HIGH SCHOOL MATHEMATICS

PLANE GEOMETRY

AND RELATED SUBJECTS

THE UNIVERSITY OF CHICAGO
AND LAIDLAW BROTHERS, INC.
HIGH SCHOOL MATHEMATICS
SERIES

High School Mathematics

Plane Geometry and Related Subjects

By

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INTRODUCTION

During the school year 1903-04 the Department of Mathematics of the University High School of the University of Chicago began the experiment of correlating the kindred elements of the separate mathematical subjects into an organic body of mathematical fundamentals and essentials. In 1906 a textbook for use in first-year high-school courses was published embodying the results of the experiment up to that date. In 1909 this textbook was revised. Again in 1915 a new edition was prepared. Advanced books were later added to the first-year book, supplying a full secondary curriculum in mathematics. Further revisions were made in 1927 and 1928. The present volume and the other volumes of the new Breslich series of high-school mathematics textbooks present the latest and maturest outcomes of the experiment which began thirty-five years ago. The exercises here used are those which long experience has shown to be most available and most successful in guiding pupils to an understanding of mathematical principles.

The chief purpose of the present series is to induce the learner to cultivate, through independent, inductive reasoning, mathematical insight and ability to use mathematical concepts. The series is organized in such a way as to facilitate in the highest possible degree systematic thinking on the part of those who use the books.

Since the experiment was initiated in 1903, the plan of teaching unified mathematics has been adopted in many schools throughout the United States. A number of textbooks have appeared. The most successful of these books which followed the plan of the original series were prepared, it is interesting to note, by teachers who received their training on the faculty of the University High School under the chairmanship of Professor Breslich.

The principle of correlation, which was first followed in the books on mathematics, is being widely accepted as the guiding principle in the organization of the materials of instruction in many fields. The social sciences are breaking down the artificial barriers between history, geography, and the study of economic society. In the natural sciences the courses in general science are firmly established, and courses in the physical sciences combining physics and chemistry are being successfully organized.

The courses presented in the Breslich series of high-school texts have been used in schools of all types, public and private, so extensively and under such diverse conditions that their success in the hands of competent teachers is beyond question. The refinements which are added to the original texts in this new series are the results of carefully conducted tests in which the reactions of pupils to all sections of the course have been carefully analyzed. The new series is offered with full confidence that its use will correct many of the difficulties which attend the teaching of conventional courses in which algebra and geometry are treated as totally unrelated disciplines.

CHARLES H. JUDD

August, 1935

THE AUTHOR'S PREFACE

After completing the first book of this series the pupil approaches the second course with a considerable knowledge of geometric facts and principles. He knows some things about every chapter, or unit, of plane geometry. He should not be required to discard what he has learned and to repeat it. Such a plan would be wasteful and result in lack of interest on the part of the learner.

To solve the problem of articulating the work of the first two years has required a great deal of study and experimentation. The results were finally presented in the form in which they appear in this volume.

The following facts about the selection and organization of materials will be of interest to the teacher.

Allowance is made for the difference in the previous geometric experiences of pupils. This course provides for three classes of pupils: those who have studied first-year algebra just before entering the tenth grade, those who have studied general mathematics in the ninth grade, and those who have come from junior high schools. Chapter I is an introductory chapter. It is intended primarily for the first group. It contains the facts and principles taught in junior high school geometry. It provides the experiences necessary to acquire clear understanding of the new concepts of geometry, to appreciate the need, purpose, and meaning of logical demonstration, and to develop skill in the use of the drawing instruments of geometry. It therefore offers a natural and easy approach to the study of demonstrative geometry.

To the other two groups chapter I serves as a summary or statement of previously studied geometry. Some of it may be entirely omitted, or briefly reviewed, or retaught.

Throughout the book the combined types of material of mathematics is taught. The emphasis is on demonstrative geometry. Arithmetic operations and laws are reviewed wherever opportunity is offered or occasion warrants, as in the evaluation of formulas and in problems of calculation. The algebraic ground gained in the preceding year is held. Skills developed are kept intact. Some algebraic topics are further developed when opportunity and need arise. The solution of the quadratic equation by means of the formula, the operations with fractions, and a certain amount of factoring are all reviews or extensions of topics begun in the preceding year.

The study of plane geometry is completed. In the preceding work the pupil has recognized the advantages of the reasoning process over the process of measuring. He is now given the secret of geometrical strategy, i.e., skill in attacking, taking possession of and exploiting a geometrical difficulty, and formulating proofs. With this in view, methods of proof are discussed and emphasized, not once for all, but throughout the course.

To cultivate versatility and system, students are taught to choose between various methods of proof, always to follow some definite plan, and not to trust to the chance of stumbling upon a proof. To this end many model proofs are given. With other proofs statements or reasons that are more or less apparent to the student are omitted, in order that he may acquire the habit of independent thought and that his powers of argumentation may be strengthened. Every theorem becomes an original exercise, and the pupil does not fall into the error of committing proofs to memory.

The instructional materials are organized in pedagogical units. This arrangement is far better adapted to the ability of high-school students than is the traditional grouping into "books," since such subject matter is grouped together as is best taught and learned together. This has been found to be economical of the student's time and energy.

Individual differences among pupils are provided for. A modern text must enable teachers to adapt instruction to the varying capacities of their pupils. The basic principles of geometry listed in the requirements of the College Entrance Board and in the Report of the National Committee on Mathematical Requirements have been marked by an asterisk or a small circle, respectively. Together with a group of subsidiary principles to be selected by the teacher they form a minimum course. For the maximum course each unit contains a sufficient amount of additional material. Finally, for the rapid workers and those who develop special interest in mathematics the author has provided materials denoted as "supplementary" in the text.

The inductive approach is used. New concepts and principles are best understood if developed inductively. Hence, the meaning of new concepts are made clear by the processes of measurement, construction, and observation. The definition is always the last step in the development. Similarly many new principles, by the same methods, are first discovered by the pupil before they are stated in the textbook. Then follows the formulation of the proof and the proof itself.

Since the usefulness of a study always appeals very strongly to a beginner, this phase is emphasized throughout the course. The importance and the significance of geometrical facts in the affairs of everyday life are impressed upon the pupil. This wins his sanction of the worth of the study to himself more fully than any other sort of appeal that the teacher of geometry can make. Numerous historical notes add to the interest of the pupil.

Functional thinking is sufficiently emphasized. The subjects of geometry and algebra are concerned to a large extent with relationships. Opportunities for the development of functional thinking are offered on almost every page. The author has availed himself freely of these opportunities to give the function concept the emphasis which it deserves.

The study of trigonometry begun in the first year is continued. It is a distinct educational loss that the strong appeal that trigonometry has for high-school pupils should not be utilized earlier than is customary. Moreover, trigonometric methods here often replace algebraic and geometric methods, giving the student the opportunity to see some of the advantages of trigonometry over algebra and geometry. The following trigonometric materials are included: the application of three trigonometric functions (sine, cosine, and tangent) to the solution of the right triangle and to a number of practical problems, and the development of some of the fundamental relations between these important functions.

No topical treatment of the theory of limits is intended. Such a treatment is believed not to be suitable to the early years of the high-school course. However, the question of the existence of incommensurable lines and numbers is raised, and examples are given.

Tests are available to supplement instruction. The author has constructed a series of brief answer tests, one for each chapter, to be used in measuring the results of instruction. They cover all the essentials of the course and may be scored objectively. The teacher will find them exceedingly helpful. The tests may be purchased from the publishers of this series.

Acknowledgments. The author makes acknowledgment to Professor Charles H. Judd, Head of the Department of Education, Professor H. C. Morrison, Professor of Education, and Professor W. C. Reavis, Professor of Education, all of the University of Chicago, for their encouragement and assistance during the period of experimentation.

The portraits used as inserts in the text have been taken from the "Philosophical Portrait Series" published by the Open Court Publishing Company.

