 10404. **27.** 40401. **28.** 11025. **29.** 41209.
30. 1002001. **31.** 1004004. **32.** 100060009. **33.** 400400100.
34. 10060·09. **35.** 2520·04. **36.** 640480·09.

EXAMPLES XXIV. Page 78

28.	9604.	**29.**	998001.	**30.**	249001.	**31.**	99980001.
32.	98·01.	**33.**	999600·04.	**34.**	809280·16.	**35.**	9994·0009.

EXAMPLES XXV. Page 81

26.	120.	**27.**	396.	**28.**	20000.	**29.**	280.
30.	999996.	**31.**	39991.	**32.**	6391.	**33.**	120·75.
34.	399·9984.	**35.**	3·9984.				

EXAMPLES XXVII. Page 87

31.	$12bc^2$.	**32.**	$7x^2y^3z$.	**33.**	$5ab^3$.	**34.**	$-7a^4b^5$.
35.	$-4lm$.	**36.**	ab^2.	**37.**	$\frac{xy^2}{3}$.	**38.**	$2x^2$.
39.	x^2y^3.	**40.**	$2a^2b$.	**41.**	x feet.	**42.**	$7xy$ yards.
43.	$6x$ yards.	**44.**	$4xy$ yards.				

EXAMPLES XXVIII. Page 88

1.	1.	**2.**	2.	**3.**	3.	**4.**	2.	**5.**	1.	**6.**	6.
7.	2.	**8.**	6.	**9.**	2.	**10.**	6.	**11.**	13.	**12.**	$3\frac{7}{15}$.
13.	61.	**14.**	23.	**15.**	4.	**16.**	6.	**17.**	16.	**18.**	6.
19.	-48.	**20.**	-6.	**21.**	36.	**22.**	48.	**23.**	-2.	**24.**	-12.
25.	3.	**26.**	0.	**27.**	5.	**28.**	1.	**29.**	1.	**30.**	0.
31.	2.	**32.**	0.	**33.**	-5.	**34.**	10.	**35.**	$6.\sqrt[4]{45}$.	**36.**	$\frac{1}{3}$.
37.	$-\frac{3}{4}$.	**38.**	-2.	**39.**	-41.	**40.**	43.	**41.**	2.	**42.**	2.
43.	-53.	**44.**	2.								

EXAMPLES XXX. Page 95

1.	6.	**2.**	10.	**3.**	4.	**4.**	$\frac{1}{6}$.	**5.**	-6.
6.	4.	**7.**	-7.	**8.**	2.	**9.**	2.	**10.**	8.
11.	0.	**12.**	6.	**13.**	8.	**14.**	28.	**15.**	18.
16.	15.	**17.**	a.	**18.**	$\frac{a-b}{3}$.	**19.**	$\frac{1}{2}$.	**20.**	$2\frac{1}{2}$.
21.	-1.	**22.**	-1.	**23.**	-3.	**24.**	5.	**25.**	-3.
26.	-10.	**27.**	5.	**28.**	21.	**29.**	7.	**30.**	-5.
31.	6.	**32.**	$3\frac{1}{3}$.	**33.**	$1\frac{1}{2}$.	**34.**	$\frac{56}{81}$.	**35.**	3·1.
36.	1·7.	**37.**	2·085.	**38.**	·483.	**39.**	-5.	**41.**	3.
42.	$-1\frac{1}{2}$.	**43.**	3.	**44.**	-13.	**45.**	5.	**46.**	$1\frac{9}{14}$.
47.	$1\frac{1}{3}$.	**48.**	2.	**49.**	-8.	**50.**	48.	**51.**	5.
52.	$\frac{a+b+c}{3}$.								

EXAMPLES XXXI. Page 97

1.	$-2\frac{2}{3}$.	**2.**	$-\frac{9}{14}$.	**3.**	12.	**4.**	12.	**5.**	$11\frac{1}{4}$.
6.	8.	**7.**	$2\frac{2}{5}$.	**8.**	$3\frac{3}{5}$.	**9.**	1.	**10.**	12.
11.	12.	**12.**	11.	**13.**	17.	**14.**	8.	**15.**	$-\frac{1}{3}$.
16.	5.	**17.**	$\frac{6}{13}$.	**18.**	$3\frac{1}{2}$.	**19.**	$\frac{71}{84}$.	**20.**	$1\frac{1}{3}$.
21.	5.	**22.**	$\frac{4}{25}$.	**23.**	11.	**24.**	5.	**25.**	18.
26.	$\frac{1}{2}$.	**27.**	34.	**28.**	$8\frac{23}{31}$.	**29.**	7.	**30.**	$-9\frac{1}{2}$.
31.	$1\frac{2}{5}$.	**32.**	$-1\frac{1}{5}$.	**33.**	11.	**34.**	$3\frac{6}{7}$.	**35.**	41.
36.	10.	**37.**	1.	**38.**	7.	**39.**	1.	**40.**	$7\frac{1}{2}$.
41.	5.	**42.**	2.	**43.**	8.	**44.**	$-1\frac{5}{37}$.	**45.**	8.
46.	9.	**47.**	1.	**48.**	7.	**49.**	·4.	**50.**	2.
51.	14.	**52.**	4.	**53.**	6.	**54.**	$7\frac{2}{11}$.	**55.**	$4\frac{1}{2}$.

EXAMPLES XXXIII. Page 105

1. (i) 12, 13, 14 ; (ii) x, $x+1$, $x+2$; (iii) 10, 11, 12 ; (iv) $x-2$, $x-1$, x.

2. (i) 11, 12, 13 ; (ii) $x-1$, x, $x+1$.

3. (i) 12, 14, 16 ; (ii) $2x$, $2x+2$, $2x+4$; (iii) 8, 10, 12 ; (iv) $2x-4$, $2x-2$, $2x$; (v) 10, 12, 14 ; (vi) $2x-2$, $2x$, $2x+2$.

4. (i) 15, 17, 19 ; (ii) $2x+1$, $2x+3$, $2x+5$; (iii) 11, 13, 15 ; (iv) $2x-3$, $2x-1$, $2x+1$; (v) 13, 15, 17 : (vi) $2x-1$, $2x+1$, $2x+3$; (vii) $2x-5$, $2x-3$, $2x-1$.

5. $2x-5$, $2x-3$, $2x-1$, $2x+1$.

6. $2x-4$, $2x-2$, $2x$, $2x+2$. **7.** (i) 32 ; (ii) $10y+x$; (iii) $10x+y$.

8. $10b+a$, $10a+b$. **9.** $6x$, $\frac{21}{2}x$.

10. $100z+10y+x$, $100x+10y+z$. **11.** $124x$, $421x$. **12.** $9x$.

EXAMPLES XXXIV. Page 106

1.	$2x=12$.	**2.**	$3x=18$.	**3.**	$\frac{x}{2}=6$.	**4.**	$\frac{x}{4}=8$.
5.	$\frac{x}{5}=15$.	**6.**	$x+5=15$.	**7.**	$x+a=b$.	**8.**	$x=10a$.
9.	$2x-10=8$.	**10.**	$4x-a=24$.	**11.**	$\frac{x}{4}-a=b$.	**12.**	$3x=2x+6$.
13.	$5x=\frac{x}{2}+18$.	**14.**	$x-5=8$.	**15.**	$3x-4=2x$.	**16.**	$4x-2x=6$.

17. $2x-20=x+6$. **18.** $x=\frac{5}{100}\times 500$. **19.** $x=\frac{5}{100}\times y$.

20. $x=\frac{z}{100}\times y$. **21.** $2x=\frac{1}{2}(5x-10)$. **22.** $2x=(5x-10)-20$.

23. $2x = (4x+20) - 40$. **24.** $(4x+20) = (5x-10) + 20$.
25. $4x + 20 = 6x$. **26.** $(4x+20) - 10 = 5(2x-10)$.
27. $2x + 10 = (5x-10) - 10$. **28.** $2x + (5x-10) = 4x + 20$.
29. $(4x+20) - 2x = 5x - 10$.
30. $(2x-5) + (5x-10-5) + (4x+20-5) = 105$.
31. $5x + 15 = 2 \times 3x$. **32.** $16x - 20 = 4 \times 3x$.
33. $3x + (5x+15) = 45$. **34.** $(5x+15) - 10 = (3x+10) - 5$.
35. $(16x-20) - 30 = 5x + 15$. **36.** $3x + (5x+15) + (16x-20) = 105$.
37. $(16x-20) = 3x + (5x+15) + 15$.

EXAMPLES XXXV. Page 110

1. (i) 160 sq. yd. (ii) 3 ft. (iii) 21 ft., (iv) 108 sq. ft., 7 yd., 20 m.
2. (i) 40 sq. ft. (ii) 16 yd. (iii) 7m. (iv) 15 sq. in., 4 ft., 16 yd.
3. (i) 132 ft. (ii) 14 mm. **4.** (i) $38\frac{1}{2}$ sq. cm. (ii) 7 yd.
5. (i) 1800 ft. (ii) $26\frac{2}{3}$ sec. (iii) 44 ft. per sec. (iv) 1320 ft., 30 ft. per sec., $\frac{x}{y}$ sec.
6. (i) 22176 cu. in. (ii) $\frac{1}{3}$ m. (iii) 1 ft. **7.** (i) 616 sq. ft. (ii) 15 ft. (iii) 8 yd. (iv) 8 m. **8.** (i) 6 cu. ft. (ii) ·5 yd. (iii) 80 m. (iv) $44\frac{4}{7}$ in. (v) 160 cu. in., 2 in., 6 cm., 160 mm., $6x^3$ cu. in.
9. (i) 20 ft. (ii) 3 ft. (iii) 12 ft. (iv) 5 ft., 12 in., 12 yd., $10x$ cm.
10. (i) Rs. 60. (ii) 4%. (iii) 5 years. (iv) Rs. 600. (v) Rs. 120., 4 years, Rs. 750, Rs. $\frac{xyn}{100}$. **11.** (i) 18. (ii) 10. (iii) 15. (iv) 120.
12. (i) 34·64. (ii) 12. (iii) 17·32. (iv) 100. **13.** (i) 96. (ii) 44.
14. (i) 5050. (ii) 40. (iii) 10. (iv) 10. **15.** (i) 30. (ii) 84.

EXAMPLES XXXVI. Page 118

1.	12.	**2.**	51.	**3.**	250.	**4.**	40 ; 120.	**5.**	70 ; 40.
6.	11.	**7.**	80.	**8.**	16.	**9.**	11.	**10.**	9.
11.	20.	**12.**	4.	**13.**	15.	**14.**	7.	**15.**	25.
16.	21.	**17.**	13.	**18.**	4.	**19.**	52.	**20.**	90.
21.	27.	**22.**	6.	**23.**	105.	**24.**	2.	**25.**	16.
26.	5.	**27.**	3.	**28.**	45.	**29.**	6.	**30.**	10.
31.	21.	**32.**	6.	**33.**	20.	**34.**	10 years.	**35.**	15 years.
36.	30 years.			**37.**	Rs. 450.	**38.**	54.	**39.**	Rs. 1200.
40.	Rs. 250.			**41.**	20.	**42.**	10 years.		

EXAMPLES XXXVII. Page 123

1. 24, 12. **2.** 36, 12. **3.** 13, 26. **4.** 11, 33. **5.** 60, 90.
6. 60, 140. **7.** 160, 120. **8.** 60, 10. **9.** 100, 50. **10.** $53\frac{1}{3}$, $133\frac{1}{3}$.
11. 30, 66. **12.** 25, 13. **13.** 52, 13. **14.** 20, 25. **15.** 68, 17.
16. 81, 59. **17.** $25\frac{1}{2}$, $24\frac{1}{2}$. **18.** 60, 40. **19.** 4, 5.
20. 50, 40. **21.** 150, 100. **22.** 16, 20. **23.** 22, 33.
24. 100, 60. **25.** 56, 88. **26.** 125, 64. **27.** 94, 63.
28. Rs. 36, Rs. 20. **29.** Rs. 11, Rs. 18. **30.** 5, 10.
31. 15, 5. **32.** 8 in., 4 in. **33.** Rs. 86, Rs. 43.
34. Rs. 84, Rs. 14. **35.** 30 yr., 25 yr. **36.** 35 yr., 15 yr.
37. 24 yr., 12 yr. **38.** 58 yr., 26 yr. **39.** 30 yr., 15 yr.
40. 15 yr., 5 yr. **41.** 33 yr., 3 yr. **42.** 25 yr., 15 yr.
43. 60 yr., 40 yr. **44.** 45 yr., 15 yr. **45.** 70 yr., 35 yr.
46. 15, 30, 45. **47.** 6, 12, 14. **48.** 1, 3, 5.
49. 63, 53, 43. **50.** Rs. 15. 10*a*. 6*p*., Rs. 6. 11*a*. 6*p*.
51. Rs. 24, Rs. 34, Rs. 42. **52.** Rs. 100, Rs. 50, Rs. 65.
53. £ 68, £ 65, £ 53. **54.** 8, 12. **55.** 6, 12, 36.
56. Rs. 15, Rs. 20, Rs. 105. **57.** 13 in., 14., 15 in.
58. 20 cm., 17 cm., 10 cm. **59.** 62°, 40°, 78°.
60. 30°, 50°, 100°. **61.** 5, 10, 15. **62.** 6, 10, 16.
63. 380, 920, 1520. **64.** 16 yr., 25 yr., 10 yr.
65. 20 yr., 10 yr., 25 yr. **66.** 26 yr., 35 yr., 40 yr.
67. 12 yr. **68.** 24 rupees, 48 eight-anna pieces, 56 two-anna pieces.
69. 50. **70.** 18 pounds, 54 shillings. **71.** 120.
72. 10 rupees, 9 eight-anna pieces, 30 four-anna pieces, 32 two-anna pieces.

EXAMPLES XXXVIII. Page 128

1. 7, 8, 9. **2.** 11, 12, 13, 14. **3.** 12, 14. **4.** 8, 10, 12.
5. 18, 20, 22, 24. **6.** 13, 15. **7.** 19, 21, 23.
8. 7, 9, 11, 13, 15. **9.** 6, 7, 8. **10.** 25, 26, 27.
11. 34. **12.** 47. **13.** 54. **14.** 64.
15. 36. **16.** 62. **17.** 48. **18.** 96.
19. 63. **20.** 36. **21.** 246. **22.** 531.

EXAMPLES XXXIX. Page 133

1. After 3 hours ; 30 miles from Cawnpore. **2.** $2\frac{1}{2}$ miles per hour.
3. 5 hours ; 190 miles from Cawnpore. **4.** 40, 20 miles per hour.

5. $2\frac{1}{2}$ hours after the start of the first train from Aligarh ; 70 miles from Aligarh. 6. 4 hours.

7. 3 hours after the start of the second man.

8. 10, $7\frac{1}{2}$. 9. 30 miles. 10. 48 miles. 11. 72 miles.

12. 175 miles. 13. $2\frac{1}{12}$. 14. 3. 15. $46\frac{2}{3}$.

16. 5. 17. 2. 18. 24 minutes.

19. Barley, 96 seers ; gram, 70 seers.

20. Silk cloth, 30 yards ; cotton cloth, 25 yards. 21. 20 lb., 25 lb.

22. 15. 23. 144 sq. yd. 24. 500 sq. ft. 25. 16 yd., 18 yd.

26. 500 yd., 200 yd. 27. 50 ft. 28. £400, £500.

29. 400. 30. 200. 31. Rs. 1200. 32. 24.

33. 5. 34. Rs. 60, Rs. 45. 35. Rs. 500, Rs. 250. 36. 33.

EXAMPLES XL. Page 141

1. $x=5\frac{1}{2}$, $y=2\frac{1}{2}$. 2. $x=18$, $y=12$. 3. $x=18$, $y=30$.

4. $x=15$, $y=16$. 5. $x=2$, $y=1\frac{1}{2}$. 6. $x=8$, $y=0$.

7. $x=0$, $y=3$. 8. $x=\frac{a+b}{2}$, $y=\frac{a-b}{2}$. 9. $x=17$, $y=30$.

10. $x=3$, $y=1$. 11. $x=1$, $y=2$. 12. $x=8$, $y=2$.

13. $x=3$, $y=2$. 14. $x=3$, $y=2$. 15. $x=5$, $y=4$.

16. $x=1$, $y=6$. 17. $x=3$, $y=-3$. 18. $x=\frac{1}{2}$, $y=4$.

19. $x=-4$, $y=-1$. 20. $x=5$, $y=3$. 21. $x=4\frac{1}{7}$, $y=-1\frac{1}{7}$.

22. $x=-5$, $y=-2$. 23. $x=13$, $y=5$. 24. $x=2$, $y=1$.

25. $x=43$, $y=33$. 26. $x=-30$, $y=-7\frac{1}{2}$. 27. $x=9$, $y=7$.

28. $x=4$, $y=3$. 29. $x=5$, $y=3$. 30. $x=1\frac{1}{3}$, $y=-2\frac{1}{3}$.

31. $x=-2\frac{1}{2}$, $y=5\frac{1}{2}$. 32. $x=4$, $y=3$. 33. $x=15$, $y=-5$.

34. $x=-1$, $y=2$. 35. $x=-1$, $y=-1$. 36. $x=-55$, $y=60$.

37. $x=a$, $y=3a$. 38. $x=a$, $y=a$. 39. $x=3{\cdot}6$, $y=-7{\cdot}2$.

40. $x=3{\cdot}5$, $y=6$. 41. $x={\cdot}3$, $y={\cdot}4$. 42. $x={\cdot}02$, $y=2{\cdot}9$.

EXAMPLES XLI. Page 146

1. $x=12$, $y=6$. 2. $x=5$, $y=5$. 3. $x=16$, $y=-24$.

4. $x=6$, $y=8$. 5. $x=-6$, $y=2$. 6. $x=2$, $y=3$.

7. $x=-2$, $y=3$. 8. $x=-2$, $y=-3$. 9. $x=-3$, $y=9$.

10. $x=4\frac{11}{83}$, $y=1\frac{77}{83}$. 11. $x=1$, $y=5$. 12. $x=2\frac{81}{83}$, $y=1\frac{69}{83}$.

13. $x=7$, $y=3$. 14. $x=-5\frac{11}{13}$, $y=-4\frac{2}{13}$.

15. $x=-1\frac{17}{57}$, $y=-2\frac{71}{251}$. 16. $x=7$, $y=10$.

17. $x=8$, $y=6$. 18. $x=10$, $y=4$. 19. $x=4$, $y=3$.

20. $x=18$, $y=48$. **21.** $x=6$, $y=-4$. **22.** $x=\frac{1}{4}$, $y=\frac{1}{3}$.
23. $x=1$, $y=2$. **24.** $x=\frac{1}{2}$, $y=1$. **25.** $x=3$, $y=4$.
26. $x=\frac{1}{6}$, $y=\frac{1}{3}$. **27.** $x=2$, $y=3$. **28.** $x=3$, $y=4$.
29. $x=3$, $y=4$. **30.** $x=\frac{1}{3}$, $y=\frac{1}{7}$. **31.** $x=1$, $y=-1$.
32. $x=\frac{1}{4}$, $y=3$. **33.** $x=\frac{1}{2}$, $y=\frac{1}{2}$. **34.** $x=-\frac{10}{19}$, $y=\frac{12}{32}$.
35. $x=\frac{1}{4}$, $y=-\frac{1}{3}$. **36.** $x=7$, $y=5$. **37.** $x=2$, $y=7$.
38. $x=4\frac{31}{33}$, $y=-3\frac{6}{35}$. **39.** $x=-\frac{5}{8}$, $y=-\frac{1}{4}$.
40. $x=2$, $y=5$.

EXAMPLES XLII. Page 150

1. 24, 16. **2.** 36, 27. **3.** 19, 11. **4.** 20, 15.
5. 19, 9. **6.** 50, 30. **7.** 26, 12. **8.** 4, 5.
9. 15, 9. **10.** 30 ft., 20 ft. **11.** 14, 21.
12. 60, 41. **13.** 50, 30. **14.** 7, 5. **15.** 27, 5.
16. 128, 98. **17.** 75, 40. **18.** Rs. 35, Rs. 28.
19. Rs. 60, Rs. 30. **20.** Rs. 32, Rs. 28.
21. 33, 7. **22.** 80, 40. **23.** 55, 44. **24.** 100.
25. 48 yr., 24 yr. **26.** 42 yr., 18 yr.
27. 30 yr., 10 yr. **28.** 33 yr., 20 yr.
29. 20 yr. **30.** 16 yr. **31.** 60 yr., 25 yr.
32. 40 yr., 18 yr. **33.** 28 yr., 15 yr.
34. 61 yr., 29 yr.

EXAMPLES XLIII. Page 156

1. 73. **2.** 65. **3.** 43. **4.** 36. **5.** 72. **6.** 42.
7. Rs. 150, Rs. 30. **8.** Rs. 12, Rs. 6. **9.** 11, 4.
10. 12*a*., 7*a*. **11.** 7000, 3000. **12.** 1800, 2200.
13. 4%, 3%. **14.** $3\frac{1}{2}$%, 3%. **15.** 4%, $3\frac{1}{2}$%.
16. 18, 24 ; $10\frac{2}{7}$. **17.** 960, 480. **18.** 50, 45.
19. 20, 13. **20.** Rs. 24, Rs. 8. **21.** 15*d*., 6*d*.
22. 7 miles per hour, 3 miles per hour.
23. 4 miles per hour, 3 miles per hour.

EXAMPLES XLIV. Page 167

1. (i) (10, 5), (5, 6), (0, 2), (−5, 5), (−10, 10), (−5, 0), (−8, −5), (0, −5), (5, −5), (8, −2), (6, −8), (10, −5).

(ii) (5, 2·5), (2·5, 3), (0, 1), (−2·5, 2·5), (−5, 5), (−2·5, 0), (−4, −2·5), (0, −2·5), (2·5, −2·5), (4, −1), (3, −4), (5, −2·5).

iii) (1, ·5), (·5, ·6), (0, ·2), (−·5, ·5) (−1, 1), (−·5, 0), (−·8, −·5), (0, −·5), (·5, −·5) (·8, −·2), (·6, −·8), (1, −·5).

5. (i) 13. (ii) 10. (iii) 13. (iv) 22. (v) 17. (vi) 44·8. (vii) 19.

6. 9·84, 9·84, 8.

7. (i) 16 sq. units. (ii) 10 sq. units. (iii) 44 sq. units. (iv) 96 sq. units.

8. (i) 50 sq. units. (ii) 52 sq. units. (iii) 42 sq. units. (iv) 50 sq. units. (v) 96 sq. units. (vi) 90 sq. units.

9. (i) $43\frac{1}{2}$ sq. units. (ii) 73 sq. units. (iii) $83\frac{1}{2}$ sq. units.

10. (i) 180 sq. units. (ii) 162 sq. units. (iii) $71\frac{1}{2}$ sq. units.

EXAMPLES XLVI. Page 179

1. $(5, 3\frac{1}{2})$. 2. $(7, -5)$. 3. $(3, 0)$. 4. $(5, 3)$.
5. $(-3, -5)$. 6. $(1, -2)$. 7. $(2, 1)$. 8. $(2, 3)$.
9. $(-\frac{2}{3}, -4)$. 10. $(5\frac{1}{2}, 4\frac{1}{2})$. 11. $(8, 2)$. 12. $(3, 5)$.
13. $(-3, 2)$. 14. $(0, 3)$. 15. $(5, 10)$.
16. $(\frac{1}{3}, 1\frac{2}{3})$, $(2\frac{1}{2}, -\frac{1}{2})$, $(-4, -7)$. 17. $(2\frac{2}{3}, 2\frac{1}{3})$, $(2, 1)$, $(3, 2\frac{1}{2})$.
18. $(2, 4)$. 19. $(5, -1)$. 20. $3x-2y=0$.
21. $3x-y+5=0$. 22. $3x+4y-12=0$. 23. $2x-5y+3=0$.
24. $2x-y+5=0$. 25. $5x-y=0$. 26. $4x-6y-5=0$.
27. $3x-5y-4=0$. 28. $x-y+3=0$. 29. $m=2$, $c=5$.
30. $m=-1$, $c=1$.

EXAMPLES XLVII. Page 182

1. $-2x^4+9ax^3-14a^2x^2+9a^3x-2a^4$.
2. $3a^5+a^4-17a^3+13a^2+16a-12$. 3. $-l^4+4m^4+4m^2n^2+n^4$.
4. $-2x^4-11x^3+2x^2+17x-6$. 5. $a^5+a^4-2a^3+a^2-5a+4$.
6. $k^2-2kl+l^2-m^2+2mn-n^2$. 7. $a^4+2a^3+a^2-b^4+2b^3-b^2$.
8. $x^3+y^3+z^3-3xyz$. 9. $a^6-3a^4b^2+3a^2b^4-b^6$.
10. $xy^3+2xyz^2+y^3z+yz^3-2xy^2z-xz^3-y^2z^2$.
11. $a^2b^2-a^2-b^2+1$. 12. $a^4-2a^2b^2+4abc^2+b^4-c^4$.
13. $a^3-b^3-c^3-3abc$. 14. $a^3+8b^3+c^3-6abc$.
15. $a^3+8b^3-27c^3+18abc$.

EXAMPLES XLVIII. Page 186

1. $4x^5+6x^4-7x^3+8x^2-29x-36$. 2. $x^6+2x^3y^3+y^6$.
3. $1+x^2-x^4-x^6$. 4. $1-8x^4-9x^6$.
5. $x^6-3x^4y^2+3x^2y^4-y^6$. 6. $1-6x^5+5x^6$.
7. x^8-1. 8. $x^8+x^6+9x^4+x^2+1$.
9. $x^6-6x^4-8x^3-1$. 10. $x^{10}-x^5y^5+y^{10}$.
11. $x^{12}+2x^6y^6+y^{12}$. 12. $x^4-\frac{13}{6}x^2+\frac{1}{3}x+\frac{1}{6}$.

13. $\frac{1}{6}x^4+\frac{1}{4}x^3-\frac{7}{8}x^2+\frac{3}{4}x-\frac{3}{16}$.
14. $\frac{3}{4}x^5-4x^4y+\frac{77}{8}x^3y^2-\frac{43}{4}x^2y^3-\frac{33}{4}xy^4+27y^5$. **15.** x^6-a^6.
16. x^8-16y^8. **17.** $8x^2-2x+3$. **18.** $-10x^2+5x-1$.
19. $5x^5+16x^4+17x^3$. **20.** $3a^6+8a^5-19a^4$.

EXAMPLES XLIX. Page 189

1. x^2-2x+3. **2.** x^2+2x-3. **3.** x^3+2x^2+3x+4.
4. x^3-7x^2-7x-3. **5.** x^2-2x+4. **6.** $1+3x-x^3-x^4$.
7. x^3+3x-2. **8.** $4-3x+2x^2-x^3$. **9.** $x^3-2x^2y-4y^3$.
10. $x^3-2a^2x+a^3$. **11.** $x^3-3x^2y+2y^3$. **12.** $\frac{1}{2}x^2-\frac{1}{3}x+\frac{1}{4}$.
13. $\frac{1}{2}x+2$. **14.** $\frac{3}{2}x^3-5x^2+\frac{1}{4}x+9$. **15.** $1-\frac{1}{2}x+\frac{1}{3}y$.
16. $x^2+\frac{1}{2}ax-2a^2$.

EXAMPLES L. Page 192

1. $x^2+y^2+1-xy+x+y$. **2.** $a^2+9b^2+4c^2-3ab+2ac+6bc$.
3. $x^2+y^2+z^2-xy+xz+yz$. **4.** $a+2b+3c$.
5. $3x^2+5y^2$. **6.** $5x-2y$. **7.** $x^4+x^3y+x^2y^2+xy^3+y^4$.
8. $x^4-x^3y+x^2y^2-xy^3+y^4$. **9.** $x^3+2x^2y+2xy^2+y^3$.
10. x^4+4x^2+8. **11.** $x^7-x^6y+x^4y^3-x^3y^4+xy^6-y^7$.
12. $x^{10}+x^8y^2+x^6y^4+x^4y^6+x^2y^8+y^{10}$. **13.** $a^8-a^6+2a^2-2$.
14. $a^7-a^6x+a^5x^2-a^4x^3+a^3x^4-a^2x^5+ax^6-x^7$, rem. $2x^8$.
15. $1+x+x^2+x^3$. **16.** $1+2x+4x^2+8x^3+16x^4+32x^5$.
17. $1+2x+2x^2+2x^3$. **18.** $1+a-a^3-a^4$.
19. $2+4x+8x^2+16x^3+32x^4$. **20.** 10. **21.** -3.
22. -10. **23.** -3. **24.** 8. **25.** 7.
26. $x^2+2xy+y^2-1$. **27.** $x+y+z+xyz$.

MISCELLANEOUS EXAMPLES II. Page 193

A

1. $3x^2+8ax-2a^2$. **2.** $100a+10b+c$. **3.** $-2b$ miles east or $+2b$ miles west of O. **4.** 0. **5.** 1. **6.** 27 ft.

B

1. a. **2.** $4x^2+2x-3$. **3.** 4. **4.** Rs. $\frac{xy}{40}$. **5.** Rs. $x\left(1+\frac{y}{100}\right)$.

C

1. $8a^3-2a-1$. **2.** $3x^4-5x^3a-12x^2a^2-xa^3+3a^4$. **4.** 1.
5. $(a+3)$ ft. **6.** 16.

D

1. (i) Rs. $\frac{xy}{12}$. (ii) Rs. $\frac{xy}{12z}$. **2.** $4x$. **3.** 14. **4.** $3x^2-2x-4$.

5. $\frac{xy}{3}$, $2z(3x+y)$.

E

1. $7a+2c-2d$. **2.** $\frac{1}{4}$. **3.** 0. **4.** $2a^2-3ax+x^2$.

5. $2a-2b+2c$. **6.** Rs. $\frac{x}{5}$, Rs. $\frac{3x}{10}$, Rs. $\frac{x}{2}$.

F

1. $4a^2+5a$; 6. **2.** $(x-y)$ yards. **3.** 4. **4.** 7.

5. $(a+b+c+3x)$ years. **6.** $127\frac{1}{2}$ sq. units.

G

1. (i) $(2x-y)$ years. (ii) $(2x-y+30)$ years. (iii) $(2x-y-30)$ years. (iv) y years. **2.** $x=2\frac{2}{5}$, $y=12$. **3.** $4x-5$.

4. $10x^2+10x+10$. **5.** $2b$. **6.** 96 sq. units.

H

1. $3x^3-3x^2+3x-9$. **2.** $3x^3+x^2-2x-5$. **3.** 12. **4.** 6.

5. $(ax+by)$ miles. **6.** 32 sq. units.

I

1. $x+\frac{y}{z}$. **2.** $6a^2-a+1$. **3.** 6. **4.** $2a-2x$. **5.** Rs. 5.

6. $x=5$, $y=6$.

J

1. $x^2-2x+15$. **2.** $x^4-x^3y+x^2y^2-xy^3+y^4$. **3.** 15.

4. -3, -6, 0, 30, 4. **5.** 4 days. **6.** 6.

TEST PAPERS. Page 197

I

1. a^2+3b^2. **2.** 216. **3.** $2a-3b$. **4.** 2. **5.** $2\frac{1}{5}$.

6. 4, 10, 2, 12.

II

1. 36 sq. units. **2.** $\frac{31}{2}b$. **3.** $\cdot 06-\cdot 3x+\cdot 2x^2-x^3$; $\cdot 031$.
4. $a^2+b^2+4c^2+2ab+2bc+2ca$. **5.** $-2\frac{1}{2}$. **6.** 138, 140.

III

1. $-\frac{3}{4}$. **2.** $100a+10b+c$. **3.** $2x+89$. **4.** -2.
5. $x=\cdot 3$, $y=\cdot 4$. **6.** 11.

IV

1. x^3-2x+6. **2.** 0. **3.** -18. **4.** 3. **5.** $x=12$, $y=-4$.
6. 40 yr., 35 yr.

V

1. $-3a^2+b^2-c^2$. **2.** -56. **3.** $\frac{1}{a}$. **4.** -2.
5. $x=2$, $y=3$. **6.** 15, 42.

VI

1. 2. **2.** $x=3\frac{1}{2}$, $y=2\frac{1}{2}$. **3.** $-4\frac{3}{5}$. **4.** 12, 5.
5. $x=2\frac{1}{2}$, $y=\frac{1}{2}$. **6.** 56, 67.

VII

1. $1-4x+4x^2$; rem. 15. **2.** 15. **3.** $x=3$, $y=-3$.
4. 2. **5.** $(x+8)$ years; $(x-7)$ years; 12 years.
6. 11 four-anna pieces, 17 pice.

VIII

1. 0. **2.** 0. **3.** $-1\frac{6}{37}$. **4.** $x=5$, $y=2$.
5. $x=-2$, $y=4$. **6.** 91.

IX

1. $2y-z$; first class, 14; second class, 67; third class, 193.
2. $5x-4a$. **3.** $x=4$, $y=3$. **4.** 2. **5.** $x=18$, $y=48$.
6. 120 miles.

X

1. 14. **2.** $x=\frac{5a+3b}{4}$, $y=\frac{5a-3b}{4}$. **3.** $ap+bq$.
4. x^2-34y^2; $2y^2$. **5.** 180. **6.** $x=5$, $y=6$.