ERNST R. BRESLICH

August, 1935

THE AUTHOR'S TALK TO THE PUPIL

If you turn the pages of this book you will recognize much that is familiar to you. You will see lines, angles, polygons and circles. Your knowledge about these figures has been acquired in your play, in your daily occupations, and in your school work. There are two types of geometry. The first is most commonly found in decorations, designs, and in practical work. Most of your former study of geometry has been of this type. Artists use it to make their work pleasing to the eye and attractive. In the home the rugs, draperies, and furniture are beautified with geometric figures. The floor plans of your home, the flower beds and walks in the parks, and plans of our cities are based on geometric designs. Nature employs geometry in leaves, flowers and crystals. The bee and the spider are experts in constructing honey combs and spider webs which are geometric in form. Surveyors in laying out land, engineers in building railroads, bridges and tunnels, architects in erecting monuments and buildings are constantly using geometry. History shows that the Ancient Egyptians and Greeks developed this type of geometry to the highest perfection. This geometry is sometimes called *practical* or *aesthetic* geometry. Furthermore, since the space you live in and move in is geometric it is evident that all who wish to be classed as cultured and intelligent should include geometry as part of their education.

Many years ago the Greeks discovered that this simple geometric material could be used with excellent results in exhibiting correct methods of thinking and reasoning. Geometry thus became a subject for training people in the use of the laws of correct reasoning. This training is something nobody should miss in school work. Much of the time in life is spent in convincing others that we are good thinkers and in listening to the arguments of others to decide whether to accept or reject their conclusions. The study of geometry offers a simple method of learning to reason and to attack problems. It took many years to develop this type of geometry. It reached a high state of development in the work of Euclid about 300 B. C. It is usually called *logical* or *demonstrative* geometry.

Both types are presented in this book but the logical received the greater attention. If you put in the required effort this course should prove to be very valuable to you.

ERNST R. BRESLICH

August, 1935

CONTENTS

CHAPTER I

INTRODUCTION TO THE STUDY OF GEOMETRY

How Geometry Developed

1. How ancient mathematicians studied geometry. The word *geometry* means *earth measure* or *land measure.* Although a considerable amount of geometry was known to other peoples, the origin of the "study" of geometry is attributed to the Egyptians who, following inundations of the river Nile, found it necessary to resurvey their fields to establish ownership. Their great buildings, such as the pyramids, also required considerable knowledge of geometric facts. Out of these needs the Egyptians discovered and collected a large body of geometric principles that were of practical value.

It is said that *Thales* (640–546 B.C.) who was one of the "seven wise men" introduced geometry from Egypt into Greece. He and his students discovered many important geometrical facts and relations that were not known to the Egyptians. Best known among the students of Thales is *Pythagoras* (580–501 B.C.), with whose famous theorem you are familiar. Students of mathematics of this period established the correctness of their discoveries largely by the method of experimentation and measurement. This is the method that is used today in the beginning of the study of geometry. The method, however, has certain limitations. It is more or less laborious, and often leads to incorrect statements because measurement is not always accurate. Furthermore, conclusions drawn from examination of a few special cases may not be true in general. Finally, relations which appear to

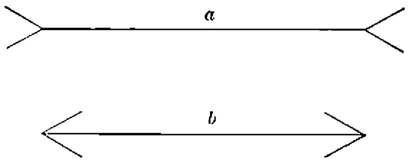


FIG. 1

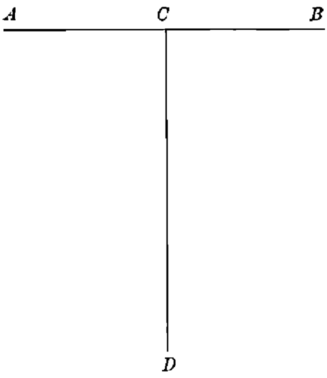


FIG. 2

the eye to be correct may be erroneous and misleading. For example, the two line segments a and b, Figure 1, and AB and CD, Figure 2, appear to be of unequal lengths, but careful measurements show that they are equal. The two lines AB and CD, Figure 3, appear to be curved, but when

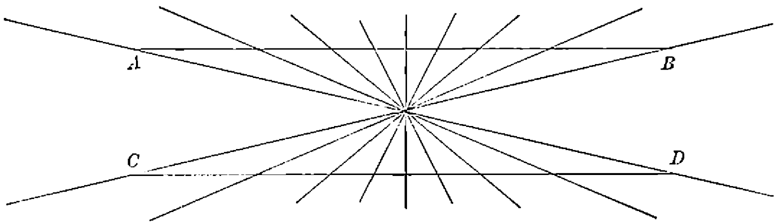


FIG. 3

they are tested with a straightedge it is seen that they are straight. As long as mathematicians failed to develop methods better than observation, their achievements had to be very limited.

Credit for introducing such methods is due to *Plato* (429–348 B.C.), one of the successors of Pythagoras, who introduced a new epoch in the development of geometry by formulating the method of *demonstrating* the truth of

PYTHAGORAS

P Y T H A G O R A S

PYTHAGORAS was born at Samos about 580 B. C. of Phoenician parents and died, probably at Metapontum, in Southern Italy, about 500 B. C. He was primarily a moral reformer and philosopher, but he was celebrated also as a mystic, a geometer, an arithmetician, and as a teacher of astronomy, mechanics, and music. His system of morals and his philosophy were founded on mathematics. He is said to have been the first to employ the word mathematics. The meaning he gave it was what we understand by general science. With him geometry meant about what high-school pupils today mean by mathematics.

He divided his mathematics into numbers absolute or arithmetic, numbers applied or music, magnitudes at rest or geometry, and magnitudes in motion or astronomy. His successors for many years regarded this as the necessary and sufficient course of study for a liberal education. It is the origin of the famous "quadrivium" that constituted higher education for 2,000 years.

After completing his studies near his home under Pherecydes of Syros and Anaximander, the latter a disciple and successor of Thales of Miletus, Pythagoras traveled extensively, studying mathematics in Egypt, Chaldea, and Asia Minor. Returning from his travels he settled at Samos and gave lectures with indifferent success until some time near 529 B. C., when he migrated to Tarentum. After a brief stay here he removed to Croton in Southern Italy, where he opened his famous school of philosophy and mathematics in 529 B. C. Here his school was attended by enthusiastic audiences.

He divided his hearers into probationers and Pythagoreans. The probationers were much the larger group. He formed the Pythagoreans into a brotherhood, like a modern fraternity. All possessions were to be held in common and all discoveries were to be referred to the founder. The chief mathematical discoveries were revealed only to the Pythagoreans. Read in some history of mathematics the story of the drowning of a Pythagorean for divulging a mathematical discovery and claiming it as his own, and other more significant facts about this secret order and its wonderful founder. The leading teachings of the brotherhood were self-command, temperance, purity, and obedience. Its secrecy and strict discipline soon gave the society such power in the state as to arouse the jealousy and hatred of certain influential classes in that democratic community. Instigated by his political opponents, a mob murdered many of Pythagoras' followers and finally, after his flight, probably to Metapontum, murdered the leader himself. After the death of their leader the Pythagoreans were dispersed over Southern Italy, Sicily, and the Grecian peninsula. They renounced secrecy, opened schools at divers centers, and they and their disciples continued publicly to teach Pythagorean doctrines for a hundred years after the death of Pythagoras.

Pythagoras' geometry consisted of the substance of what is contained in the first chapter of our school geometries about triangles, parallels, and parallelograms, together with some few isolated theorems about irrational magnitudes. His reasoning was often not rigorous, e.g., he sometimes assumed the converse to be true without a proof. His most original work was in the theory of numbers, called by the Greeks *Arithmetica*. Pythagoras left no books or other writings, so that what we know of him is traditional. He himself believed, not in publicity, but in secrecy.

geometric principles by a process of exact reasoning. Thus, a fact is not only discovered, but it is also proved by a method which does not involve measurement and which is therefore wholly independent of the degree of accuracy in the drawing of figures. One of the purposes of the study of geometry is to give the student training in accurate and careful reasoning.