PART II

EXAMPLES LI. Page 202

1. $a(b+c)$. 2. $a(x-y)$. 3. $x(y^2-z)$. 4. $3(x+y)$.
5. $3(x-2y)$. 6. $x(a+2)$. 7. $3x(1-3x)$. 8. $x(x+1)$.
9. $x^2(x-1)$. 10. $p^3(p-5)$. 11. $a(a-b)$. 12. $5a^2(a-5b)$.
13. $13(3-7x)$. 14. $l^2(l^2-6m^4)$. 15. $y^2(xz-3)$. 16. $-a(3a-2)$.
17. $bc(a+d)$. 18. $m(ln-pq)$. 19. $lmn^2(mn-l^2)$.
20. $2ax^2(2-3ax^2)$. 21. $-2xy(x^2-2y^2)$. 22. $a(b+c-d)$.
23. $x(y-z+w)$. 24. $-a(l-m+n)$. 25. $la(la-xb+xyb)$.
26. $lmn(n+m-l)$. 27. $7lmn(ln^2-3l^2m+5m^2n)$.
28. $-xyz(y-z+a-x)$.

EXAMPLES LII. Page 205

1. $(m+n)(x+y)$. 2. $(a-b)(l-m)$. 3. $(c+d)(x-y)$.
4. $(x-y)(3-a)$. 5. $(l+m)(a+b)$. 6. $(l-m)(x^2+1)$.
7. $(x+y)(a-b)$. 8. $(a^2+b^2)(2x-3y)$. 9. $(c+d)(ab+x)$.
10. $(x+3)(x-4)$. 11. $(x-2)(x+a)$. 12. $(x+1)(x^2+1)$.
13. $(y^2+1)(y-1)$. 14. $(b+1)(a+1)$. 15. $(x^2+1)(3x-7)$.
16. $(x^2+1)(6x-7)$. 17. $(x^2+1)(3x-1)$. 18. $(x+1)(x^4+1)$.
19. $(1+z)(x^2-y)$. 20. $(a+1)(a-b^2)$. 21. $(l^2+4)(3l-1)$.
22. $(x+y)(x^2-yz)$. 23. $(x-y)(c-3)$. 24. $(a-5)(a^2-5)$.
25. $(c-1)(a^2b-1)$. 26. $(a-c)(2a-7b)$. 27. $(a+b+c)(x+y)$.
28. $(x^2+x+1)(a+b)$. 29. $(l-m-n)(x-y)$. 30. $(x^2+y^2+xy)(a+3)$.

EXAMPLES LIV. Page 210

1. $(x+2)(x+4)$. 2. $(x+2)(x+5)$. 3. $(x-3)(x-4)$.
4. $(x+1)(x+6)$. 5. $(x-1)(x-7)$. 6. $(a-1)(a-4)$.
7. $(a+2)(a+7)$. 8. $(y-4)(y-5)$. 9. $(k-4)(k-6)$.
10. $(x-3)(x-7)$. 11. $(x-1)(x-9)$. 12. $(x-5)^2$.
13. $(n+8)(n+12)$. 14. $(a+3x)(a+4x)$. 15. $(x-y)(x-7y)$.
16. $(x-10y)(x-12y)$. 17. $(a+11b)(a+13b)$. 18. $(xy+3)(xy+6)$.
19. $(cd-4)(cd-10)$. 20. $(pq-1)(pq-17)$. 21. $(3+x)(7+x)$.
22. $(1-x)(17-x)$. 23. $(8-n)(12-n)$. 24. $(7-a)^2$.

25. $(a+5)(a+13)$. **26.** $(x^2+5)(x^2+7)$. **27.** $(a^2+1)(a^2+11)$.
28. $(a^2+8)(a^2+17)$. **29.** $(c^3-5)^2$. **30.** $(x+\frac{1}{2})^2$.
31. $(p-\frac{1}{2})^2$. **32.** $(y-\frac{1}{3})^2$. **33.** $(y+\frac{1}{2})(y+\frac{1}{4})$.
34. $(x+\frac{1}{3})(x+\frac{1}{4})$. **35.** $(x+a)(x+b)$. **36.** $(x-l)(x-m)$.
37. $(y-2a)(y-b)$. **38.** $(y+2a)(y+3b)$. **39.** $(x-p)(x-5q)$.
40. $(x+m^2)(x+n^2)$.

EXAMPLES LV. Page 212

1. $(x-3)(x+4)$. **2.** $(x+1)(x-2)$. **3.** $(x+1)(x-3)$.
4. $(x-2)(x+3)$. **5.** $(x+2)(x-3)$. **6.** $(y-7)(y+9)$.
7. $(y+8)(y-9)$. **8.** $(a-7)(a+8)$. **9.** $(a+7)(a-11)$.
10. $(x-2)(x+12)$. **11.** $(b+1)(b-21)$. **12.** $(c+6)(c-13)$.
13. $(x-3a)(x+5a)$. **14.** $(x+4a)(x-7a)$. **15.** $(x+5y)(x-6y)$.
16. $(l-10m)(l+11m)$. **17.** $(xy-1)(xy+5)$. **18.** $(cd+7)(cd-10)$.
19. $(ab+4)(ab-12)$. **20.** $(ax-5)(ax+7)$. **21.** $(ab+4c)(ab-9c)$.
22. $(x+7yz)(x-9yz)$. **23.** $(x+2pq^2)(x-5pq^2)$. **24.** $(x-\frac{1}{2})(x+\frac{1}{3})$.
25. $(a+\frac{1}{2})(a-\frac{1}{4})$. **26.** $(n-\frac{1}{2})(n+\frac{1}{5})$. **27.** $(x+a)(x-b)$.
28. $(x-l)(x+m)$. **29.** $(y-a)(y+2b)$. **30.** $(y-3a)(y+2b)$.
31. $(x-2p)(x+5q)$. **32.** $(x^2+1)(x^2-3)$. **33.** $(x^2+1)(x^2-5)$.
34. $(a^2+2)(a^2-3)$. **35.** $(y^2+1)(y^2-2)$. **36.** $(x^2-2)(x^2+5)$.
37. $(x^2-2y^2)(x^2+4y^2)$. **38.** $(l^2-7m^2)(l^2+9m^2)$.
39. $(p^2+6q^2)(p^2-11q^2)$. **40.** $(a^3-2)(a^3+8)$.

EXAMPLES LVI. Page 214

21. $(x^2+y^2)(x+y)(x-y)$. **22.** $(a^2+1)(a+1)(a-1)$.
23. $(1+a^2)(1+a)(1-a)$. **24.** $(l^2+2m^2)(l^2-2m^2)$.
25. $(p^2+4q^2)(p+2q)(p-2q)$. **26.** $(4a^2+b^2)(2a+b)(2a-b)$.
27. $(5a^2+3b^2)(5a^2-3b^2)$. **28.** $(a^2+10x^2)(a^2-10x^2)$.
29. $(9a+b^2)(9a-b^2)$. **30.** $(8a^3+b^2)(8a^3-b^2)$.
31. $(100x^2+1)(10x+1)(10x-1)$. **32.** $(25+y^2)(5+y)(5-y)$.
33. $(9x^2+16y^2)(3x+4y)(3x-4y)$. **34.** $(11l+pq^2)(11l-pq^2)$.
35. $(p^2q^2+2rs)(p^2q^2-2rs)$. **36.** $(y^3+2x^5)(y^3-2x^5)$.
37. $(a^2b^5+c^4)(a^2b^5-c^4)$. **38.** $(x^4+y^6)(x^2+y^3)(x^2-y^3)$.
39. $(20+ab^2)(20-ab^2)$. **40.** $\left(\frac{x}{y}+1\right)\left(\frac{x}{y}-1\right)$.
41. $\left(x+\frac{a}{b}\right)\left(x-\frac{a}{b}\right)$. **42.** $\left(\frac{2x}{3}+5\right)\left(\frac{2x}{3}-5\right)$

43. $\left(1+\frac{5x}{4}\right)\left(1-\frac{5x}{4}\right)$.
44. $\left(\frac{4a}{5}+3\right)\left(\frac{4a}{5}-3\right)$.
45. $\left(\frac{x}{y}+\frac{a}{b}\right)\left(\frac{x}{y}-\frac{a}{b}\right)$.
46. $\left(\frac{2x}{3y}+\frac{5a}{4b}\right)\left(\frac{2x}{3y}-\frac{5a}{4b}\right)$.
47. $\left(\frac{2x}{5a}+\frac{7y}{4b}\right)\left(\frac{2x}{5a}-\frac{7y}{4b}\right)$.
48. $(a+b+c)(a+b-c)$.
49. $(a-b+2x)(a-b-2x)$.
50. $(x+y+z)(x-y-z)$.
51. $(x+y-z)(x-y+z)$.
52. $(a-b+1)(a-b-1)$.
53. $(1+l-m)(1-l+m)$.
54. $(2m+n+p)(2m-n-p)$.
55. $(x+y+a+b)(x+y-a-b)$.
56. $(x-y+a-b)(x-y-a+b)$.
57. $4ab$.
58. $-4ab$.
59. $24xy$.
60. $-7(a+b)(3a+b)$.
61. $4x$.
62. $(ab+xy)(ab-xy-2)$.
63. $(x+y+z+a+b+c)(x+y+z-a-b-c)$.
64. $4z(x+y)$.
65. $-4z(x-y)$.
66. $(a+b-c+l-m+n)(a+b-c-l+m-n)$.
67. $(2a-2b+3c-3d)(2a-2b-3c+3d)$.
68. $(7a+7b+4c+4d)(7a+7b-4c-4d)$.
69. $(5a+5b-5c+1)(5a+5b-5c-1)$.
70. $(10a+10b-10c+x-y+z)(10a+10b-10c-x+y-z)$.
71. $(17x-5y-5z)(17y+17z-5x)$.
72. $(x+\frac{5}{6})(x+\frac{1}{6})$.
73. $(x+\frac{5}{6})(x-\frac{1}{6})$.
74. $x(x-\frac{2}{3})$.
75. $(y+\frac{3}{2})(y-\frac{1}{2})$.
76. $y(y+\frac{3}{2})$.
77. $ab(ab-\frac{4}{3})$.
78. $(k+\frac{2}{5}l)(k-\frac{6}{5}l)$.
79. $\left(4p-\frac{3q}{2r}+\frac{7}{4}\right)\left(4p-\frac{3q}{2r}-\frac{7}{4}\right)$.
80. $(x+y+a)(x+y-a)$.
81. $(x-y+z)(x-y-z)$.
82. $(a-b+c)(a-b-c)$.
83. $(l+2m+n)(l+2m-n)$.
84. $(a+x+y)(a-x-y)$.
85. $(x+a-b)(x-a+b)$.
86. $(4a+x+2y)(4a-x-2y)$.
87. $(3a-b+9c)(3a-b-9c)$.
88. $(a+b+x+y)(a+b-x-y)$.
89. $(l-m+p-q)(l-m-p+q)$.
90. $(x-y+a-b)(x-y-a+b)$.
91. $(a-b+c+d)(a-b-c-d)$.
92. $(2x^2+5x-1)(2x^2-5x+1)$.

EXAMPLES LVII. Page 217

1. $(2x+1)(x+2)$.
2. $(2a+1)(a+1)$.
3. $(2a-1)(a+1)$.
4. $(2a+1)(a-1)$.
5. $(y+1)(3y+2)$.
6. $(2y-1)(y+2)$.
7. $(5y-2)(y-2)$.
8. $(a-2)(3a-1)$.
9. $(3x+2)(x-2)$.
10. $(2m+1)(m+9)$.
11. $(2m+5)(m+2)$.
12. $(3a-2)(a-3)$.
13. $(3x-4)(4x+5)$.
14. $(x-4)(4x+3)$.
15. $(2x-7)(3x-1)$.
16. $(k-2)(3k-7)$.
17. $(2k-1)(2k-3)$.
18. $(2b-3)(b-1)$.

19. $(2b-7)(b+4)$. **20.** $(3n-2)(n+3)$. **21.** $(6n+11)(n+4)$.
22. $(2c+5)(7c-3)$. **23.** $(7c+1)(7c+2)$. **24.** $(1+2y)(5+y)$.
25. $(1-2x)(3-2x)$. **26.** $(2+5x)(3+x)$. **27.** $(4-5x)(5-4x)$.
28. $(5-2x)(3-x)$. **29.** $(3-2y)(1-3y)$. **30.** $(3-4z)(4-3z)$.
31. $(8-7z)(1+z)$. **32.** $(2x+3y)(6x-5y)$. **33.** $(2x+y)(x-3y)$.
34. $(3x-2y)(2x-y)$. **35.** $(x+7y)(2x-3y)$. **36.** $(a+2b)(7a-8b)$.
37. $(5a-9b)(2a+b)$. **38.** $(ab+10)(ab-6)$. **39.** $(ab-9)(ab+8)$.
40. $(4xy-15)(8xy+9)$. **41.** $(5xy+4)(4xy-5)$. **42.** $(3-xy)(5+2xy)$.
43. $(2x^2+1)(x^2+3)$. **44.** $(3x^2+1)(x^2+2)$. **45.** $(3x^2+1)(x^2-7)$.
46. $(5x^2-1)(x^2-8)$. **47.** $(2x^2-1)(3x^2-5)$. **48.** $(3x^2-11)(7x^2-8)$.
49. $(7a-ab+7b)(a+7ab+b)$. **50.** $(a+2ab+b)(3a-ab+3b)$.
51. $(2a+3b+3c)(3a-b-c)$. **52.** $(x+y-3a-3b)(3x+3y+2a+2b)$.
53. $2(3x-2y)(x+3y)$. **54.** $(ax+b)(x+c)$.
55. $(ax-b)(x+c)$. **56.** $(lx-m)(nx+1)$.
57. $(px-1)(q-1)$. **58.** $(ax+2)(bx+3)$.
59. $(ax+b)(cx+d)$. **60.** $(lx-my)(nx-py)$.

EXAMPLES LVIII. Page 219

32. $(a+b-c)(a^2+b^2+c^2+2ab+ac+bc)$.
33. $(c-a+b)(c^2+a^2+b^2-2ab+ac-bc)$.
34. $(lm+l+m)(l^2m^2+l^2+m^2-l^2m-lm^2+2lm)$.
35. $(2x+y)(4x^2+16xy+19y^2)$. **36.** $y(12x^2-6xy+y^2)$.
37. $-2y(3x^2+y^2)$. **38.** $2(3x^2+1)$. **39.** $4q(3p^2+4q^2)$.
40. $(x^2+y^2)(x^4-x^2y^2+y^4)$. **41.** $(x^2+1)(x^4-x^2+1)$.
42. $(1+x)(1-x)((1+x+x^2)(1-x+x^2)$.
43. $(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$.
44. $(x+3)(x-3)(x^2+3x+9)(x^2-3x+9)$.

EXAMPLES LIX. Page 222

31. $(x+2)(x+4)$. **32.** $(x-3)(x-5)$. **33.** $(x+8)(x-4)$.
34. $(x-9)(x+3)$. **35.** $(x+1)(x+4)$. **36.** $(x-5)(x-8)$.
37. $(x+12)(x-5)$. **38.** $(x+5)(x-4)$. **39.** $(x-12y)(x+11y)$.
40. $(x-11y)(x+14y)$. **41.** $(2x+1)(2x+3)$. **42.** $(3x+1)(3x-2)$.
43. $(5x-2)(5x+3)$. **44.** $(4x-3y)(4x+5y)$. **45.** $(x+3)(3x-1)$.
46. $(8x-1)(x-8)$. **47.** $(2x+3y)(3x+2y)$. **48.** $(5x-2y)(7x+11y)$.
49. $(14x-13y)(9x+2y)$. **50.** $(12x-13y)(17x+11y)$.

EXAMPLES LX. Page 223

1. $(a^2+ab+b^2)(a^2-ab+b^2)$.
2. $(a^2+a+1)(a^2-a+1)$.
3. $(x^2+2x+3)(x^2-2x+3)$.
4. $(x^2+2x+4)(x^2-2x+4)$.
5. $(4x^2+2x+1)(4x^2-2x+1)$.
6. $(a^2+3a+1)(a^2-3a+1)$.
7. $(x^2+xy+2y^2)(x^2-xy+2y^2)$.
8. $(3a^2-3ab+2b^2)(3a^2+3ab+2b^2)$.
9. $(3a^2+4ab+5b^2)(3a^2-4ab+5b^2)$.
10. $(1+3y-2y^2)(1-3y-2y^2)$.
11. $(9x^2+3xy+y^2)(9x^2-3xy+y^2)$.
12. $(x^2+3xy+4y^2)(x^2-3xy+4y^2)$.
13. $(x^2+3xy-y^2)(x^2-3xy-y^2)$.
14. $(3x^2-5xy+4y^2)(3x^2+5xy+4y^2)$.
15. $(7a^2+13ax+11x^2)(7a^2-13ax+11x^2)$.
16. $(x^2+2x+2)(x^2-2x+2)$.
17. $(2x^2+2xy+y^2)(2x^2-2xy+y^2)$.
18. $(x^2+4x+8)(x^2-4x+8)$.
19. $(1+4a+8a^2)(1-4a+8a^2)$.
20. $(a^2b^2+2ab+2)(a^2b^2-2ab+2)$.
21. $(2k^2+6k+9)(2k^2-6k+9)$.
22. $(2l^2+10l+25)(2l^2-10l+25)$.
23. $(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$.
24. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.
25. $(13x^2-14xy+5y^2)(5x^2-14xy+13y^2)$.