A third epoch in the development of geometry began with the work of *Euclid* (about 300 B.C.). He organized all the discoveries of his predecessors into one whole and wrote a book called *Euclid's Elements*, which has formed the basis of instruction in geometry for over 2,000 years. Indeed, until very recently his work has been the leading textbook on geometry in the schools of England. Euclid aimed to establish geometric truths by a strictly logical method and attempted to arrange all the principles selected in a logical order, not using any fact unless it had been previously established by proof. Within a period of 300 years the Greeks transformed the practical geometry of the Egyptians into a logical system in which every new fact established depends upon and is proved by means of some or all of the preceding facts.

2. Modern geometry. The modern course in geometry aims to accomplish several purposes. By studying figures the student is to become familiar with many geometric principles and with a number of constructions in which compass and straightedge are used. Furthermore, the study of geometry teaches how to establish the truth of geometric statements by a process of reasoning, and trains the student to reason correctly, to think logically, and to prove new principles. It is not necessary, as Euclid aimed to do, to insist that every fact be proved before it may be used. A considerable number of statements will be admitted as true without proof. However, the student must understand clearly what the assumptions of the course are, and what are the facts established by proof.

Measurement of Line Segments

3. How to measure line segments with a ruler. To measure the length of a line segment, as AB, Figure 4,

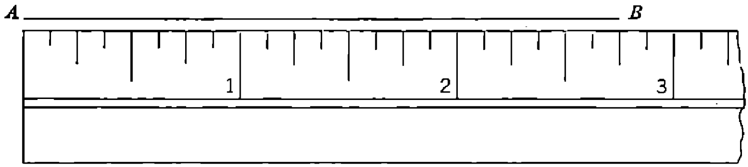


FIG. 4

with a ruler, place the marked edge of the ruler along the segment AB with the zero mark directly under A. Then take the reading below B. This is the length of AB. Thus, the length of AB is $2\frac{3}{4}$, if the *inch* is used as *unit of*

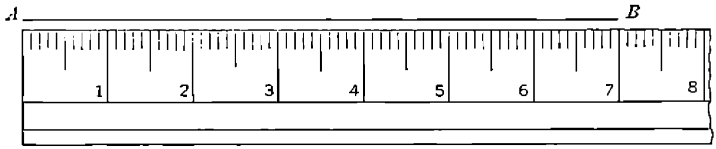


FIG. 5

length. The length of AB, Figure 5, is approximately 7, if the *centimeter* is used as unit of length.

EXERCISES

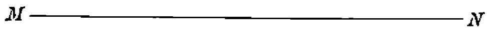


FIG. 6

1. Find the length of MN, Figure 6, in inches; in centimeters. If you cannot find the *exact* length, estimate it as well as you can.

2. A figure formed by three straight lines is a *triangle.* The symbol for triangle is $\triangle$. Measure the sides AB, BC, and CA of $\triangle ABC$, Figure 7, in inches; in centimeters.

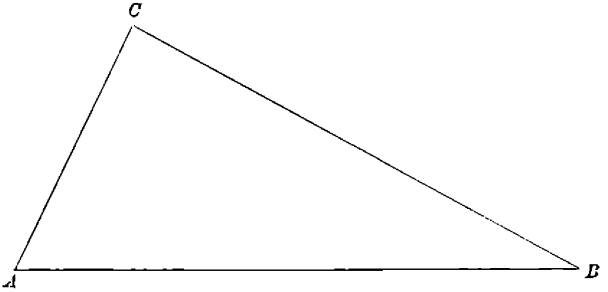


FIG. 7

3. Measure the length of the page of your textbook in inches; in centimeters.

4. With a yardstick measure the length and width of your classroom; of your desk; of the door.

4. Geometric lines. Some of the line segments measured in the foregoing exercises have length and thickness. When lines are studied in *geometry* such properties as width, color, and thickness are not considered. **Geometric lines** have only *length*. A straight *line* extends infinitely. A *line segment* is a part of a line bounded by two points.

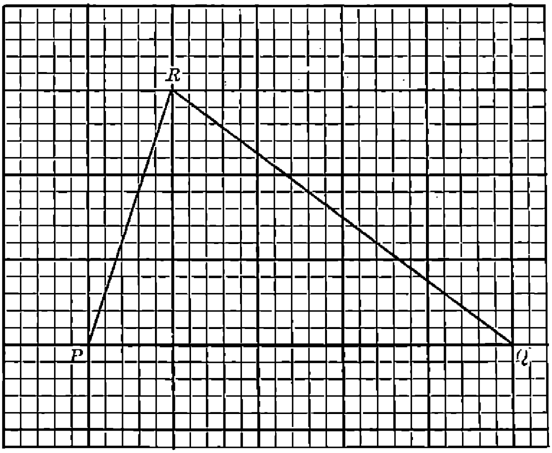


FIG. 8

5. How to measure segments with compass and squared paper. It is often convenient to use squared paper for measuring segments. Each of the sides of the large squares, Figure 8, is one centimeter long. Show that the length of *PQ* is 5, if the centimeter is used as a unit, and that it is 25 if one-fifth of a centimeter is used.

To measure *PR*, place the points of your compass at *P* and *R*. Then place the metal point on one of the corners of a large square and, with the pencil point, make a mark on one of the heavy lines passing through that corner. This enables you to read off the length of *PR* along the heavy line. Measure *RQ* similarly.

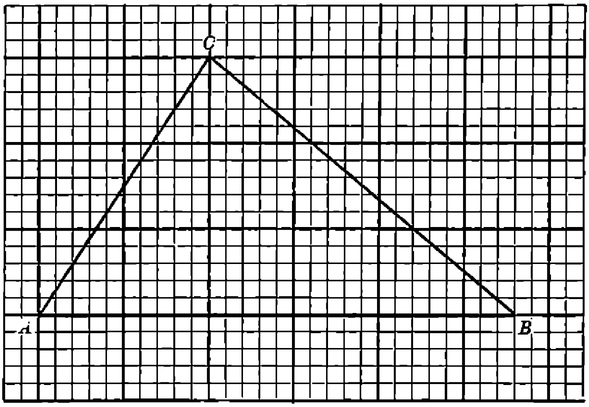


FIG. 9

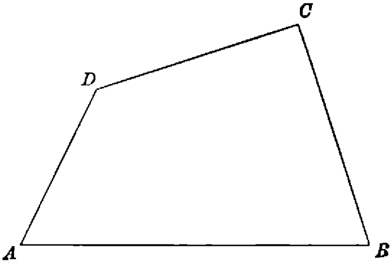


FIG. 10

EXERCISES

1. Measure the sides *AB*, *BC*, and *CA*, Figure 9, of triangle *ABC*, using as a unit an inch; a centimeter; one-fifth of a centimeter.

2. Measure the sides of *ABCD*, Figure 10, using as a unit an inch; a centimeter; one-fifth of a centimeter.

3. Draw line segments *AC* and *BD*, Figure 10, and measure each, using as a unit an inch; a centimeter; one-fifth of a centimeter.

Measurement of Angles

6. **What is meant by an angle?** A figure formed by two lines meeting in a point, Figure 11, is called an **angle.** The straight lines BA and BC are the **sides** of the angle. The point B is the **vertex.**

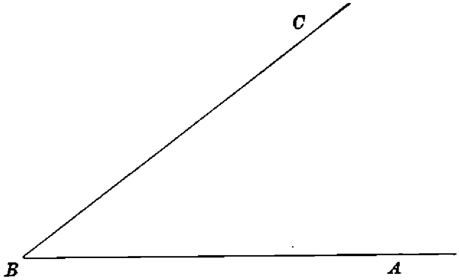


FIG. 11

EXERCISES

1. Point out several angles in the classroom.
2. Draw an angle on paper; on the blackboard.
3. Draw a triangle and point out the angles in it.

7. **Meaning of size of an angle.** An angle may be formed by keeping point B of a given line BA, Figure 12, fixed, and then turning BA to another position, such as BC. The **size** of an angle is the *amount of turning* required to carry a moving line from one side of the angle to the other. The size of the angle is not determined by the lengths of the sides. Thus angle A, Figure 13, is greater than angle B. Why?

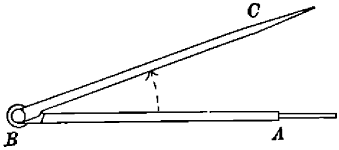


FIG. 12

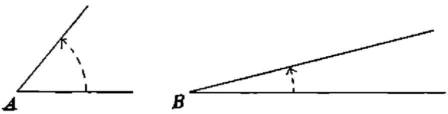


FIG. 13

8. Classification of angles. When the amount of rotation is less than a quarter of a turn, the angle is called **acute** (sharp) as angle B, Figure 14.

When the amount of rotation is equal to a quarter-turn, as angle B_1, Figure 14, the angle is called a **right** angle.

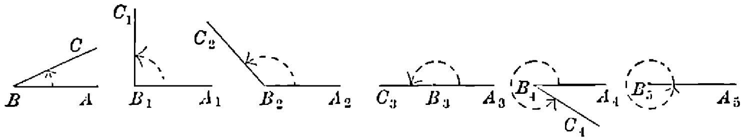


FIG. 14

When the amount of rotation is more than a quarter-turn but less than a half-turn, the angle is called **obtuse** (blunt). Thus angle B_2, Figure 14, is obtuse.

When the amount of rotation is equal to a half-turn the angle is a **straight** angle as angle B_3, Figure 14.

When the amount of rotation is greater than a half-turn but less than a complete turn, the angle is a **reflex** angle, as angle B_4.

When the amount of rotation is a complete turn as angle B_5, Figure 14, the angle is a **perigon.**

EXERCISES

1. What kind of angle is formed by a vertical pole and its shadow?

2. In the classroom point out angles that are acute, obtuse, right, reflex.

9. Notation for angles. In the preceding discussion the vertex letter was used to denote an angle. The symbol for angle is $\angle$. The symbol for angles is $\measuredangle$. Thus, "the angle whose vertex is B" is written briefly $\angle B$.

Sometimes three letters are used to denote angles. Thus, $\angle B$ is changed to $\angle ABC$. When this notation is used the *vertex* letter is always the *middle* letter. Thus in triangle ABC, Figure 15, $\angle A$ means the same as

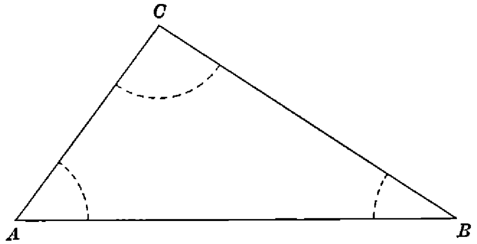


FIG. 15

$\angle BAC$; $\angle B$ means the same as $\angle CBA$; and $\angle C$ means the same as $\angle BCA$.

10. Measuring angles with a protractor. Angles are measured with an instrument called a *protractor*, Figure 16.

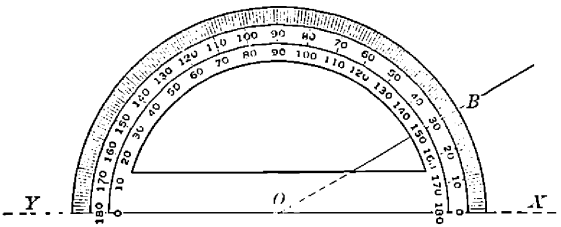


FIG. 16

Note that the curved rim of the protractor is divided into 180 equal parts. If lines are drawn from the midpoint O to the marks on the rim, 180 equal angles are formed whose vertices lie at O. Each of these is a **degree.** Thus, a degree is the 180th part of a straight angle and the 90th part of a right angle.

Draw an angle as XOB, Figure 16. To **measure** $\angle XOB$ place the center of the protractor on the vertex O, place the zero mark on one side, and take the reading on the other side. The reading is the number of degrees contained in the angle, or the *measure* of $\angle XOB$.

EXERCISES

1. What is the measure $\angle XOB$, Figure 16?

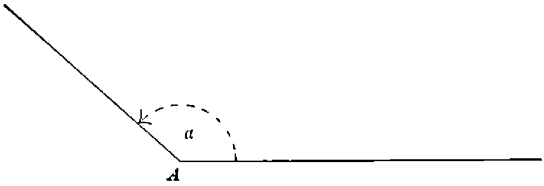


FIG. 17

2. The unknown number of degrees in $\angle A$, Figure 17, is denoted by a. Find the value of a by measuring $\angle A$.

3. Using the protractor, draw angles equal to 40°; 80°; 120°; 160°.

4. Draw an angle and divide it into two equal parts. The angle is then said to be **bisected.**

11. Opposite angles. Draw two intersecting lines, as AB and CD, Figure 18. The angles denoted by a and a' are **opposite** or **vertical** angles.

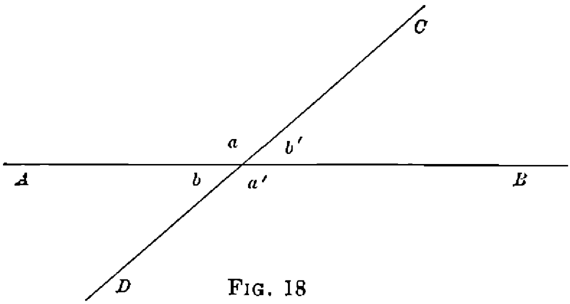


FIG. 18

Find a and a' by measuring.

How do the two opposite angles compare as to size?

The preceding shows that *if two lines intersect the opposite angles are equal.*

EXERCISES

1. Show by measurement (Figure 18), that $b=b'$.

2. Using the principle above find x, a, and a', if $a=8x+15$, and $a'=16x-9$.

3. By means of equations determine x in the following table if a and a' denote the number of degrees in two opposite angles:

a	a'
$7x+17$	$4x+47$
$\frac{1}{3}x+\frac{5}{2}x$	$\frac{3x}{2}+42$
$\frac{7x}{4}-\frac{x}{6}$	$\frac{2x}{3}+66$
r	$tx+s$
y	$mx+b$

12. Complementary angles. Measure the angles denoted by x and y, Figure 19. Find the sum of x and y.

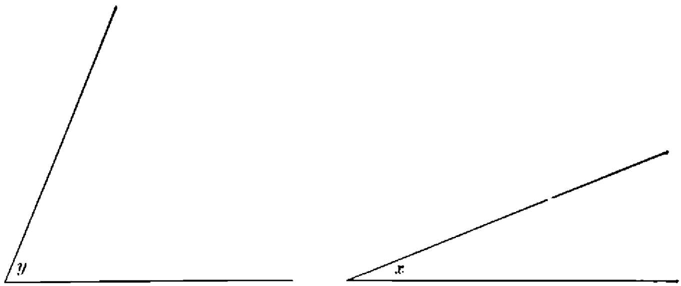


FIG. 19

If the sum of two angles is 90°, or a right angle, the angles are **complementary**. Either angle is the *complement* of the other.

EXERCISES

1. Show that the angles denoted by a and b, Figure 20, are complementary angles.

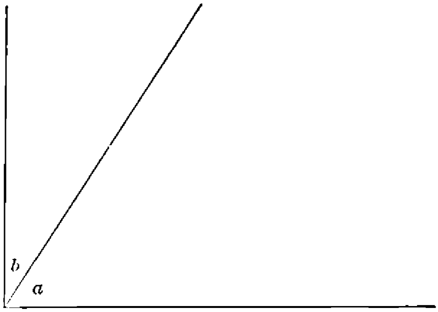


FIG. 20

2. Using the protractor, draw two complementary angles.

3. Find the complement of 30°; 45°; 20°; 18°; 54° 20′; 38° 16′; 70° 18′ 34″; 21° 54′ 3″.

4. State by means of an equation that 18° is the complement of a°.

5. If $y+40=90$, what is the complement of y? Of 40?

6. One of two complementary angles is 32° larger than the other. Find the angles by means of an equation.

7. The difference of two complementary angles is 78°. Find them.

8. Divide a right angle into two parts so that one is $\frac{2}{5}$ as large as the other.

9. Two angles, x and y, are complementary. One is 28° less than the other. Form two equations in x and y and solve them

10. Two complementary angles, x and y are to each other as 5:7. Find x and y.

11. State by means of an equation that x and y are complementary. Make a graph of the equation. From the graph find y when x is 60°; 78°; 24°.