EXAMPLES LXI. Page 225

1. $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.
2. $(x+y-z)(x^2+y^2+z^2-xy+yz+zx)$.
3. $(x-y-z)(x^2+y^2+z^2+xy-yz+zx)$.
4. $(x+y+1)(x^2+y^2-xy-x-y+1)$.
5. $(1-a+b)(1+a^2+b^2+a-b+ab)$.
6. $(a+b-2c)(a^2+b^2+4c^2-ab+2bc+2ca)$.
7. $(a-2b+3c)(a^2+4b^2+9c^2+2ab+6bc-3ca)$.
8. $(a-b-3c)(a^2+b^2+9c^2+ab+3ac-3bc)$.
9. $(2a-b+4c)(4a^2+b^2+16c^2+2ab+4bc-8ca)$.
10. $(x^2+y^2-z^2)(x^4+y^4+z^4-x^2y^2+y^2z^2+z^2x^2)$.
11. $\left(a+b-\frac{c}{2}\right)\left(a^2+b^2+\frac{c^2}{4}-ab+\frac{ac}{2}+\frac{bc}{2}\right)$.
12. $\left(a-\frac{b}{3}-c\right)\left(a^2+\frac{b^2}{9}+c^2+\frac{ab}{3}-\frac{bc}{3}+ac\right)$.
13. $\left(a+3b-\frac{c}{3}\right)\left(a^2+9b^2+\frac{c^2}{9}-3ab+bc+\frac{ac}{3}\right)$.

14. $\left(a-\frac{b}{2}+\frac{c}{5}\right)\left(a^2+\frac{b^2}{4}+\frac{c^2}{25}+\frac{ab}{2}+\frac{bc}{10}-\frac{ac}{5}\right)$.

15. $2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

16. $(3x-a-b-c)(a^2+b^2+c^2-ab-bc-ca)$.

17. $2(c-b)(3a^2+b^2+c^2-3ab+bc-3ca)$.

EXAMPLES LXII. Page 227

1. $-(a-b)(b-c)(c-a)$.
2. $-(a-b)(b-c)(c-a)$.
3. $(x-y)(y-z)(z-x)$.
4. $(a+b)(b+c)(c+a)$.
5. $-(x-y)(y-z)(z-x)(x+y+z)$.
6. $-(a-b)(b-c)(c-a)(ab+bc+ca)$.
7. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$.
8. $-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)$.
9. $-(a-b)(b-c)(c-a)(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+abc)$.
10. $-(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)$.

EXAMPLES LXIII. Page 229

1. $a(x-a)(x^2+xa+a^2)$.
2. $3ax(x-a)(x^2+xa+a^2)$.
3. $x^2(x+y)(x^2-xy+y^2)$.
4. $x\left(x+\frac{1}{3y}\right)\left(x^2-\frac{x}{3y}+\frac{1}{9y^2}\right)$.
5. $-2b(a+b)$.
6. $(b-1)(a-1)$.
7. $(x+y-3z)(x+y-3z-1)$.
8. $(a-b)(a+b-1)$.
9. $(a-3)(a+3)(a-4)(a+4)$.
10. $(x-1)(x+1)(x-3)(x+3)$.
11. $8ax(x^2+a^2)$.
12. $2(x+1)(x+3)(x+4)$.
13. $(x+y)(1+2x+2y)(1-2x-2y+4x^2+8xy+4y^2)$.
14. $9(x+1)(x^2+x+1)$.
15. $(x+y)(xy-yz+zx)$.
16. $(a-b)(a-b-1)$.
17. $(ax+1)(bx+1)$.
18. $(x-1)(x+1)(y-1)(y+1)$.
19. $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$.
20. $(a-b)(a+b)(a^2+b^2)(a^2-ab+b^2)(a^2+ab+b^2)(a^4-a^2b^2+b^4)$.
21. $(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$.
22. $(a-3b)(a+3b)(3a-b)(3a+b)$.
23. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)(x^8-x^4+1)$.
24. $(a^2+b^2)(a^4+b^4-a^2b^2-1)$.
25. $(x-1)(x-2)(x+3)(x+4)$.
26. $(x+2)(x-3)(x+4)(x-5)$.
27. $(x+y-7a)(x+y-5a)$.
28. $(5a+b+c)(4a+b+c)$.
29. $(2x+3a+3b)(3x+2a+2b)$.
30. $(5a-4b)(2b-7a)$.
31. $(x+y-a)(x+y-b)$.
32. $(x^2-xy+y^2)(x^2+y^2+xy+x+y)$.
33. $(x^2+y^2)(x^2+2xy-y^2)$.
34. $2(a-y)(a+b+x+y)$.

35. $(x^2-2y^2+xyz)(x^2-2y^2-xyz)$. **36.** $(1+x+y-z)(1+x-y+z)$.
37. $(a+b-c+d)(a-b-c-d)$. **38.** $(ax-by)(bx-ay)$.
39. $(ax+ay+bx-by)(ax-ay-bx-by)$. **40.** $(a+b)(b+c)(c+a)$.
41. $(ac+bd)(ad+bc)$. **42.** $(x+1)(x+6)(x^2+7x+16)$.
43. $(x^2-5x+7)(x^2-5x-3)$. **44.** $(2x^2-3x+5)(2x^2-3x-7)$.

EXAMPLES LXIV. Page 233

1. 9. **2.** 21, 1. **3.** 10. **4.** -9, -15, $-7\frac{1}{2}$, -9, -10.
5. 0, 63. **6.** p^4-p^3-p+1, p^4+p^3+p+1. **7.** 0, $-8a^3$.
8. 0. **9.** 0. **10.** 0. **11.** 0. **19.** 8. **20.** 3.
21. -5. **22.** $-\frac{1}{2}$. **23.** 8. **25.** 5, 6. **26.** 6, -3.

EXAMPLES LXV. Page 236

1. $(x-1)(x-2)(x-3)$. **2.** $(x-1)(x-2)(x+3)$.
3. $(x+1)(x+2)(x+3)$. **4.** $(x+1)(x+2)(x-3)$.
5. $(a-1)(a-2)(a-4)$. **6.** $(a+1)(a+3)(a+4)$.
7. $(a-1)(a-2)(a+4)$. **8.** $(a+1)(a+3)(a-4)$.
9. $(x+2)(x+3)(x-4)$. **10.** $(x+2)(x+3)(x-5)$.
11. $(x-3)(x+5)(x-6)$. **12.** $(x-5)(x+5)(x+6)$.
13. $(x-5)(x+5)^2$. **14.** $(x+1)(x-4)(x+4)$.
15. $(x-1)(x+1)(x-2)(x+2)$. **16.** $(x+1)^2(x+2)(x+3)$.
17. $(x-2)(x-4)(2x-1)$. **18.** $(2x-1)(2x+1)(3x-1)$.
19. $(2a-1)(2a+1)(5a+1)$. **20.** $(a+3)(a+4)(2a-1)(2a-3)$.
21. $(a+b)^2(a+2b)$. **22.** $(a-2b)(5a^2+7ab+14b^2)$.
23. $(x+y)(x-y)(x+3y)$. **24.** $(x+y)(x+2y)(x+5y)$.

EXAMPLES LXVI. Page 239

1. $-(a-b)(b-c)(c-a)$. **2.** $(a-b)(b-c)(c-a)$.
3. $-(a-b)(b-c)(c-a)(a+b+c)$. **4.** $(a-b)(b-c)(c-a)(a+b+c)$.
5. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$.
6. $-(a-b)(b-c)(c-a)(ab+bc+ca)$.
7. $5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$.
8. $-(a-b)(b-c)(c-a)(a^3+b^3+c^3+a^2b+a^2c+b^2a+b^2c+c^2a+c^2b+abc)$.

EXAMPLES LXVIII. Page 243

1. $x-2$. **2.** $a+b$. **3.** $x+3y$. **4.** $x+y$.
5. $x+y$. **6.** $a-b$. **7.** $a^2(a+b)$. **8.** $xy(x+y)$.
9. $x^2(x-y)^3$. **10.** $(a+b)^2$. **11.** $x-a$. **12.** $x+y$.

13. $a-2b$. 14. $a+1$. 15. $a+b$. 16. $a(a^2+b^2)$.
17. $a^2b^2(b+2a)$. 18. $6(a^2+b^2)$. 19. $a(b-a)$.
20. $ax(a-x)^2$. 21. $y(y^2+y+1)$. 22. $y-5$.
23. $y+1$. 24. $x+4$. 25. x^2-xy+y^2.
26. $2x-1$. 27. $a^2x^2(a-2x)$. 28. $a+2$.
29. $a+9$. 30. x^2-xy+y^2. 31. $a(a+2b)$.
32. $x(x+3)$. 33. $x-1$. 34. $x-1$.
35. x^2-y^2. 36. $x-2$. 37. $x-1$.
38. $a-1$. 39. $ax(a-x)$. 40. $a+b+c$.
41. $(x-1)(x+3)$. 42. $x-3$. 43. x^2+2x+5.
44. $x-y$. 45. $(2x-1)(3x+1)$.

EXAMPLES LXIX. Page 248

1. $x+1$. 2. $x+2$. 3. $x+1$. 4. x^2+x+1.
5. x^2+2x+5. 6. $2x^2-4x+3$. 7. $x-1$. 8. $x-2$.
9. $x-2$. 10. x^2+x-2. 11. $x(x+2)$. 12. $x-1$.
13. $2x^2-x-3$. 14. $2x(x+2)(x+3)$. 15. $2x^2-7$.
16. $x+y$. 17. x^2+xy+y^2. 18. $x-3$. 19. $3x+1$.
20. $2x^2+5x+6$.

EXAMPLES LXXI. Page 251

1. x^2-y^2. 2. $(a+b)(a^3-b^3)$. 3. $a^2(a+b)(a-b)$.
4. $(a-b)^2(a+b)$. 5. $(a-b)(a-2b)(a-3b)$. 6. $12a^2b^2(a+b)^2(a-b)$.
7. $a^2bc(a+d)$. 8. $x^2y^2(x^4-y^4)$. 9. $(a-b)^2(a+b)^2$.
10. $(a+b)^2(a-b)$. 11. $(a+2b)^2(a-2b)^2$. 12. a^6-b^6.
13. $xy^2(x^2-y^2)$. 14. $x(x-3y)(x+3y)$. 15. $y(x^2-y^2)$.
16. $xy(x+y)(x-y)(x^2+y^2)$. 17. $60(x^3-1)$.
18. $12x^2(x-y)(x+y)$. 19. $24x^2y^3(x^2-y^2)(x^3-y^3)$.
20. $x(x+1)(x+2)(x+3)$. 21. $(x-3)(x-4)(x-6)$.
22. $(x-1)(x+2)(x-3)$. 23. $(x-5)(x+5)(x-7)$.
24. x^6-1. 25. x^4-1. 26. $(x-2)(x+2)(x^2+4)$.
27. $(x+1)(x+2)(x-3)(x-4)$. 28. $x^2(x^3+1)(x^3-1)$.
29. $(x^2-9)(x^2+3x+9)$.
30. $30a(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
31. $(2x-y)(2x+y)(4x^2+2xy+y^2)$.
32. $12(x+y)(x-y)(x^2+y^2)(x^2+xy+y^2)$.
33. $42a^3b^3c^2(a-d)(a+d)$. 34. $(x-a)(x+a)(x+b)$.
35. $(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)$.

EXAMPLES LXXII. Page 253

1. $x^4+4x^3-x^2-16x-12$.
2. $x^5+5x^4-6x^3+8x^2+40x-48$.
3. $2x^4-3x^3+2x^2+3x-4$.
4. $4x^4+6x^3-16x^2+9x+9$.
5. $x(9x^4-36x^3-x^2+2x+8)$.
6. $x^4+x^3-36x^2+37x-35$.
7. $x^4+2x^3-4x^2-2x+3$.
8. $x^5-6x^4-28x^3+56x^2+288x+256$.

EXAMPLES LXXIII. Page 255

17. $\frac{a}{a+b}$.
18. $\frac{b}{a-b}$.
19. $\frac{y^2}{1+y^2}$.
20. $\frac{1}{x-2y}$.
21. $\frac{a-b}{a+b}$.
22. $\frac{a}{a+b}$.
23. $\frac{x+1}{x-1}$.
24. $\frac{3a}{5b}$.
25. $\frac{1-b}{1+b}$.
26. $\frac{a}{a-3}$.
27. $\frac{a^2}{a^2-1}$.
28. $\frac{5}{x+6}$.
29. $\frac{x^2-xy+y^2}{x+y}$.
30. $\frac{x^2+y^2}{x^2}$.
31. $-\frac{1}{x+3}$.
32. $-\frac{1}{x+5}$.
33. $\frac{1}{7-x}$.
34. $-\frac{x^2y^2}{x^2y^2+z^2}$.
35. $-\frac{x}{x+4a}$.
36. $\frac{a^2+1}{a^4+a^2+1}$.
37. $\frac{x+3}{x+1}$.
38. $\frac{x-1}{x+7}$.
39. $\frac{x+3}{x+1}$.
40. $\frac{x+3}{x+4}$.
41. $\frac{x-1}{x+1}$.
42. $\frac{x+2y}{x-2y}$.
43. $\frac{1-2x}{1-5x}$.
44. $\frac{2x+1}{7x-1}$.
45. x^2-xy+y^2.
46. $\frac{x^2+y^2}{x^2+xy+y^2}$.
47. $\frac{x^3+y^3}{x^2+y^2}$.
48. $\frac{x+b}{x-b}$.
49. $\frac{a+b-c-d}{a-b-c+d}$.
50. $\frac{a+b-2c}{a-b-2c}$.

EXAMPLES LXXIV. Page 257

1. $\frac{x+5}{3x+2}$.
2. $\frac{x+2}{x(2x+1)}$.
3. $\frac{x+2}{2x+1}$.
4. $\frac{x-2}{x^2+1}$.
5. $\frac{x^2+2x-3}{x^2-7x+22}$.
6. $\frac{x^2+3}{x^2+x+4}$.

EXAMPLES LXXV. Page 258

1. $\frac{bx}{3a}$. 2. $\frac{c}{a}$. 3. $\frac{3ac}{2}$. 4. $\frac{ab}{c^2}$. 5. $\frac{by}{ax}$.

6. 1. 7. $\frac{2a}{3b^2}$. 8. $\frac{ad}{bc}$. 9. $\frac{a}{d}$. 10. 1.

11. 1. 12. $\frac{5b}{7y}$. 13. $\frac{18x}{z}$. 14. $\frac{y}{3}$. 15. $-\frac{c^2x^2}{a^2z^2}$.

16. $-\frac{c^2x^2}{3a^2z^2}$. 17. $\frac{2m^2n^4}{3pqr}$. 18. $\frac{4yz^5}{3x^3}$. 19. $\frac{l^2n}{6m^2}$.

20. $\frac{1}{x^2}$. 21. $\frac{y}{x^3}$. 22. $\frac{2x+1}{2y-1}$.

23. $\frac{(x+2)(x+3)}{(x+5)(x-3)}$. 24. $\frac{x+1}{x+4}$. 25. 1.

26. $\frac{x^2+xy+y^2}{x-2y}$. 27. $y(x+y)$. 28. $\frac{(x+2)^2}{x^2}$.

29. $\frac{a^2+ab+b^2}{a^2-ab+b^2}$. 30. 1. 31. $\frac{1}{x-3}$. 32. $\frac{1}{x-2}$.

33. $\frac{x-6}{x+3}$. 34. 1. 35. $\left(\frac{b+c-a}{c+a-b}\right)^2$.

36. $\frac{1}{x^2-1}$. 37. $\frac{a-5}{a-4}$. 38. x^2-x+1. 39. $\frac{x-1}{x+3}$.

40. $2(x^2+2x+4)$. 41. $\frac{(a+b)^2}{a^2+b^2}$. 42. $\frac{a^2+b^2}{(a+b)^2}$.

43. $\frac{a+b-c}{b-a-c}$. 44. $\frac{(x+y-z)(y+z-x)(z+x-y)}{(x+y+z)^3}$.

EXAMPLES LXXVI. Page 261

1. $\frac{13x}{12}$. 2. $\frac{3x^2}{8}$. 3. $\frac{11}{6x}$.

4. $\frac{yz+zx+xy}{xyz}$. 5. $\frac{5a}{6b}$. 6. $\frac{ayz+bzx+cxy}{xyz}$.

7. $\frac{ayz-bzx-cxy}{xyz}$. 8. $\frac{13x+23}{12}$. 9. $\frac{31a-53}{30}$.

10. $\frac{a^2+b^2+c^2}{abc}$. 11. $\frac{x^2y^2+y^2z^2+z^2x^2}{xyz}$. 12. $\frac{bx+ay+z}{a^2b^2}$.