13. Supplementary angles. Measure the angles denoted by a and b, Figure 21. Find the sum.

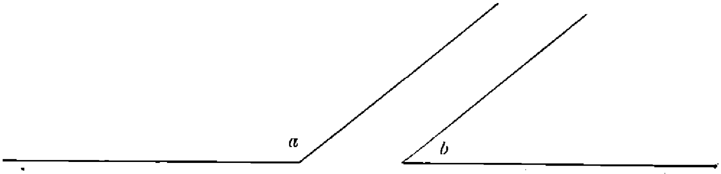


FIG. 21

Two angles whose sum is 180° are **supplementary** angles. Either is the **supplement** of the other.

EXERCISES

1. Draw two supplementary angles.

2. Find the number of degrees in the supplement of 50°; 88°; 120°; 142°; 98° 14′; 111° 19′ 23″; 61° 43′ 15″.

3. State by means of an equation that a and b are supplementary. Make a graph of the equation. From the graph find a when b is 20°; 82°; 48°.

4. One of two supplementary angles is $\frac{1}{8}$ as large as the other. Find the two angles.

5. One of two supplementary angles is 98° less than the other. Find the angles.

6. The sum of an angle and $\frac{1}{3}$ of the supplement is 90°. Find the angle.

7. $x-\frac{x}{7}$ and $\frac{3x}{4}+90$ are supplementary. Find x and each angle.

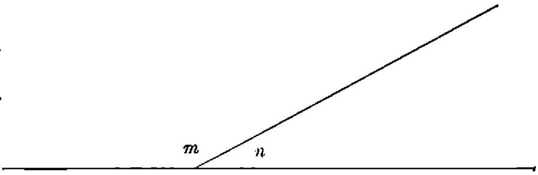


FIG. 22

8. $3x+a+60$ and $2b-6a+80$ denote supplementary angles. Express x in terms of a and b.

9. Angles m and n, Figure 22, are supplementary adjacent angles. If n is $\frac{1}{5}$ of m, find each angle.

10. Draw two intersecting lines making one of two adjacent angles 4 times as large as the other.

Suggestion: Find the angles by means of an equation.

11. Show without measuring that *the opposite angles formed by two intersecting lines are equal.*

Suggestion:

Show that $a+b=180$, Figure 23,

and that $b+a'=180$.

Hence $a+b=b+a'$.

For if each of the sums is equal to 180° they must be equal to each other.

Subtracting b from each member of the equation you have $a=a'$.

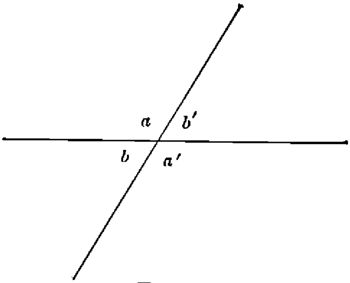


FIG. 23

14. Adjacent angles. Two angles having a common vertex and a common side between them are **adjacent angles.** Thus, $\angle a$ and $\angle b'$, Figure 23, are adjacent angles.

15. Symbol for therefore. The symbol ∴ is used to denote "hence" or "**therefore.**"

16. Perpendicular lines. If two straight lines form *equal adjacent* angles they are **perpendicular** to each other.

Thus, the statement that *CD is perpendicular to AB*, Figure 24, is expressed briefly by means of the equation $m=n$.

The symbol ⊥ also means "*is perpendicular to.*" With this symbol the statement *CD is perpendicular to AB* is written $CD \perp AB$.

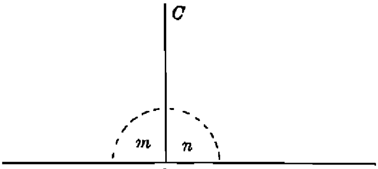


FIG. 24

EXERCISES

1. Point out perpendicular lines in the classroom.

2. Make a sketch of two lines perpendicular to each other.

The Study of Triangles

17. The sum of the angles of a triangle. Draw a triangle, as $\triangle ABC$, Figure 25.

Estimate the number of degrees in $\angle A$. Check your estimate by carefully measuring $\angle A$ with a protractor.

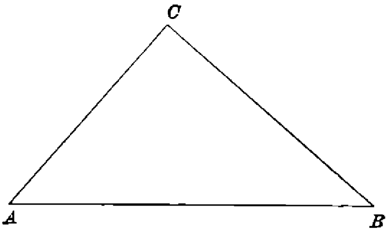


FIG. 25

Similarly, estimate the number of degrees in angles B and C. Then measure angles B and C.

Find the *sum* of the number of degrees in the angles of $\triangle ABC$.

What do you find to be the number of degrees in the sum of angles A, B, and C?

If your drawing and measuring is done carefully and correctly you will find that *the sum of the angles of a triangle is 180°.* In symbols this statement may be expressed by means of the equation $a+b+c=180$, where a, b, and c denote the number of degrees in the angles of the triangle. The equation expresses a relation between the three angles which is useful to the surveyor. For if two of the angles of a triangle are known, he can find the third by solving the equation $a+b+c=180$.

EXERCISES

1. Two angles of a triangle are 20° and $62\frac{1}{2}°$. State the equation for finding the third angle, and solve it.

2. Draw a triangle two of whose angles are 38° and 54°. Measure the third angle, and check the accuracy of your drawing by finding the third angle by means of the relation $a+b+c=180$.

3. One angle of a triangle is three times the second and the third is six times the second. Find each angle.

4. Show that the acute angles of a triangle having one right angle (right triangle) are complementary.

5. The acute angles of a right triangle are denoted by mx, and n. Find x in terms of m and n.

6. The angles of a triangle are ax, b, and $180-c$. Find x in terms of a, b, and c.

7. Draw a triangle having two equal angles. Measure the sides opposite the equal angles. If your drawing is made correctly the two sides should be equal. This fact may be used as a check of accuracy.

8. Draw a right triangle one of whose acute angles is 30°. Measure the side opposite the 90°-angle (hypotenuse), and the side opposite the 30°-angle. What percentage is the second side of the first?

9. Without measuring, show that *if two angles of one triangle are equal to two angles of another, the third angles are equal.*

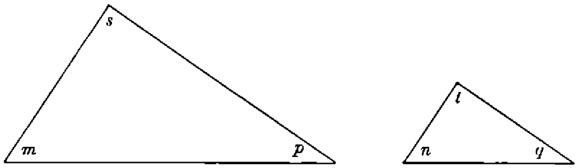


FIG. 26

Suggestion: Let $m=n$, Figure 26, and $p=q$.

Then $m+p=n+q$. For, if equal numbers are added to equal numbers, the sums are also equal.

But $m+p+s=180$,
and $n+q+t=180$. For the sum of the angles of a triangle is 180°.

$\therefore\ m+p+s=n+q+t$. By substituting $n+q+t$ in place of 180 in $m+p+s=180$.

Subtracting $m+p=n+q$,
you have $s=t$, which was to be shown.

18. To draw a triangle when the lengths of two sides and the angle included between them are given. Let the given angle be 50° and let the lengths of the given sides be 8 centimeters and 6 centimeters, respectively.