13. $\frac{11x-6}{36}$. 14. $\frac{ac-ab}{bc}$. 15. $\frac{2xz-yz-xy+2x^2}{xyz}$.

16. $\frac{2x-1}{2x}$. 17. $\frac{a+b}{b}$. 18. $\frac{b-a}{b}$.

19. $\dfrac{ab+a}{b}$. 20. $\dfrac{ab-a}{b}$. 21. $\dfrac{a^2-b^2}{a}$.

22. $\dfrac{a^2b+a^3+b^3}{ab}$. 23. 0. 24. 0.

25. $\dfrac{5a^3+8a^2b-5ab^2+4b^3}{4a^2b^2}$.

EXAMPLES LXXVII. Page 263

1. $\dfrac{2}{x-1}$. 2. $\dfrac{2x}{x+y}$. 3. $\dfrac{-2y}{x+y}$. 4. $\dfrac{x(2x+3)}{(x+1)(x+2)}$.

5. $\dfrac{(x-y)z}{(x+z)(y+z)}$. 6. $\dfrac{2ab}{a^2-b^2}$. 7. $\dfrac{2}{a^2-b^2}$.

8. $\dfrac{5}{6(b-a)}$. 9. $\dfrac{4xy}{x^2-y^2}$. 10. $\dfrac{20x}{x^2-25}$.

11. $\dfrac{12ab}{a^2-9b^2}$. 12. $\dfrac{-2ab}{a^2-b^2}$. 13. $\dfrac{3}{(x-3)^2}$.

14. $\dfrac{-3y}{x^2-9y^2}$. 15. $\dfrac{x^2}{x^3-1}$. 16. $\dfrac{x(x^2+y^2)}{x^2-y^2}$.

17. $\dfrac{a^2}{(a-2x)^2}$. 18. $\dfrac{x^2-2xy-y^2}{(x+y)^2(x-y)}$. 19. $-2b$. 20. 0.

21. 0. 22. $\dfrac{7a^2+13ab+6b^2}{3a(a^2-b^2)}$. 23. $\dfrac{2x^3}{x^2-y^2}$. 24. 0.

25. $\dfrac{y^4+2xy^3+2x^3y-x^4}{x^6-y^6}$. 26. $\dfrac{3x^2-x-1}{x^2+x-6}$.

27. $\dfrac{-5x^2+3x+5}{x^2-9}$. 28. $\dfrac{x^2-y^2+x-3y}{x^2-y^2}$. 29. $\dfrac{x(x+a)}{a(x-a)}$.

30. $\dfrac{1}{x}$. 31. $\dfrac{x^2}{(a+x)^2}$. 32. $\dfrac{4x^2}{x^2-y^2}$.

33. 0. 34. $\dfrac{2}{x(x+1)}$. 35. 2. 36. 0.

37. $\dfrac{x^2+7x+2}{x^2-1}$. 38. $\dfrac{20ax}{4a^2-25x^2}$. 39. $\dfrac{2(x+7y)}{x-7y}$.

40. $\dfrac{x-23}{(x-2)(x-3)(x-7)}$. 41. $\dfrac{2(x-6)}{(x-1)(x-3)(x-5)}$.

42. $\dfrac{3}{(x-2)(x-3)}$. 43. $\dfrac{3x^2+10x+9}{(x+1)^2(x+2)(x+3)}$.

44. $\dfrac{x^2-6x+7}{(x-1)^2(x-2)(x-3)}$. 45. $\dfrac{18}{(x-1)(x+2)(x+5)}$.

46. $\dfrac{x^2-a^2+2x+1}{x^2-a^2}$.

EXAMPLES LXXVIII. Page 266

1. 0. **2.** 0. **3.** $\dfrac{2(ab+bc+ca-a^2-b^2-c^2)}{(a-b)(b-c)(c-a)}$. **4.** 1.
5. 2. **6.** 0. **7.** 0. **8.** 1. **9.** $a+b+c$. **10.** 0.
11. 0. **12.** 1. **13.** $\dfrac{1}{abc}$. **14.** $\dfrac{1}{(x-a)(x-b)(x-c)}$.
15. $\dfrac{x^2}{(x-a)(x-b)(x-c)}$.

EXAMPLES LXXIX. Page 268

1. $\dfrac{1}{ab-c}$. **2.** $\dfrac{a+1}{a-1}$. **3.** $\dfrac{a^2-b^2}{a^2+b^2}$. **4.** $\dfrac{3(4a+1)}{2(6a-1)}$.
5. $\dfrac{x^2-1}{x^2+1}$. **6.** $3+2x$. **7.** $\dfrac{x}{2}$. **8.** $b(a+b)$.
9. 1. **10.** $\dfrac{1}{18x^2-1}$. **11.** $\dfrac{x^2+1}{2x}$. **12.** $\dfrac{4ab(a^2+b^2)}{a^4+6a^2b^2+b^4}$.
13. $\dfrac{4y^2}{x^2}$. **14.** $\dfrac{x-8}{x-2}$. **15.** $\dfrac{2(x-2)}{2x-5}$.
16. $\dfrac{a}{a^3-1}$. **17.** 1.

EXAMPLES LXXX. Page 270

1. $\dfrac{c}{ac-b}$. **2.** $\dfrac{2a+1}{a+1}$. **3.** $\dfrac{a^3+2a}{a^2+1}$. **4.** $\dfrac{a^2-1}{a^2}$. **5.** $\dfrac{b^2}{a^2}$.
6. $\dfrac{(x+1)^3}{(x^3+3x^2+2x+1)}$. **7.** $\dfrac{177x-168}{95x-88}$. **8.** $\dfrac{a^4+4a^3+5a^2-1}{a^3+4a^2+6a+2}$.
9. $\dfrac{2a(a+1)}{2a+1}$. **10.** $\dfrac{2x^2}{1+x^2+x^4}$.

EXAMPLES LXXXI. Page 272

1. $\dfrac{4}{1-4x^2}$. **2.** $\dfrac{6(2-x)}{1-9x^2}$. **3.** 0. **4.** $\dfrac{2a^2}{1-a^4}$.
5. $\dfrac{12}{(81a^4-4)(81a^4-1)}$. **6.** $\dfrac{2a}{a+b}$. **7.** $\dfrac{2x^4}{x^6-1}$.
8. $\dfrac{1+2x}{1+x^2+x^4}$. **9.** $\dfrac{2(a+x)}{a^2+ax+x^2}$. **10.** $\dfrac{2x^3+2xy^2-x^2-y^2+1}{x^4-y^4}$.
11. 0. **12.** 1. **13.** $\dfrac{2(x^4+x^2-1)}{x^4-1}$. **14.** $\dfrac{4(x^2+y^2)}{xy}$.
15. $\dfrac{2(x^2+1)}{x(x^2-1)}$. **16.** 0. **17.** $\dfrac{1}{xy}$.

18. $x(x+1)$. **19.** x. **20.** a^2+b^2.

21. $\frac{1}{x+y}$. **22.** $\frac{(x-y)^2}{8x(x+y)^2}$. **23.** $\frac{(x^2+1)^2}{(x+1)^4}$. **24.** 1.

25. 1. **26.** $\frac{y-z}{1-yz}$. **27.** $\frac{4xy(x^2+y^2)}{x^4+x^2y^2+y^4}$.

28. $\frac{4xy}{x^2-y^2}$. **29.** $\frac{(x+1)^2}{2x^3-x^2+1}$. **30.** $x+y+z$. **31.** 1.

32. -1. **33.** 1. **34.** $5(xy+yz+zx-x^2-y^2-z^2)$. **35.** 4.

EXAMPLES LXXXIII. Page 281

1. 8.	**2.** 7.	**3.** $\frac{1}{3}$.	**4.** 12.	**5.** 12.
6. -5.	**7.** 8.	**8.** 7.	**9.** 2.	**10.** $-\frac{2}{7}$.
11. $-1\frac{5}{8}$.	**12.** 1.	**13.** $-1\frac{10}{11}$.	**14.** -2.	**15.** 3.
16. $-2\frac{1}{2}$.	**17.** -3.	**18.** $1\frac{13}{21}$.	**19.** 4.	**20.** $2\frac{2}{3}$.
21. 3.	**22.** $1\frac{1}{2}$.	**23.** 5.	**24.** $1\frac{1}{2}$.	**25.** 5.
26. $1\frac{1}{6}$.	**27.** $-2\frac{5}{7}$.	**28.** 5.	**29.** $-1\frac{1}{2}$.	**30.** $-2\frac{1}{2}$.
31. -5.	**32.** $-2\frac{1}{2}$.	**33.** $3\frac{1}{2}$.	**34.** -3.	**35.** 4.
36. $-2\frac{1}{2}$.	**37.** $8\frac{1}{2}$.	**38.** $-2\frac{1}{2}$.	**39.** $-5\frac{1}{2}$.	**40.** $-\frac{1}{2}$.
41. -2.				

EXAMPLES LXXXIV. Page 287

1. $x=1,\ y=2,\ z=4$.	**2.** $x=1,\ y=2,\ z=3$.
3. $x=1,\ y=0,\ z=-1$.	**4.** $x=3,\ y=4,\ z=1$.
5. $x=2,\ y=3,\ z=4$.	**6.** $x=3,\ y=\frac{1}{2},\ z=-1$.
7. $x=-1,\ y=-2,\ z=-3$.	**8.** $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}$.
9. $x=2,\ y=5,\ z=10$.	**10.** $x=1,\ y=1,\ z=1$.
11. $x=5,\ y=7,\ z=9$.	**12.** $x=12,\ y=12,\ z=12$.
13. $x=2,\ y=3,\ z=-1$.	**14.** $x=12,\ y=12,\ z=12$.
15. $x=12,\ y=24,\ z=36$.	**16.** $x=1,\ y=-8,\ z=12\frac{2}{3}$.
17. $x=1,\ y=2,\ z=3$.	**18.** $x=-2,\ y=1,\ z=3$.
19. $x=3,\ y=3,\ z=3$.	**20.** $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}$.
21. $x=\frac{1}{4},\ y=\frac{1}{3},\ z=\frac{1}{2}$.	**22.** $x=1\frac{1}{6},\ y=-3\frac{1}{2},\ z=2\frac{1}{10}$.
23. $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}$.	**24.** $x=1\frac{5}{7},\ y=2\frac{2}{5},\ z=-12$.
25. $x=1,\ y=1,\ z=-2$.	

EXAMPLES LXXXV. Page 290

1. $\frac{ab}{a+b}$. **2.** $\frac{lm}{m-l}$. **3.** 1. **4.** $\frac{b-d}{a-c}$.

5. $a-b.$ **6.** $a+b.$ **7.** $a+b.$ **8.** $\frac{6}{7}a.$

9. $abc.$ **10.** $\frac{ab-cd}{a+b-c-d}.$ **11.** $\frac{b(b-a)}{a+3b}.$ **12.** $a+b.$

13. $b.$ **14.** $\frac{c}{a+b}.$ **15.** $-a.$ **16.** $-\frac{a^2}{2b}.$

17. $a-2b.$ **18.** $\frac{ab}{a+b}.$ **19.** $\frac{a+b}{2}.$ **20.** $-\frac{a+b}{2}.$

21. $\frac{b}{a}(a+c-b).$ **22.** $-(a+b).$ **23.** $a+3b+5c.$ **24.** $-(a+b+c).$

EXAMPLES LXXXVI. Page 292

1. $x=\frac{m-b}{am-bl},\ y=\frac{l-a}{lb-am}.$ **2.** $x=\frac{bc}{a+b},\ y=\frac{ac}{a+b}.$

3. $x=b,\ y=a.$ **4.** $x=b,\ y=a.$

5. $x=l+m,\ y=0.$ **6.** $x=\frac{a}{a+b},\ y=\frac{b}{a+b}.$

7. $x=p-q,\ y=p-q.$ **8.** $x=2q-p,\ y=2p-q.$

9. $x=a,\ y=b.$ **10.** $x=\frac{ab}{a+b},\ y=\frac{ab}{a+b}.$

11. $x=a+b,\ y=a+b.$ **12.** $x=\frac{n^2-m^2}{bn-am},\ y=\frac{n^2-m^2}{an-bm}.$

13. $x=\frac{3b+a}{4},\ y=\frac{3b-a}{4}.$ **14.** $x=\frac{ab(3a-b)}{a^2-b^2},\ y=\frac{ab(a-3b)}{a^2-b^2}.$

15. $x=\frac{a}{a^2-b^2},\ y=-\frac{b}{a^2-b^2}.$ **16.** $x=\frac{12abc}{a+b},\ y=\frac{c(a-b)(7b-5a)}{a+b}.$

17. $x=1,\ y=0.$

18. $x=\frac{a+b+2c}{4},\ y=\frac{a+2b+c}{4},\ z=\frac{2a+b+c}{4}.$

19. $x=\frac{1}{2a},\ y=\frac{1}{2b},\ z=\frac{1}{2c}.$

20. $x=\frac{m+n-l}{2l},\ y=\frac{n+l-m}{2m},\ z=\frac{l+m-n}{2n}.$

21. $x=\frac{a}{2},\ y=\frac{b}{2},\ z=\frac{c}{2}.$ **22.** $x=\frac{2}{a},\ y=\frac{2}{a},\ z=\frac{2}{a}.$

23. $x=\frac{1}{a},\ y=\frac{1}{b},\ z=\frac{1}{c}.$ **24.** $x=2l,\ y=2m,\ z=2n.$

25. $x=\frac{1}{a},\ y=\frac{1}{b},\ z=\frac{1}{c}.$

26. $x=\frac{bc}{(a-b)(a-c)},\ y=\frac{ca}{(b-c)(b-a)},\ z=\frac{ab}{(c-a)(c-b)}.$

27. $x=a,\ y=b,\ z=c.$

EXAMPLES LXXXVII. Page 298

1. $x=-2,\ y=-1$.
2. $x=5,\ y=7$.
3. $x=-3,\ y=4$.
4. $x=a,\ y=b$.
5. $x=1,\ y=2,\ z=3$.
6. $x=-1,\ y=-2,\ z=-3$.
7. $x=5,\ y=15,\ z=10$.
8. $x=2,\ y=4,\ z=-6$.
9. $x=1,\ y=2,\ z=3$.
10. $x=2,\ y=6,\ z=-7$.
11. $x=\dfrac{a^2bc}{(a-b)(a-c)},\ y=\dfrac{b^2ca}{(b-c)(b-a)},\ z=\dfrac{c^2ab}{(c-a)(c-b)}$.
12. $x=\dfrac{1}{(a-b)(a-c)},\ y=\dfrac{1}{(b-a)(b-c)},\ z=\dfrac{1}{(c-a)(c-b)}$.
13. $x=b-c,\ y=c-a,\ z=a-b$.
14. $x=bc(c-b),\ y=ca(a-c),\ z=ab(b-a)$.
15. $x=\dfrac{bc(a+b+c)}{(a-b)(a-c)},\ y=\dfrac{ca(a+b+c)}{(b-a)(b-c)},\ z=\dfrac{ab(a+b+c)}{(c-a)(c-b)}$.
16. $x=b^2-c^2,\ y=c^2-a^2,\ z=a^2-b^2$.

EXAMPLES LXXXVIII Page 303

1. $\frac{2}{3}$.
2. $\frac{17}{21}$.
3. $\frac{3}{10}$.
4. $\frac{30}{41}$.
5. $\frac{2}{5}$.
6. $\frac{5}{7}$.
7. $\frac{1}{2}$.
8. 40.
9. 1 : 2.
10. 2 : 7.
11. (i) $\dfrac{a}{a+b}$. (ii) $\dfrac{b}{a+b}$. (iii) $\dfrac{ax}{a+b}$. (iv) $\dfrac{bx}{a+b}$.
12. 15 sr. ; 20 sr., 29 sr.
13. 30 sr. ; 42 sr., 50 sr.
14. Rs. 1000 ; 7 men.
15. Rs. 120 ; 20 men.
16. 24 ; $1\frac{1}{2}$ anna per orange.
17. Wheat, 9 sr. a rupee ; gram, 12 sr. a rupee.
18. Wheat, Rs. 2. 8a. a md. ; barley, Re. 1. 8a. a md.
19. Rs. 288, Rs. 144.
20. 431.
21. 359.
22. 253.
23. 435.
24. *A*, Rs. 500 ; *B*, Rs. 600 ; *C*, Rs. 700.
25. Rama, 50 ; Gopal, 70 ; Krishna, 100.
26. 5005 ; 1430.

EXAMPLES LXXXIX. Page 307

1. 50, 80.
2. 110.
3. $17\frac{1}{2}$.
4. 15.
5. 480, 600.
6. (i) $\dfrac{11}{6x}$. (ii) $\dfrac{3}{x}$.
7. 30, 20, 60.
8. 9, 18.
9. 5.
10. Mohan, 12 days ; Sohan, 6 days ; Rohan, 4 days.
11. 152.

12. (i) $\frac{a+b}{ab}$. (ii) $\frac{x(a+b)}{ab}$. (iii) $\frac{ab}{a+b}$.

13. (i) A, $\frac{1}{a}$; B, $\frac{1}{b}$; C, $\frac{1}{c}$. (ii) A, $\frac{x}{a}$; B, $\frac{x}{b}$; C, $\frac{x}{c}$.

(iii) $\frac{ab+bc+ca}{abc}$. (iv) $\frac{abc}{ab+bc+ca}$.

14. 3, $4\frac{1}{2}$. **15.** 12. **16.** 47. **17.** $2\frac{1}{2}$. **18.** $1\frac{43}{46}$.

19. (i) $\frac{1}{p}$. (ii) $\frac{p+q}{pq}$. (iii) $\frac{x(p+q)}{pq}$. (iv) $\frac{pq}{p+q}$.

20. (i) $\frac{abc}{bc+ca-ab}$. (ii) $\frac{c(ab-10a-10b)}{bc+ca-ab}$. (iii) $\frac{10c(a+b)}{ab}$.

EXAMPLES XC. Page 311

1. 600 miles ; 24 miles an hour. **2.** 720 miles.
3. $31\frac{1}{2}$ miles ; $3\frac{1}{2}$ miles an hour. **4.** 5 miles an hour ; 15 miles.
5. 8 hours. **6.** 30 miles an hour ; 120 miles.
7. 200 miles ; 25 miles an hour. **8.** $18\frac{3}{4}$ and $6\frac{1}{4}$ miles an hour.
9. 3 and 8 miles an hour. **10.** $1\frac{1}{2}$ and $\frac{3}{8}$ miles an hour.
11. (i) $(a+b)$ miles an hour. (ii) $(a-b)$ miles an hour.
(iii) $\frac{x}{a+b}$. (iv) $\frac{x}{a-b}$. **12.** 4 yd. ; 5 yd.
13. 120, 130. **14.** 100 miles.

EXAMPLES XCI. Page 314

13. $\frac{a}{4}-b$. **14.** $\frac{a}{b}+\frac{b}{a}$. **15.** $\frac{2x}{3y}-\frac{3y}{2x}$. **16.** $\frac{a^4}{b^4}-\frac{b^4}{a^4}$.

17. $a+b+1$. **18.** $a-2x+1$. **19.** $10x^2-y-z$. **20.** $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

21. $a+3b-\frac{1}{2}$. **22.** $\frac{a}{b}-3$. **23.** $2a$. **24.** $-2x+11y$.

25. $2(a^2+b^2)$. **26.** $a-b+c$. **27.** $a+b+1$. **28.** $x+y-2z$.

29. $x+2y-3z$. **30.** $x-2y+\frac{1}{2}z$. **31.** $x+\frac{1}{x}+4$. **32.** $x+\frac{1}{x}+5$.

33. $x+\frac{1}{x}+2$.

EXAMPLES XCII. Page 319

1. x^2+x+1. **2.** x^2-2x+3. **3.** $2x^2+x-1$.
4. $2x^2+3x+4$. **5.** $3x^2-5x-9$. **6.** $a^2-2ab+b^2$.
7. x^2-xy-y^2. **8.** $2a^2+2ab-b^2$. **9.** $x^2(2x^2-x+1)$.

10. $x^3(x^2+x+2)$.
11. a^3-4a^2-3a+2.
12. $a^3-4a^2b+2ab^2-3b^3$.
13. $x^2-xy-yz-zx$.
14. $\frac{x^2}{2}+2x+3$.
15. $x^3+x-\frac{1}{2}$.
16. $\frac{x^2}{2}-\frac{2x}{3}-\frac{3}{4}$.
17. $\frac{x}{y}-2+\frac{y}{2x}$.
18. $\frac{2x}{3y}+1-\frac{3y}{2x}$.
19. $\frac{3x}{4y}-\frac{1}{2}-\frac{2y}{9x}$.
20. $x^2+1+\frac{3}{x^2}$.
21. $x^2+5x+\frac{1}{x^2}$.
22. $\frac{x^2}{y^2}-1+\frac{y^2}{x^2}$.
23. $\frac{1}{3}x^3-3x^2+2x-\frac{1}{2}$.
24. $x^2-z^2+\frac{yz}{2}$.
25. $\frac{2x}{3y}-\frac{4x}{5z}-\frac{3y}{4z}$.
26. 2.
27. $3x+1$.
28. $-8xy^3$.
29. -3.
30. 49.
31. 3.

EXAMPLES XCIII. Page 322

12. $\pm a$.
13. $\pm 2b$.
14. ± 3.
15. ± 5.
16. ± 9.
17. ± 2.
18. $2\pm\sqrt{3}$.
19. $-3\pm\sqrt{11}$.
20. 3, -5.
21. 0, 14.

EXAMPLES XCIV. Page 324

30. 1, 2.
31. 2, 3.
32. 1, 5.
33. -3, 4.
34. 3, -3.
35. 1, -7.
36. 2, $\frac{1}{2}$.
37. -3, $\frac{1}{2}$.
38. $\frac{1}{2}$, $\frac{1}{3}$.
39. $-\frac{1}{2}$, $-\frac{1}{3}$.
40. $\frac{1}{3}$, $1\frac{1}{2}$.
41. $\frac{1}{6}$, $-1\frac{1}{2}$.
42. 3, -5.
43. -3, 4.
44. 5, -2.
45. 2, 6.
46. -4, 8.
47. $\frac{1}{2}$, $-\frac{1}{3}$.
48. 5, $-3\frac{1}{2}$.
49. 4, 11.
50. 0, 3.
51. 0, 21.
52. 3, $-3\frac{1}{3}$.
53. 3, 12.
54. 2, -5.
55. 1, 5.
56. 7, $1\frac{2}{3}$.
57. $1\frac{1}{3}$, $-1\frac{2}{3}$.
58. 3, $\frac{3}{11}$.
59. 4, $-2\frac{1}{4}$.
60. 6, $3\frac{1}{11}$.

EXAMPLES XCV. Page 326

1. 1, 3.
2. 5, -3.
3. 9, -5.
4. $5\pm\sqrt{46}$.
5. 2, -1.
6. 4, -1.
7. 3, -6.
8. 11, -9.
9. $\frac{1\pm\sqrt{5}}{2}$.
10. 9, -8.
11. 13, -12.
12. 8, -1.
13. 40, -5.
14. 3, $\frac{1}{3}$.
15. 2, $-5\frac{1}{3}$.
16. 1, $-\frac{1}{3}$.
17. $\frac{1}{2}$, $-\frac{2}{3}$.
18. $\frac{3}{4}$, -6.
19. $\frac{-2\pm\sqrt{19}}{3}$.
20. $\frac{5}{3}$, -3.
21. $\frac{3}{4}$, $-\frac{1}{5}$.
22. $2\frac{1}{6}$, $-\frac{2}{3}$.
23. $1\frac{1}{3}$, $5\frac{2}{3}$.
24. $2a$, $-6a$.
25. $9a$, $-4a$.
26. $\frac{1}{4}k$, $\frac{1}{4}k$.
27. k, $-\frac{2}{3}k$.
28. 13, -7.

29. 2, −12. **30.** 3, −25. **31.** $\dfrac{-8\pm\sqrt{189}}{5}$.

32. $\dfrac{1\pm\sqrt{-527}}{12}$. **33.** $\dfrac{-1\pm\sqrt{5}}{2}$. **34.** 2, −1.

35. $\frac{1}{3}$, $\frac{1}{3}$. **36.** 2, $\frac{1}{2}$. **37.** −1, 2. **38.** 1, $10\frac{2}{3}$.

39. ±5. **40.** $\dfrac{-1\pm\sqrt{-15}}{2}$. **41.** 0, −33.

42. 4, $1\frac{1}{3}$. **43.** $2\pm\sqrt{3}$. **44.** $17\pm\sqrt{193}$. **45.** 5, $-\frac{1}{6}$.

46. 4, $1\frac{1}{5}$. **47.** 4, $5\frac{1}{2}$. **48.** 2, $\frac{1}{2}$. **49.** 5, $-2\frac{1}{3}$.

50. 3, $-4\frac{2}{3}$. **51.** 13, $\frac{2}{3}$. **52.** $2\pm2\sqrt{3}$.

EXAMPLES XCVI. Page 329

1. $-a$, $-b$. **2.** a, b. **3.** $-a$, $\dfrac{1}{a}$. **4.** $\dfrac{-m\pm\sqrt{m^2-4ln}}{2l}$.

5. $\dfrac{-b\pm\sqrt{b^2-4c}}{2}$. **6.** $\dfrac{q\pm\sqrt{q^2-4pr}}{2p}$. **7.** $\dfrac{q\pm\sqrt{q^2+4pr}}{2p}$.

8. c, $-\dfrac{1}{c}$. **9.** m, $\dfrac{l^2}{m}$. **10.** b, $\dfrac{a^2}{b}$. **11.** $\pm\dfrac{a}{\sqrt{2}}$.

12. $\dfrac{1}{a}$, $-\dfrac{c}{b}$. **13.** q, $q-2p$. **14.** $-\dfrac{a}{b}$, $-\dfrac{b}{a}$.

15. $\dfrac{l+m}{l-m}$, $\dfrac{m-l}{m+l}$. **16.** 0, $\pm\sqrt{ab}$.

EXAMPLES XCVII. Page 331

1. Rational. **2.** Equal. **3.** Imaginary.

4. Real and unequal. **5.** Equal, but opposite in sign.

6. Rational. **7.** Sum, 3 ; product, 5.

8. Sum, 1 ; product, 1. **9.** Sum, −1 ; product, −6.

10. Sum, 2 ; product, $-\frac{7}{2}$. **11.** Sum, $-b$; product, c.

12. Sum, $\dfrac{m}{l}$; product, $\dfrac{n}{l}$. **13.** $x^2-5x+6=0$.

14. $x^2-x-12=0$. **15.** $x^2+5x=0$.

16. $4x^2-1=0$. **17.** $x^2-2ax+a^2-b^2=0$.

18. $9x^2+3px-2p^2=0$. **19.** 3, −2. **20.** 1, −3.

21. 12, −16. **22.** 39, −20. **23.** a, b.

24. $2a-b$, b. **25.** 6, −6.

EXAMPLES XCVIII. Page 334

1. $x=1, 2$; $y=2, 1$.
2. $x=3, 4$; $y=4, 3$.
3. $x=4, -2$; $y=2, -4$.
4. $x=5, -4$; $y=4, -5$.
5. $x=3, 7$; $y=-7, -3$.
6. $x=2, -3$; $y=8, 13$.
7. $x=1, 2\frac{1}{3}$; $y=2, 1\frac{1}{3}$.
8. $x=1, 1$; $y=3, 3$.
9. $x=1, 2$; $y=2, 1$.
10. $x=5, -3$; $y=3, -5$.
11. $x=2, -1$; $y=-1, 2$.
12. $x=1, 3, -1, -3$; $y=3, 1, -3, -1$.
13. $x=1, 2, -1, -2$; $y=2, 1, -2, -1$.
14. $x=7, 4, -4, -7$; $y=4, 7, -7, -4$.
15. $x=1, -1\frac{1}{5}$; $y=2, -1\frac{2}{3}$.
16. $x=2, 1\frac{1}{2}$; $y=1, 1\frac{1}{3}$.
17. $x=10, 5$; $y=5, 10$.
18. $x=11, -8$; $y=8, -11$.
19. $x=10, -9$; $y=9, -10$.
20. $x=\frac{1}{p}$; $y=\frac{1}{q}$.
21. $x=2, 5$; $y=5, 2$.
22. $x=2, -5$; $y=5, -2$.

EXAMPLES XCIX. Page 337

1. 8. **2.** 7, 3. **3.** 10. **4.** 12, 3. **5.** 6. **6.** 3, 4.
7. 7, 5. **8.** 11. **9.** 4, 6. **10.** 7, 9. **11.** 8, 10. **12.** 7, 4.
13. 4, 6. **14.** 9, 3. **15.** $1\frac{1}{2}$. **16.** 36. **17.** 45.
18. 3·708 inches ; 2·292 inches. **19.** (i) 4·392 inches ; (ii) 3·516 inches.
20. 10 ft., 12 ft. **21.** 40 yd. ; 30 yd. **22.** 48 sq. ft.
23. 6 inches. **24.** 22 yd. ; 7 yd. **25.** 9 ft., ; 12 ft.
26. 5 inches ; 12 inches. **27.** $\frac{432abc}{xyz}$. **28.** 14 ft. ; 12 ft. ; 9 ft.
29. 20 ft. ; 15 ft. ; 12 ft. **30.** 10 ft. ; 16 ft.
31. 2 yd. **32.** 21 ft. ; 15 ft. ; 12 ft.

EXAMPLES C. Page 340

1. 12 ; Rs. 20. **2.** 20. **3.** 15 miles an hour.
4. $\frac{200}{x}$ hr. ; $\frac{200}{x+5}$ hr. ; 20. **5.** 20 and 35 miles an hour.
6. 50 miles an hour. **7.** 5 miles an hour.
8. 60 and 40 miles an hour. **9.** 25 and 20 miles an hour.
10. 25 and 30 miles an hour. **11.** 6 ft. ; 4 ft. **12.** $\frac{5280}{x}$; 8.
13. 10 miles an hour. **14.** 10 and 12 miles an hour.

15. 12 annas. **16.** 15. **17.** 80. **18.** 25. **19.** Rs. 2.
20. 40. **21.** 18. **22.** 20 min. ; 30 min. **23.** 20.
24. 20 days, 16 days ; Re. 1. 4a., Re. 1. **25.** 50.

26. $\frac{(x-y)z}{x}$ yards per second. **27.** 32, 28. **28.** Rs. 40.

29. 3a. 4p. **30.** Rs. 20 ; Rs. 80. **31.** Rs. 60. **32.** Rs. 78. 2a.

33. $z\left(\frac{x^2}{y^2}-1\right)$. **34.** Rs. 25 ; 16.

MISCELLANEOUS EXAMPLES III. Page 342

A

1. $(x-3)(x-4)$; $(3x-2)(2x+3)$; $(x-2)(x^2+2x+4)$.

2. $6x^3, 3x^4y, 6x^2y^2$; $2x, 3x^2$. **3.** ± 30. **4.** $-\frac{x+y}{xy}$.

5. $x=y=\frac{1}{a+b}$. **6.** $\frac{15fh}{22m}$.

B

1. $(3x+2)(x-4)$; $(9a^2-12ab+8b^2)(9a^2+12ab+8b^2)$; $(3x+y)(3x-y)(x+3y)(x-3y)$. **2.** (i) $x+2y-3$.
(ii) $x^2+4y^2-4xy+10x-20y+25$. **4.** $\frac{a+4}{a+5}$.

5. $x=1, y=-2, z=3$. **6.** (i) $\frac{5280}{a}$ min. (ii) $20a$. (iii) $\frac{ab}{88}$.

C

1. $(13x-11)(3x+2)$; $(x^2+2x+3)(x^2-2x+3)$; $(x^2+2y^2)(x^2-2y^2)(x^2+2xy+2y^2)(x^2-2xy+y^2)$.

2. (i) and (iii). **3.** $\frac{x}{y}-\frac{y}{2x}-\frac{1}{2}$. **4.** $\frac{a}{b}$. **5.** $4, -1\frac{1}{2}$. **6.** $\frac{7}{8}$.

D

2. No ; Yes. **3.** $a+\frac{1}{a}-1$. **4.** $\frac{b(a-b)}{a(a+b)}$. **5.** 3. **6.** $4a-3b$.

E

3. $\frac{2x^3}{x^4-1}$. **4.** $-\frac{4}{5}$. **5.** $x=\frac{ac}{a^2-b^2}, y=\frac{bc}{b^2-a^2}$. **6.** 24.

F

1. $(a+b)(a+5b)$, $(a-b)^2(a+b)$, $(a-b)(a+6b)$; $(a+b)(a-b)^2(a+5b)(a+6b)$.

2. $x-2$. **3.** x^2+3x+1. **4.** -1. **5.** $2, -3$. **6.** 63.

G

1. $(xy+1)(xy-1)(x-y+1)$. **2.** $6(x^2-1)(x^2-4)$. **4.** 3.

5. $-1\frac{1}{2}$. **6.** $\frac{pna}{mq}$ days.

H

1. $(x-y)(a+b)(a-b)$. **2.** x^6-a^6.

3. $\frac{2x}{x^2-1}$. **4.** $-\frac{1}{2}(a+b)$.

5. $x=4,\ y=5,\ z=6$. **6.** 72.

I

2. -6. **3.** 1. **4.** 2, $4\frac{6}{17}$.

5. $x=1,\ y=\frac{1}{3},\ z=\frac{1}{2}$. **6.** $\frac{3(y+z)}{x}$.