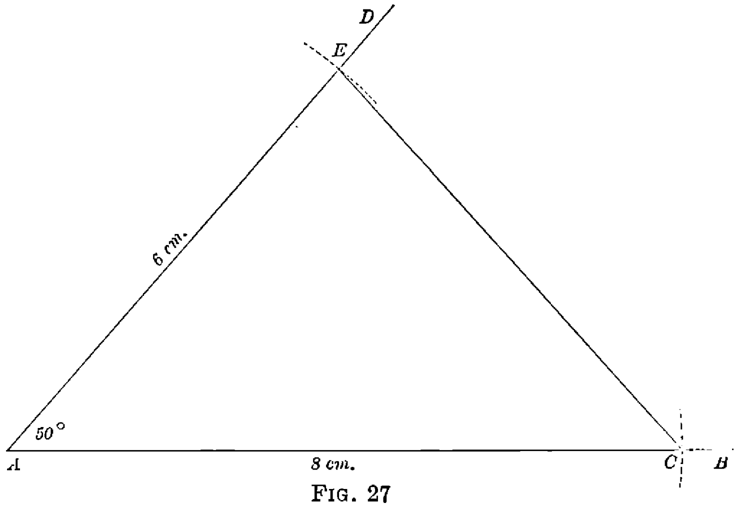


FIG. 27

Draw a line, as AB, Figure 27. Using a compass, lay off on AB a length AC, equal to 8 centimeters. With a protractor draw $\angle BAD=50°$. On AD lay off AE equal to 6 centimeters. Draw EC.

$\triangle AEC$ is the required triangle.

Cut out $\triangle AEC$, write your name on it, and pass it on to the teacher.

The following may be used as a test of accuracy of drawing:

If the triangles constructed by the pupils of the class are carefully drawn they can be made to coincide if placed together.

19. Congruent triangles. Triangles that can be made to fit are called **congruent** triangles. The preceding discussion illustrates the following important geometric principle: *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.*

This principle may be used by surveyors to find unknown distances, as follows: If it is required to find the distance AB, Figure 28, a convenient point, C, is chosen so that straight lines may be drawn from A and B through C.

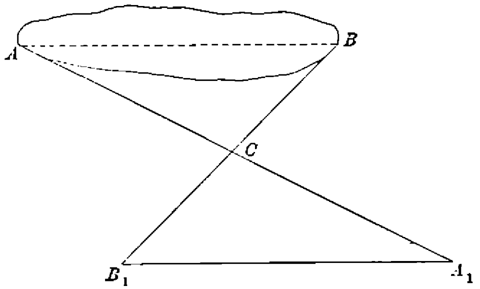


FIG. 28

Then a distance CB_1 is marked off equal to CB, and a distance CA_1 equal to CA.

A_1B_1 is drawn.

Show by the principle above that triangles ABC and A_1B_1C are congruent.

Show that AB may be found by measuring A_1B_1.

The method employed in the preceding problem is called the *congruent triangle method* of finding unknown distances.

20. Symbol for congruence. Congruence is expressed by the symbol $\cong$. Thus the statement "triangle ABC

is congruent to triangle $A_1B_1C_1$" is written briefly $\triangle ABC \cong \triangle A_1B_1C_1$.

21. Meaning of the term "theorem." A **theorem** is a geometric principle to be proved.

22. Isosceles and equilateral triangles. A triangle having two equal sides is called **isosceles**, Figure 29. The angle included between the equal sides, $\angle ACB$, is the **vertex angle.** The other two angles, A and B, are the **base angles.** If *three* sides of a triangle are equal it is an **equilateral** triangle.

23. Theorem: ⊛ *The base angles of an isosceles triangle are equal.*

Let $\triangle ABC$, Figure 29, be isosceles, i.e., $a = b$.

It is to be shown that $\angle A = \angle B$.

Suggestions: Draw the helping line, CD, bisecting $\angle C$, i.e., making $x = y$.

Show by using the principle of § 19, that $\triangle ADC \cong \triangle BDC$.

It follows that $\angle A = \angle B$, because corresponding angles of congruent triangles are equal.

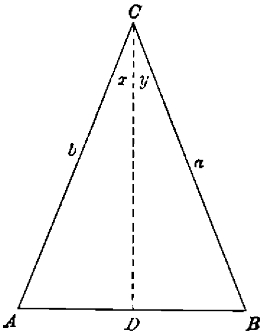


FIG. 29

EXERCISES

Show that the following statements are true:

1. An equilateral triangle is equiangular.

Suggestion: Apply the principle of § 23.

2. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

Suggestion: Use the congruent triangle method, as in § 23.

3. *All points on the perpendicular bisector of a line segment are equidistant from the end points.*

24. To draw a triangle when the length of one side and the sizes of two angles are known. Let the length of one side of the triangle to be drawn be 5 centimeters, and let the angles be 42° and 68°, respectively.

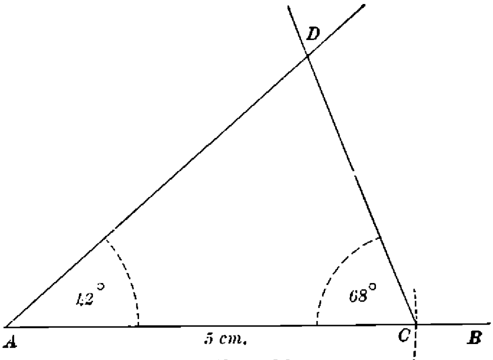


FIG. 30

Draw a line, as AB, Figure 30.

On AB lay off AC equal to 5 centimeters.

With the protractor, draw angles at A and C equal to 42° and 68°, respectively. Make the sides intersect at D. $\triangle ACD$ is the required triangle.

Note that if one of the two given angles is to lie opposite the given side, the remaining angle must first be found by § 17.

Cut $\triangle ACD$ from the paper, write your name on it, and pass it on to the teacher.

Those triangles that the pupils of the class have constructed carefully and correctly can be made to fit. They are *congruent* triangles.

The preceding discussion illustrates the following geometric principle:

If two angles and a side of one triangle are equal to the corresponding angles and side of another, the triangles are congruent.

This principle may be used in surveying, as follows:

Let it be required to find the distance AC, Figure 31, C being a point in a lake and A being a point on the shore.

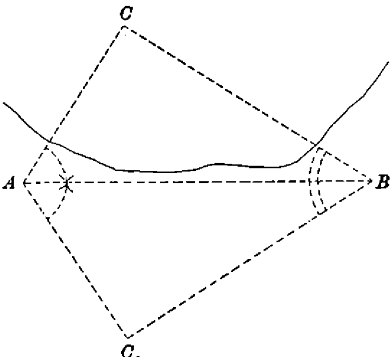


FIG. 31

A base line, AB, of convenient length is laid off. The angles CAB and ABC are measured with a surveyor's transit.

Then AC_1 is laid off, making $\angle C_1AB = \angle CAB$, and BC_1 making $\angle C_1BA = \angle CBA$.

Finally AC_1 is measured.

Since $\triangle C_1AB \cong \triangle CAB$, having two angles and a side equal respectively, it follows that the length of AC_1 is also the required length of AC. For corresponding sides of congruent triangles are equal.

EXERCISES

1. Show that *if two angles of a triangle are equal, the sides opposite them are equal.*

Suggestion: Bisect angle C, Figure 32.

Show that $\triangle ADC \cong \triangle BDC$ (§ 24).

It then follows that $AC = BC$. Why?

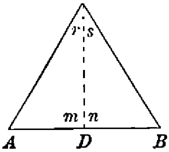


FIG. 32

2. If in $\triangle ABC$ it is known that $r = s$ and that the adjacent angles at D are equal, show that $AD = DB$.

Constructions with Ruler and Compass

25. What is meant by a circle. A **circle**, Figure 33, is a closed curved line every point of which has a fixed distance from a point within called the **center.** A segment drawn from the center to a point on the circle is the **radius.** A segment passing through the center and intercepted (cut off) by the circle is a **diameter.** A part of a circle is an **arc.** A line segment joining two points of a circle is a **chord.**

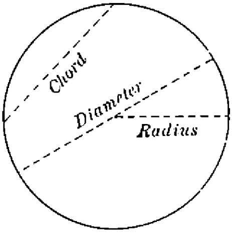


FIG. 33

26. To construct an equilateral triangle.

Construction: Draw a line segment, as AB, Figure 34.