J

1. $(x-y)^2(x+y)$. **2.** $x=b+c-a,\ y=c+a-b,\ z=a+b-c$.

3. $\frac{1}{x+y}$. **4.** 0, $\frac{1}{2}(a+b)$.

5. $x=3,\ y=2,\ z=1$. **6.** $4x-\frac{y}{6}$.

K

1. $(a^2-b^2-1-2b)(a^2-b^2-1+2b)$. **2.** $x+3$. **3.** 1.

4. 3. **5.** $x=\frac{a^2+b^2}{2ab},\ y=\frac{b^2-a^2+2ab}{2ab}$. **6.** 600.

L

1. 2, -58, -4. **2.** $x-2$; $\frac{x^3+2x^2-11x+6}{2x^2+4x-7}$.

3. 0. **4.** 8.

5. $x=2,\ y=1$. **6.** 5 miles.

M

1. $(x+1)(x-2)(x-3)$; -1, 2, 3. **2.** $(x-1)^3(x+1)$. **4.** 1.

5. $x=5,\ y=-2$. **6.** $\frac{5}{6}$.

N

1. $(a+b)(b-c)(b^2+bc+c^2)$. **2.** $2x^2+2x+3$. **3.** 4, -7.

5. 4. **6.** $\frac{ax+by+cz}{a+b+c}$ rupees.

O

1. $3(a-b)(b-c)(c-a)$.
2. H.C.F. $=x^2+x-3$; L.C.M. $=(x^2+x-3)^2(x^2-x+3)(x^2-x+3)$.
4. $\frac{x^2}{abc}$. 5. $-\frac{2ab}{c}$. 6. 4 for 3d. ; 512.

EXAMPLES CI. Page 355

1. Re. 1. 8a. ; 10 sr. ; $x=4y$.
2. 4s. 6d. ; Rs. 2.
3. 2·56 cm. ; 1·36 in.
4. 24 ; 13a. 4p. ; $2x=3y$.
5. 10 miles ; $37\frac{1}{2}$ min.
6. $83\frac{1}{3}$ yd. ; 33 sec.
7. Rs. 17. 8a.
8. Re. 1. 4a. ; 4 ft.
9. 4·27 cu. in. ; 49·2 cu. cm.
10. 4 hours.
11. (i) 25 miles. (ii) 3 hr. 24 min.
12. $1\frac{1}{2}$ hours after the start of the first man ; $5\frac{1}{4}$ miles.
13. 2 hours after the start of the first train ; 50 miles from Allahabad.
14. (i) After 10 hours ; 310 miles from Bombay.
   (ii) $\frac{1}{2}$ hour before and after their passing.
15. (i) 1 hr. 39 min. after the start of the first man ; $13\frac{1}{5}$ miles from Bareilly.
   (ii) 15 minutes before and after their meeting.
16. $3\frac{1}{2}$ hours after A's start ; 40 miles.
17. (i) 5.40 p.m. ; 180 miles from Lucknow. (ii) $\frac{1}{2}$ hr. before.
18. $6\frac{1}{4}$ miles an hour.
19. (i) $4\frac{1}{2}$ hours after P's first start ; 12 miles from A.
   (ii) $\frac{1}{2}$ hour before and after their meeting.
20. (i) 4 p.m. ; 16 miles from Moradabad. (ii) 3.30 p.m.
21. 6.38 a.m. ; $7\frac{1}{2}$ miles.
22. 10.50 a.m. ; $30\frac{1}{2}$ miles.

EXAMPLES CII. Page 361

1. 2. 2. $7\frac{1}{2}$. 3. $2\frac{1}{2}$ min. 4. 34. 5. 20.
6. (i) 28·4. (ii) 54·4 (iii) 10.
7. 15 and 27 yards behind winning post.
8. Rs. 3500 ; Rs. 500.
9. Rs. 1350 ; Rs. 450.
10. (i) Rs. 5600. (ii) 2800.
11. (i) Sixth year. (ii) Rs. 100, Rs. 125.

EXAMPLES CIII. Page 366

6. Rs. 24, Rs. 50.
7. $46\frac{1}{2}$ lb., 86 lb.
8. 27·1 and 30·5 crores.
9. Rs. 5. 6a. ; Rs. 3. 1a. 9p.
13. 5·24 millions ; 1905.
14. 11 yr. 4 mo. ; 14 yr. 11 mo.
15. 1929.

EXAMPLES CIV. Page 377

12. (i) 1, -1. (ii) 2, 3. (iii) -3, -4. (iv) 3, -5. (v) $\frac{1}{2}$, $\frac{1}{3}$.
(vi) 3, $-\frac{1}{2}$.
13. (i) -2, 3. (ii) -4, 5. (iii) $-\frac{1}{2}$, $1\frac{1}{2}$. (iv) 0, 1. (v) $-\frac{1}{3}$, $1\frac{1}{3}$.
14. (0, 0), (2, 12).
15. (3, 4), $(-3, -4)$.
16. (6, 8), (8, 6), $(-8, -6)$, $(-6, -8)$.
17. (i) (2, 3), (3, 2). (ii) (4, 0), (0, 4). (iii) (3, 8), $(-2, 3)$.
(iv) $(\frac{1}{2}, 1)$, $(8, -4)$. (v) (1, 1).
(vi) (3·09, ·64), (·64, 3·09), $(-3·09, -·64)$, $(-·64, -3·09)$.
(vii) (0, 0), (2, 4).

EXAMPLES CV. Page 382

1. 3 : 4.	**2.** 3 : 5.	**3.** 13 : 21, 4 : 7, 2 : 5.	**4.** 6 : 17.
5. 2 : 3.	**6.** 1 : 2.	**7.** $a : c$. **8.** 1.	**9.** $x+y : x-y$.
10. 4 : 9.	**11.** 64 : 125.	**12.** $x^3 : y^3$.	**13.** 3 : 5.
14. 4 : 3.	**15.** 32 : 45.	**16.** 8 : 21.	**17.** $2x : 3y$.
18. 1 : 5.	**19.** 2 : 3.	**20.** $\frac{3}{4}$.	**21.** $\frac{7}{2}$.
22. 1 : 3.	**23.** $\frac{5}{2}$.	**24.** $\frac{2}{1}$, $\frac{3}{1}$.	**25.** 4.
26. 5.	**27.** 9.	**28.** Re. 1. 1a. ; Rs. 3. 3a.	

29. Rs. 5. 6a. ; Rs. 8. 1a.
30. $\frac{20a}{a+b}$, $\frac{20b}{a+b}$.
31. $\frac{5}{1+k}$, $\frac{5k}{1+k}$.
32. $\frac{2a}{5}$, $\frac{3a}{5}$.
33. $\frac{pa}{p+q}$, $\frac{qa}{p+q}$.
34. 12 ft. ; 6 ft.
35. 30 yd., 36 yd., 42 yd.
36. 30°, 60°, 90°.
37. 5.

EXAMPLES CVI. Page 388

1. 10.	**2.** 7.	**3.** $\frac{bc}{a}$.	**4.** $6ac$.
5. a^2-b^2.	**6.** 12.	**7.** a^2.	**8.** b^2.
9. $(a-b)^2$.	**10.** 4.	**11.** $12x^2y$.	**12.** $9a^2b^2$.
13. $14(a^2-b^2)$.			

EXAMPLES CVIII. Page 391

1. 24, 32. **2.** 30, 40. **3.** 27, 48.
4. 300, 500. **5.** 50. **6.** $\frac{1}{72000}$, 3·872 sq. in.
7. *A*, Rs. 1600 ; *B*, Rs. 2400 ; *C*, Rs. 3000 ; *D*, Rs. 3500.
8. 5 : 9. **9.** 8. **10.** 2 sr., 12 sr.

EXAMPLES CIX. Page 396

23. $\frac{1}{x^2}$. **24.** $\frac{2}{x^3}$. **25.** $\frac{3}{x^{\frac{1}{2}}}$. **26.** $\frac{2}{a^{\frac{2}{5}}}$.

27. $2a$. **28.** $5b^3$. **29.** $\frac{a^2}{3}$. **30.** $\frac{x^{\frac{1}{2}}}{5}$.

31. $\frac{4x^{\frac{1}{3}}}{3}$. **32.** $a^{\frac{1}{n}}$. **33.** $\frac{3a^3y^3}{4x^2b^2}$. **34.** $\frac{a^3b^4}{x^5y^7}$.

35. $\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}}{5}$. **36.** $\sqrt{x}$. **37.** $\frac{1}{\sqrt{x}}$. **38.** $\frac{2}{\sqrt[5]{x}}$.

39. $\frac{3}{\sqrt[n]{a}}$. **40.** $\frac{1}{3\sqrt[3]{a}}$. **41.** $2\sqrt[3]{x^2}$. **42.** $\sqrt[n]{x}$.

43. $\sqrt[n]{a^2}$. **44.** $\sqrt{a^n}$. **45.** $\frac{1}{\sqrt[6]{a^5}}$. **46.** $\frac{2}{\sqrt[12]{a^7}}$.

47. $\frac{2}{3a^4}$. **48.** $\frac{2}{\sqrt{x}}$. **49.** $\sqrt[6]{a^5}$. **50.** $\frac{3}{4\sqrt{a}}$.

51. $\frac{1}{5\sqrt[3]{a^4}}$. **52.** $\frac{1}{x}$. **53.** $\sqrt[6]{a^{13}}$. **54.** $\sqrt[3]{a^n}$.

55. 2. **56.** 2187. **57.** $\sqrt[30]{5^{31}}$. **58.** $\sqrt[12]{a^{13}}$.

59. $\frac{1}{343}$. **60.** $\sqrt[60]{a^{77}}$. **61.** $\frac{1}{\sqrt[60]{a^{77}}}$. **62.** $\sqrt{2}$.

63. 3. **64.** 4. **65.** $\sqrt[12]{x^{25}}$. **66.** $x^2y^2z^2$.

67. $\frac{1}{5}$. **68.** $\frac{1}{572}$. **69.** $\sqrt[12]{x^5}$. **70.** $\sqrt[12]{x^{11}}$.

71. 1. **72.** $\sqrt{a^3b^3c^3}$. **73.** xy. **74.** 15.

75. 0. **76.** 1. **77.** -17. **78.** $724\frac{9}{14}$.

79. $24\sqrt{2}$. **80.** 2.

EXAMPLES CX. Page 400

25. a^2. **26.** a^2. **27.** a^2. **28.** $a^{\frac{3}{2}}b^{\frac{3}{4}}$.

29. $\frac{8}{a^3}$. **30.** $8a^9$. **31.** $\frac{1}{4a^6}$. **32.** $27x^{12}$.

33. $\frac{x^{12}}{27}$. **34.** $x^{\frac{1}{5}}$. **35.** $x^{\frac{3}{5}}y^{\frac{2}{5}}$. **36.** $\frac{x^{\frac{5}{3}}}{y^{\frac{9}{16}}}$.

37. $a^{\frac{7}{6}}x^{\frac{4}{3}}$. **38.** $\frac{a}{b^5}$. **39.** $\frac{a^{\frac{1}{2}}}{x^{\frac{2}{3}}}$. **40.** $a^{12}b^{12}$.

41. $\frac{b^4}{256a^4}$. **42.** $\frac{3x}{2y}$. **43.** $\frac{x^{\frac{1}{4}}}{y^{\frac{1}{4}}}$. **44.** $\frac{x}{y}$.

45. x^6y^9. **46.** $\frac{1}{y^{2n+3m}}$. **47.** $\frac{1}{2a^{\frac{1}{2}}x^{\frac{1}{2}}}$. **48.** x^{n-1}.

49. a^{m^2}. **50.** $\frac{a^{\frac{3}{4}}y}{b^{\frac{1}{2}}}$. **51.** $\frac{1}{a^{\frac{2}{3}}b^{\frac{1}{2}}}$. **52.** 1.

53. x^{6n}. **54.** x^{3mn}. **55.** $x^{\frac{1}{9}}$. **56.** $\frac{1}{x^{\frac{1}{m}}}$.

57. $(x+y)^{\frac{13}{6}}$. **58.** $x^{c-b}y^{a-c}z^{b-a}$. **59.** $\frac{1}{(a^2-b^2)^{2n}}$.

60. $\frac{a+b}{(a-b)^{\frac{1}{2}}}$. **61.** x^2. **62.** $\frac{b^2}{a^3}$.

EXAMPLES CXI. Page 403

1. $7-\sqrt{35}$. **2.** $6+\sqrt{2}$. **3.** $x-\sqrt{xy}$. **4.** 2.
5. $x-y$. **6.** $5+2\sqrt{6}$. **7.** $x+y+2\sqrt{xy}$.
8. $x+6y+5\sqrt{xy}$. **9.** $2b$. **10.** b. **11.** x^4+1+x^{-4}.
12. $21x+x^{\frac{2}{3}}+x^{\frac{1}{3}}+1$. **13.** $15a-3a^{\frac{1}{3}}-2a^{-\frac{1}{3}}+8a^{-1}$.
14. $6x^{\frac{5}{3}}-4x^{\frac{4}{3}}-20x-7x^{\frac{2}{3}}-20x^{\frac{1}{3}}+10$. **15.** $a+6a^{\frac{3}{4}}+13a^{\frac{1}{2}}+12a^{\frac{1}{4}}+4$.
16. $x^{\frac{1}{3}}-4$. **17.** $3x^{\frac{1}{3}}-5y^{\frac{1}{3}}$. **18.** $x^{\frac{1}{3}}+1$. **19.** $x^{\frac{1}{2}}-x^{\frac{1}{4}}+1$.
20. $x^{\frac{3}{5}}-2x^{\frac{2}{5}}+x^{\frac{1}{5}}+1$. **21.** $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{1}{3}}a^{\frac{1}{3}}$.

EXAMPLES CXII. Page 405

1. $7\sqrt{2}$. **2.** $5\sqrt{6}$. **3.** $12\sqrt{2}$. **4.** $4\sqrt[3]{4}$.
5. $7\sqrt[3]{3}$. **6.** $-4\sqrt[3]{7}$. **7.** $7a\sqrt{a}$. **8.** $2ab\sqrt[3]{3b}$.
9. $5ax^3\sqrt{3ax}$. **10** $2a\sqrt[3]{4a}$. **11.** $\frac{3}{2}$. **12.** $\frac{5}{2}\sqrt[4]{3}$.

13. $-3x^2y.\sqrt[3]{4y}$. **14.** xy^2. **15.** $x^2y.\sqrt[n]{y}$. **16.** $(a+b)\sqrt{a+b}$.
17. $(a-b)\sqrt{a}$. **18.** $\sqrt{50}$. **19.** $\sqrt{63}$. **20.** $\sqrt{192}$.
21. $\sqrt[3]{40}$. **22.** $\sqrt[3]{5000}$. **23.** $\sqrt{2a^3}$. **24.** $\sqrt{75x^5}$.
25. $\sqrt[3]{8x^8}$. **26.** $\sqrt{3a^3b^5}$. **27.** $\sqrt[3]{54a^6b^4}$. **28.** $\sqrt[n]{a^{n+1}}$.
29. $\sqrt[n]{a^{2n}+2}$. **30.** $\sqrt[12]{a^6}$. **31.** $\sqrt[12]{x^{18}}$. **32.** $\sqrt[12]{64x^{30}}$.
33. $\sqrt[12]{x^{16}}$. **34.** $\sqrt[12]{81x^8}$. **35.** $\sqrt[12]{625x^4y^{20}}$.
36. $\sqrt[12]{27x^6}$. **37.** $\sqrt[12]{125x^9y^{21}}$. **38.** $\sqrt[12]{8x^6y^9}$. **39.** $\sqrt[12]{4x^{10}}$.
40. $\sqrt[12]{a^6b^{15}c^{21}}$. **41.** $\sqrt[n]{x^{\frac{n}{2}}}$. **42.** $\sqrt[n]{a^{\frac{n}{3}}}$.
43. $\sqrt[n]{2^{\frac{n}{4}}x^{\frac{n}{2}}}$. **44.** $\sqrt[n]{x^{\frac{2n}{7}}y^{\frac{3n}{7}}}$. **45.** $\sqrt[n]{2^{\frac{n}{5}}x^{-\frac{3n}{5}}y^{-\frac{4n}{5}}}$.
46. $\sqrt[6]{a^3}$, $\sqrt[6]{a^4}$. **47.** $\sqrt[12]{a^4}$, $\sqrt[12]{a^6}$. **48.** $\sqrt[10]{a^5}$, $\sqrt[10]{a^4}$.
49. $\sqrt[12]{a^6}$, $\sqrt[12]{a^4}$, $\sqrt[12]{a^3}$. **50.** $\sqrt[60]{a^{40}}$, $\sqrt[60]{a^{45}}$, $\sqrt[60]{a^{48}}$.
51. $\sqrt[12]{a^6x^{18}}$, $\sqrt[12]{a^8x^4}$, $\sqrt[12]{a^9x^6}$. **52.** $\sqrt[6]{x^3y^3}$, $\sqrt[6]{x^4y^4}$, $\sqrt[6]{x^5y^7}$.