With A and B as centers and with radius AB draw arcs intersecting at C.

Draw AC and BC.

$\triangle ABC$ is the required triangle.

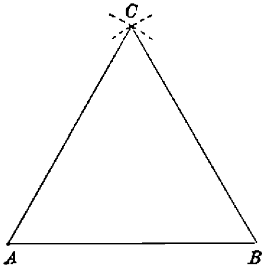


FIG. 34

27. To construct an angle of 60°.

Construction: Draw a line segment as AB, Figure 35.

With A and B as centers and with radius AB draw arcs intersecting at C.

Draw AC.

Then $\angle BAC$ is the required angle.

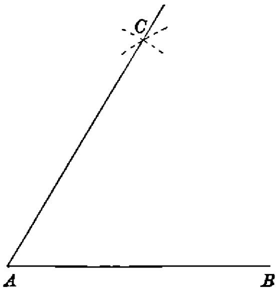


FIG. 35

Check the accuracy of your construction by measuring $\angle BAC$ with the protractor.

28. To construct a triangle whose sides are equal to the sides of a given triangle.

Construction: Draw a triangle as $\triangle ABC$, Figure 36.

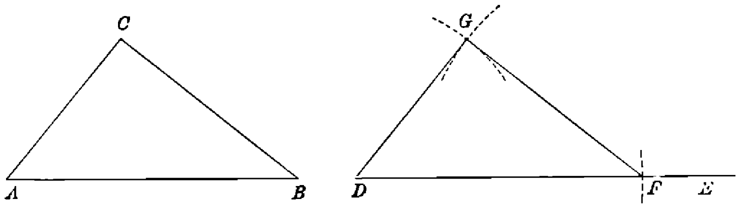


FIG. 36

Draw a line, as DE.

With D as center and radius equal to AB draw an arc cutting DE at F.

With D and F as centers and radii equal to AC and BC, respectively, draw arcs intersecting at G.

Draw DG and FG.

$\triangle DFG$ is the required triangle.

Cut $\triangle DFG$ from the paper and place it on $\triangle ABC$. If your construction is correct and carefully made, it should be possible to make the triangles fit exactly.

The preceding construction illustrates the following geometric principle:

Two triangles are congruent if the sides of one are equal to the sides of the other.

29. To bisect an angle.

Construction: Let $\angle ABC$, Figure 37, be an angle to be divided into two equal angles (to be bisected).

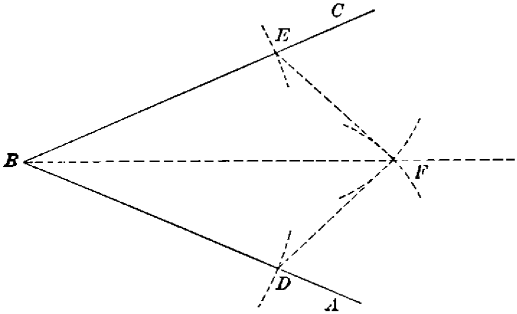


FIG. 37

With B as center and a convenient radius draw arcs intersecting BA at D and BC at E.

With E and D as centers and equal radii, draw arcs intersecting at F.

Draw BF.

BF divides $\angle ABC$ into two equal parts.

Check the accuracy of your construction by measuring angles ABF and CBF with a protractor.

The correctness of the preceding construction can be easily established by reasoning about it as follows:

Draw EF and DF.

Then $\triangle BEF \cong \triangle BDF$, since the sides of one triangle are equal to the sides of the other by construction.

Hence $\angle ABF = \angle CBF$, because corresponding angles of congruent triangles are equal.

Exercises

1. Draw an obtuse angle and bisect it.
2. Draw a triangle and bisect each angle.
3. Divide an angle into four equal parts.
4. Bisect the angle between two walls of the classroom.
5. Bisect a straight angle.

30. To bisect a line segment.

Construction: Let AB, Figure 38, be the segment to be divided into two equal parts.

With A and B as centers, draw four arcs intersecting above AB at C, and below AB at D.

Place your ruler so that the straightedge passes through C and D.

Mark the point at which it intersects AB and name it E.

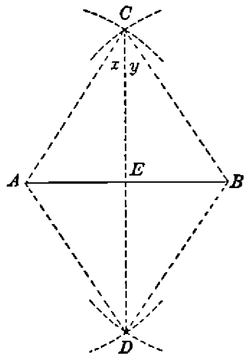


FIG. 38

Then AE and EB are the required equal parts of AB.

Check the accuracy of your construction by measuring AE and EB.

You may show without measuring that $AE = EB$, as follows:

Draw AC, AD, BC, BD, and CD.

Then $\triangle CAD \cong \triangle CBD$, since three sides of one are equal to three sides of the other.

$\therefore \quad x = y$, because corresponding angles of congruent triangles are equal.

Now it follows that $\triangle AEC \cong \triangle BEC$, since two sides and the included angle of one are equal to the corresponding parts of the other.

Therefore $AE = EB$, because corresponding sides of congruent triangles are equal.

EXERCISES

1. Draw a triangle and bisect each side.
2. Draw a four-sided figure and bisect each side.

31. At a point on a given line construct a perpendicular to the line.

Construction: Let AB, Figure 39, be the given line, and C a given point on AB.

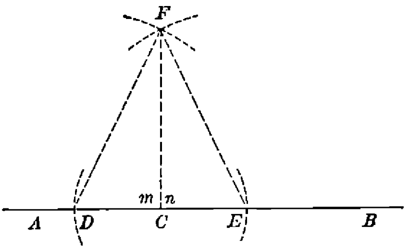


FIG. 39

With C as center and a convenient radius, draw arcs intersecting AB at points D and E.

With D and E as centers and a convenient radius, draw arcs intersecting at F.

Draw FC.

FC is the required perpendicular.

Check the accuracy of your construction by measuring angles DCF and ECF. These angles should be equal.

To show without measuring that FC is perpendicular to AB, draw DF and FE.

Show $\triangle DCF \cong \triangle ECF$.

Then $m = n$, because corresponding angles of congruent triangles are equal.

Hence FC is perpendicular to DE, and therefore to AB. Why?

Exercises

1. On the floor of the classroom erect a perpendicular to the line of intersection of the floor with the wall.

2. Erect a perpendicular to the edge of a sheet of notebook paper.

32. From a point not on a given line, to draw a perpendicular to the given line.

Construction: Let AB, Figure 40, be the given line, and let C be a point not on AB.

With C as center and a convenient radius, draw an arc intersecting AB at D and E.

Using D and E as centers and the same radius, draw arcs intersecting at F.

Draw CF.

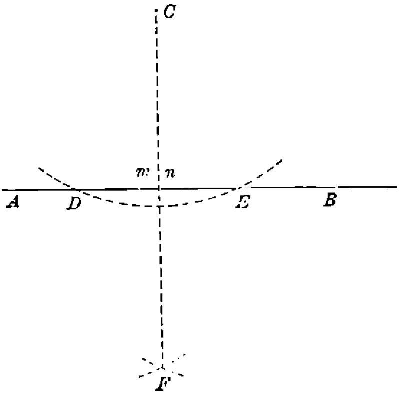


FIG. 40

CF is the required perpendicular.

To check your construction, show by measuring that $m = n$.

Exercise

Draw a triangle and from each vertex construct a perpendicular to the opposite side.

33. To construct an angle equal to a given angle.

Construction: Let $\angle ABC$, Figure 41, be the given angle.

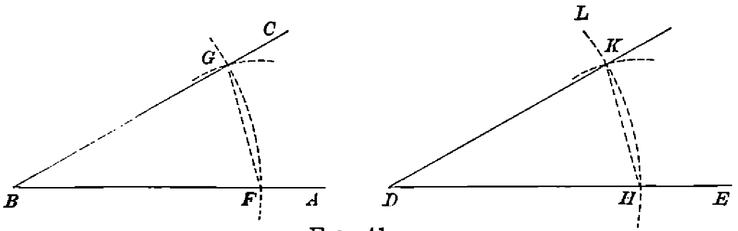


Fig. 41

Draw DE.