EXAMPLES CXIII. Page 409

1. $7\sqrt{2}$. **2.** $2\sqrt{2}$. **3.** $10\sqrt{a}$. **4.** $9\sqrt{3}$. **5.** $6a\sqrt{x}$.
6. $4\sqrt{2}$. **7.** $10\sqrt[3]{2}$. **8.** $\sqrt{ab}(a+b)$. **9.** $\sqrt[3]{ab}(a-b)$.
10. 0. **11.** $9\sqrt{2}$. **12.** $12\sqrt[3]{2}$. **13.** $-2x\sqrt{5x}$. **14.** $5ab\sqrt{ab}$.
15. $\frac{2}{5}\sqrt{10}$. **16.** $\frac{13}{3}\sqrt{3a}$. **17.** $(3x-y)\sqrt{z}$.
18. $\frac{1}{\sqrt{x}}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)$. **19.** 180. **20.** 1080. **21.** 4320.
22. 120. **23.** $14.\sqrt[3]{9}$. **24.** $\sqrt[6]{a^5}$. **25.** $\sqrt[12]{a^{13}}$. **26.** $\sqrt[12]{x^4y^3}$.
27. $12.\sqrt[12]{6912}$. **28.** $10.\sqrt[6]{500}$. **29.** $\sqrt[mn]{a^{m+n}}$.
30. $\sqrt[4]{(a+b)^2(a-b)}$. **31.** $\frac{5\sqrt{2}}{2}$. **32.** $\frac{5\sqrt{7}}{7}$. **33.** $2-\sqrt{3}$.
34. $\sqrt{5}+1$. **35.** $3(\sqrt{2}-1)$. **36.** $\frac{1+\sqrt{x}}{1-x}$. **37.** $\sqrt{3}+\sqrt{2}$.
38. $24+17\sqrt{2}$. **39.** $5+2\sqrt{6}$. **40.** $\frac{a+b-2\sqrt{ab}}{a-b}$.
41. $\frac{a+\sqrt{a^2-b^2}}{b^2}$. **42.** $\frac{a+\sqrt{a^2-b^2}}{b}$. **43.** $\frac{2-x+2\sqrt{1-x}}{x}$.
44. $\sqrt{x^2+y^2}-y$. **45.** $\frac{1+\sqrt{1-x^4}}{x^2}$.

EXAMPLES CXIV. Page 412

1. $\sqrt{3}+1$. 2. $\sqrt{2}+1$. 3. $\sqrt{5}-\sqrt{3}$. 4. $\sqrt{7}+\sqrt{5}$.
5. $\sqrt{3}-\sqrt{2}$. 6. $3+\sqrt{2}$. 7. $\frac{5}{\sqrt{2}}+\frac{\sqrt{7}}{\sqrt{2}}$. 8. $\sqrt[4]{2}(\sqrt{3}-1)$.
9. $\sqrt[4]{5}(\sqrt{2}-1)$. 10. $\sqrt[4]{5}(\sqrt{3}+1)$. 11. $\sqrt{x-1}-1$.
12. $\sqrt{a+x}+\sqrt{a-x}$.

EXAMPLES CXV. Page 413

1. 10. 2. 19. 3. 23. 4. 1. 5. 4.
6. 4. 7. $\frac{2}{5}$. 8. 0, 2. 9. $1\frac{1}{3}$. 10. $\frac{1}{2}$
11. $1\frac{2}{3}$. 12. $\frac{1}{1-a}$. 13. 27. 14. $2a$.
15. $2\sqrt[4]{a^2-4}$. 16. $\frac{24}{25}$. 17. $\frac{4a}{5}$. 18. $2\sqrt{a-1}$.
19. $\frac{q(\sqrt{p}+2\sqrt{q})^2}{p^2}$ 20. $\frac{369}{625}a$.

MISCELLANEOUS EXAMPLES IV. Page 414

A

1. (i) $(2x-1)(3x+4)$. (ii) $(a-b)(a-b-1)$. 3. $2x^2-3xa-5a^2$.
4. (i) $\frac{2ab}{a^2-b^2}$. (ii) 0. 5. (i) $x=b+a$, $y=b-a$. (ii) $b+c$.
6. 10 miles.

B

1. (i) $(2a+3b)(4a^2-6ab+9b^2)$.
(ii) $\{(a+b)x-(a-b)y\}\{(a-b)x+(a+b)y\}$.
2. (i) b. (ii) $x=\frac{c}{1+c}$, $y=-\frac{c}{1+c}$. 3. $\frac{a}{b}-\frac{3}{2}+\frac{b}{a}$.
4. (i) $\frac{a}{b}$. (ii) $\frac{x+6}{(x+2)(x+4)}$. 5. 17 miles an hour nearly. 6. 8.

C

1. (i) $(x-1)(2x^2+2x-1)$. (ii) $(x+d)(x-c-d)$.
2. (i) $x-2a$. (ii) $3ax(x^2-a^2)$. 4. (i) $\frac{x^3-2x^2+4x-2}{(x-1)^3}$. (ii) 1.
5. (i) $x=\frac{c(a+b)}{a^2+b^2}$, $y=\frac{c(a-b)}{a^2+b^2}$.
(ii) $a-2b$. 6. 8 and 3 miles an hour.

D

1. (i) $(x+4)(x-5)$. (ii) $(x^2+2a^2)(x^2-2a^2)(x^2+2xa+2a^2)(x^2-2xa+2a^2)$.
(iii) $(a^2+ax+x^2)(a^2-ax+x^2)(a^4-a^2x^2+x^4)$.

2. (i) $x=\dfrac{c(b-c)}{a(b-a)}$, $y=\dfrac{c(a-c)}{b(a-b)}$. (ii) $3\frac{1}{2}$. 3. $x+2$.

4. (i) $\dfrac{10x^2-24}{(2x+3)(x+1)(x-1)}$. (ii) 1. 5. 6.30 p.m.; 27 miles from P.

6. $\frac{3}{4}$d. an orange ; 400.

E

1. x^2-x+1. 3. (i) $x=3$, $y=4$. (ii) $\frac{2}{3}$. 6. 2 min. ; 2 min. 10 sec.

F

1. (i) $(9a^2+12ab+8b^2)(9a^2-12ab+8b^2)$.
(ii) $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.
(iii) $(b+c)(c+a)(a+b)$.

2. (i) $\dfrac{a+b}{2}$. (ii) ± 3. 3. $\dfrac{2a-3b+c}{2a-3c}$. 4. (i) $8x^3$. (ii) ac.

6. 10 yards ; 7 yards.

G

1. (i) $2(a+b)(2a-b)$. (ii) $(a^2+bc)(a^4-4a^2bc+7b^2c^2)$.
(iii) $(c+d+a-b)(c+d-a+b)$.

2. (i) $-a$. (ii) $x=\dfrac{a+b+2c}{4}$, $y=\dfrac{a+2b+c}{4}$, $z=\dfrac{2a+b+c}{4}$.

5. $2\frac{5}{6}$ miles from P. 6. Rs. 22 ; Rs. 24.

H

1. (i) $(x+y)(x+1)(x-1)$. (ii) $(1-xz)(1+xz+x^2y)$.
(iii) $(x-1)(x+1)(x^2-2)(x^2-3)$. 3. (i) $\dfrac{2(a-b)(b-c)}{c-a}$.
(ii) $x=1$, $y=2$, $z=3$. 4. $\dfrac{a^3}{(a^3+x^3)(a-x)}$.

5. 0. 6. $485\frac{1}{7}$, $348\frac{6}{7}$.

I

1. (i) $(x+2)(x+6)(x^2+8x+10)$. (ii) $(x^2-y^2+axy)(x^2-y^2-axy)$.

2. $a+b+c$. 3. $ax+by+cz$.

5. (i) 4. (ii) $x=\frac{1}{4}$, $y=\frac{1}{3}$, $z=\frac{1}{2}$. 6. 90.

J

1. $x-2-\frac{1}{x}$. 2. $(x-1)^2$.

3. (i) $\frac{ab-cd}{a+b-c-d}$. (ii) $x=\frac{bc'-b'c}{ab'-a'b}$, $y=\frac{a'c-ac'}{ab'-a'b}$.

4. (i) $\frac{(a-b)(a-c)}{2a-b-c}$. (ii) $\frac{x+1}{(x-1)(2x-1)}$.

6. 5 miles an hour ; 15 miles.

K

2. x. 3. (i) $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$. (ii) $\frac{a}{3}$, $3a$.

4 0. 5. (3, 4). 6. 8 and $7\frac{1}{3}$ yards per second.

L

1. $\frac{2a}{\sqrt{x+a}}$. 2. (i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

(ii) $x=\frac{b^2+c^2-a^2}{bm-an-cn}$, $y=\frac{a^2-c^2-b^2}{am-cm-bn}$. 4. 4.

5. $\frac{x^2}{y^2}-\frac{x}{y}+1-\frac{y}{x}+\frac{y^2}{x^2}$. 6. 275.

M

1. $(x-5q)$; $6(x+3p)(x-3p)(x-5q)$. 2. (i) $\frac{ab-cd}{a+b-c-d}$.

(ii) $x=a$, $y=b$, $z=c$. 3. $\frac{3x+a+b+c}{x+d}$. 5. 1.

6. 9 gallons ; 6 gallons.

N

1. (i) $2(a-d)(a+b+c+d)$. (ii) $(a+b-3c)(a+b-3c-1)$. 2. 1.

3. (i) $a-b$. (ii) $x=\frac{b}{a(b-a)}$, $y=\frac{a}{b(a-b)}$. 6. $\frac{8}{12}$.

O

1. (i) $ab(a-b)$. (ii) $(x+p)(x-q)(x-r)$. 2. 1. 3. (i) $\frac{ac+b^2}{b^2+c^2}$.

(ii) $x=\frac{1}{2}(\sqrt{a^2+2b^2}+\sqrt{a^2-2b^2})$, $y=\frac{1}{2}(\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2})$.

4. 0. 5. Rs. $\frac{100(x-y)^2}{xy}$.

6. (i) 20 days, 60 days, 30 days. (ii) 10 days.

EXAMINATION PAPERS

Board of High School & Intermediate Education, United Provinces.

High School Examination

1931

5. (a) $(5x+4)(4x^3+16x^2-3x-45)$. (b) $3a^2-3a+2$.

6. (a) ·5, −5·5. (b) 0. **7.** (a) 6. (b) 36 years, 27 years.

8. 23°. *Or*, (3, 2), $(2, \frac{4}{5})$.

1932

4. (a) $5, \frac{3}{13}$. (b) x. **5.** (i) $(x^2-4b)(x^4+4bx^2+16b^2)$.
(ii) $(x+a+1)(x-a-1)$. (iii) $(a^2+b^2)(a^2-b^2+2ab)$.

6. (a) 20. (b) 110. **7.** (0, 0), (−5, −3), (−3, 3).

1933

5. (i) $(x-y)(x+y+3)$. (ii) $-(x-y)(y-z)(z-x)(x+y+z)$.
(iii) $x^2y^2(5x-3y)(25x^2+15xy+9y^2)$. **6.** (a) 0. (b) −4.

7. (a) $x=\frac{bC-Bc}{aB-bA}$, $y=\frac{aC-Ac}{Ab-aB}$. (b) 8, 14, 55.

8. $3x+2y-6=0$; (1, 0), (6, 0), $(1, -3\frac{1}{3})$.

1934

5. (i) $(a+b)(b+c)(c+a)$. (ii) $(2x+7y)(15x-4y)$.
(iii) $(a^2+b^2)(x^2+y^2)$. **6.** (a) 0. (b) $x=3, y=2$. **7.** 1500.

8. 4 hours, 12 minutes after A's start ; $13\frac{1}{2}$ miles ; $5x-20y+6=0$.
Or, $x=3$, $y=2$, $4x+3y-12=0$.

1935

5. (i) $3(2x-3y)(3y-z)(z-2x)$. (ii) $-(a-b)(b-c)(c-a)$.
(iii) $(x-a)\left(x-\frac{1}{a}\right)$. **6.** (i) 2·621, −·254. (ii) $x=\frac{1}{2}, y=\frac{1}{3}$.

7. Tin, 126 lb. ; Lead, 144 lb. **8.** 5.

1936

5. (i) $(x^2+2x+2)(x^2-2x+2)$. (ii) $(x+a)(ax+1)$.
(iii) $-(a+b+c)(a-b)(b-c)(c-a)$.

6. (i) $\frac{1}{2}\frac{3}{2}$. (ii) $x=4, y=6, z=8$. **7.** 72.

8. $3\frac{3}{7}$ hr.

Board of High School Education, Central Provinces.

High School Certificate Examination

1931

5. (i) $-(x-b)(b-c)(c-x)$. (ii) $(x+y)(x-2y+1)$.
(iii) $(x+1)(x+2)(x-3)$.
6. 3 hours after A's start ; 12 miles. 7. (i) $x=\frac{1}{3}$, $y=\frac{1}{2}$.
(ii) 4, $1\frac{1}{2}$. 8. (b) 220 and 176 yards per minute.

1932

5. (i) $\frac{2a}{a+b}$. (ii) $\frac{1}{x^2+y^2}$. 6. $(x^2+2x+2)(x^2-2x+2)$; x^2+x+1.
7. (a) $x=2, 1\frac{1}{2}$; $y=1, 1\frac{1}{2}$. (b) $x=\pm 4$, $y=\pm 3$.
9. (a) A's income, Rs. 350 ; B's income, Rs. 490.
(b) Rs. 38. 2a. 4·8p. ; 220.

1933

5. (i) $(x-1)(x+1)(x+4)(x+6)$. (ii) $(x-1)^2(2x^2-x+2)$.
(iii) $3(x-y)(y-z)(z-x)$. 6. (i) 2. (ii) $\pm\sqrt{2}$.
(iii) $x=\frac{2}{3}$, $y=3\frac{1}{2}$. 7. (*a*) Rs. 105. (*b*) Rs. 185. 9. A, 40 ; B, 60.

1934

5. (*a*) 818 ; $(x-1)(x-3)(2x^2-3x+2)$.
(*b*) (i) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$. (ii) $(b-c)(ab+bc+ca)$.
6. (*a*) $\frac{2-x}{5-x}$; 8. (*b*) 2. 7. (i) $0, -\frac{3}{4}$. (ii) $x=6$, $y=2$.
8. A, Rs. 46 ; B, Rs. 30 ; C, Rs. 16. 9. (*b*) 17.

1935

4. $\frac{1}{x-3}$. 5. (i) $-\frac{a+b}{2}$. (ii) $x=-2$, $y=7$.
8. 19 and 35 miles an hour ; 95 and 41 miles.
9. (i) $(a+b)(b+c)(c+a)$. (ii) $(x^2+3x+1)^2$.
10. (ii) 18. 11. x^2-3x+1.

1936

1. (a) $(a-1)(a+3)(a^2+2a+2)$. (b) $\frac{4a^2x}{a^4-x^4}$.
2. (a) -4. (b) $x=1$, $y=2$.
3. 11-25 a.m. ; $8\frac{2}{3}$ miles from A.

Board of High School and Intermediate Education, Rajputana, Central India and Gwalior, Ajmer.

High School Examination

1934

6. (a) (i) $(x^2+3)(x^4-3x^2+9)$. (ii) $(x-1)(x+1)(1-y)(1+y)$. (iii) $3(x-5)(5x+1)$. **7.** (a) $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$. (b) 10. **8.** (i) $x=7, y=1$. (ii) $1\frac{2}{3}, -9$. (iii) 3. **9.** 36 miles.
10. (a) Rs. 5. 7a. ; Rs. 4. 8a. ; Rs. 3. 2a. (b) $(-1, 4)$; $(0, 2)$, $(1, 0)$.

1935

6. (a) (i) $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$. (ii) $(x-2y)(x^2+xy+y^2)$. (b) 1. **7.** (b) 3. **8.** (i) 3. (ii) $x=7, y=2$. (iii) $2, -3$. **9.** 45 miles ; 5 miles an hour.
10. (a) 4 ft. 9 in. ; 14 yr. 7 mo. (b) $(0, -4)$, $(0, \frac{8}{3})$, $(-5, -1\frac{1}{2}.)$

1936

6. (a) (i) $(a+b-4c)(a-b+4c)$. (ii) $(a+1)^2(a^2+2a-2)$. (iii) $(x-2y+3z)(x^2+4y^2+9z^2+2xy-3xz+6yz.)$

7. (a) $\dfrac{ab}{b-a}$; $x=4, y=10$. (b) $\frac{1}{2}, 5\frac{1}{3}$.

8. (a) $2x$. (b) $x-\dfrac{1}{x}-2$. **9.** (a) a^{m-1}. (b) 1100

10. Boy, 58·3 lb. ; girl, 55 lb. ; 12 yr. 7 mo., and 15 yr. 5 mo. $(3, 4)$, $(-3, 2)$, $(4, 1)$.