With B as center and a convenient radius draw an arc intersecting BA and BC at points F and G.

With D as center and the same radius draw arc HL.

With radius equal to FG and H as center draw an arc intersecting arc HL at K.

Draw DK.

Then $\angle EDK$ is the required angle.

Test the accuracy of your construction by measuring angles ABC and EDK.

To show the correctness of the construction without measuring, draw line segments FG and HK.

Then $\triangle FBG \cong \triangle HDK$, since three sides of one are equal to three sides of the other.

Hence $\angle FBG = \angle HDK$, because corresponding angles of congruent triangles are equal.

Logical Demonstration

34. Methods of proof. In the preceding pages some geometric facts have been established by process of reasoning, or *proof*. There is no *one specific* method by which *all* theorems or problems of geometry may be attacked or proved. However, certain *general directions and methods* as to the way of attacking problems and proving theorems may be stated. A knowledge of these methods is of greatest importance, as they will keep you from groping about blindly for a proof, wasting your time and energy. The directions follow:

1. Read the problem carefully, get it clearly in mind, and keep it in mind while at work on it. Most problems need more than one reading before you can state them without referring to the book.

2. If the problem is a geometric theorem or exercise, make a careful drawing of the figure. Thus, if the theorem refers to a triangle, draw a triangle with unequal sides, not an equilateral, or isosceles, or right traingle. This will keep you from committing the error of proving a theorem only for a *special case.*

3. Write down in symbols what is given (**the hypothesis**) and what is to be proved (**the conclusion**), referring all statements to the figure.

4. If a proof does not readily suggest itself to you, think of all the things you have learned that are like the problem you are trying to work out, e.g., recall the theorems that seem like the task before you.

35. Method of proof by superposition. This method was used in establishing some of the theorems on congruent triangles, §§ 18, 24. It consists of placing one figure over another and then showing that all parts of the one coincide with the corresponding parts of the other.

The method, although practical when the elements involved in the proof are few and specific, is not considered a good theoretical test by the mathematician. For, the axioms validating superposition are usually not given in full detail. The result is that the student is in danger of drawing rashly the conclusion which is to be established by the superposition of the one figure upon the other. The method of superposition will be used only in a few cases.

36. Arrangement of proof. To avoid errors, the statements made in a proof and the authorities for these statements must be carefully arranged to make it as easy as possible to follow the reasoning. The formal demonstration of a theorem consists of three main parts: the *hypothesis*, the *conclusion*, and the *proof*. Each statement of the proof must be supported by one of the following authorities: (1) a definition, (2) the hypothesis, (3) an axiom, (4) a theorem which has been established previously.

The final step in the proof must always be the same as the conclusion.

EXERCISES

Draw a figure and state in symbols the hypothesis and conclusion for each of the following theorems:

1. Two triangles are congruent if three sides of one are equal to three sides of the other.

2. Two triangles are congruent if two sides and the included angle of one are equal to two sides and the included angle of the other.

3. If three sides of a triangle are equal, the angles are also equal.

4. If two angles of a triangle are equal, the triangle is isosceles.

5. A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

37. Historical development of the proof. The processes of proving theorems were developed by the Greeks. Greece was indebted to Egypt for its beginnings in geometry. However, the Egyptians carried geometry no farther than was necessary for the practical needs of life. They may have *felt* the truth of some theorems; but the Greeks formulated these geometric truths into scientific language and subjected them to proof (see Ball's *History of Mathematics*, pp. 16–19). The Greeks also recognized that it is impossible to prove *everything* in geometry, and that *some* simple statements have to be assumed.

Euclid (about 300 B.C.) used the term *common notion* in the sense in which in modern mathematics we use *axiom*, i.e., a general statement admitted to be true without proof. Thus, the statement: "If equals are added to equals, the sums are equal" is an *axiom* because it holds in mathematics in general, i.e., in arithmetic, algebra, or geometry.

In modern mathematics, a statement referring to geometry only and admitted to be true without proof is called a *postulate*. Thus, the statement "Two points determine a straight line" is a postulate. Some textbook writers use the word *axiom* or *assumption* to denote postulates as well as axioms.

38. Illustrations of the method of superposition. The discussions establishing the theorems of congruent triangles (§§ 18, 24, 28) will be arranged on the next pages in the form of "formal" demonstrations.

I. **Theorem:** ⊛*If two sides and the included angle of one triangle are equal to two sides and the included angle of another, the triangles are congruent.*

Given $\triangle ABC$ and $A_1B_1C_1$, Figure 42, having $AB = A_1B_1$, $AC = A_1C_1$, $\angle A = \angle A_1$.

To prove $\triangle ABC \cong \triangle A_1B_1C_1$.

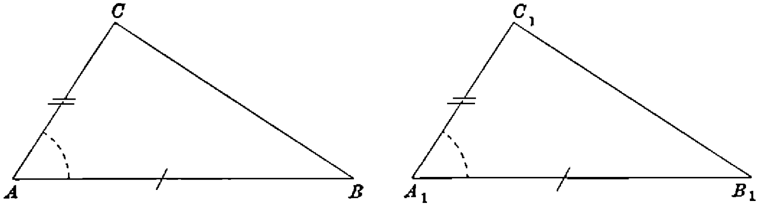


FIG. 42

STATEMENTS	AUTHORITIES
Imagine $\triangle A_1B_1C_1$ placed on $\triangle ABC$ so that $\angle A$ coincides with $\angle A_1$.	It is assumed that a geometric figure may be moved in space without changing shape or size of parts, and $\angle A$ was given equal to $\angle A_1$.
Since $AB = A_1B_1$,	Given.
point B_1 must fall on point B.	If two segments are equal they can be made to coincide throughout.
Since $AC = A_1C_1$,	Given.
point C_1 must fall on point C.	If two segments are equal they can be made to coincide.
B_1C_1 must then coincide with BC.	Only one straight line can be drawn through two points.
$\triangle ABC \cong \triangle A_1B_1C_1$.	They have been made to coincide.

II. Theorem:[⊛] *If two angles and a side of one triangle are equal to the corresponding parts of another the triangles are congruent.*

Given $\triangle\triangle DEF$ and $D_1E_1F_1$, Figure 43, having $\angle D = \angle D_1$, $\angle E = \angle E_1$, and $DE = D_1E_1$.

To prove $\triangle DEF \cong \triangle D_1E_1F_1$.

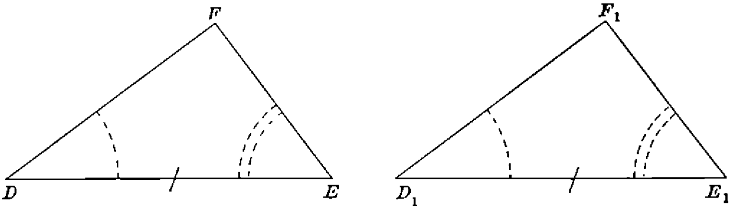


FIG. 43

Note that in Figure 43 the angles adjacent to the equal sides are used. If $\angle F$ is given equal to $\angle F_1$, you know from Exercise 9, § 17, that $\angle E = \angle E_1$.

Imagine $\triangle D_1E_1F_1$ placed on $\triangle DEF$, making D_1E_1 coincide with DE.	For, a geometric figure can be moved about in space without changing its size, and DE was given equal to D_1E_1.
Since $\angle D = \angle D_1$,	Given.
D_1F_1 falls along DF, and point F_1 falls somewhere on DF or its extension.	If two angles are equal, their sides can be made to fall together.
Since $\angle E = \angle E_1$,	Given.
E_1F_1 falls along EF, and F_1 somewhere on EF or its extension.	If two angles are equal, their sides can be made to fall together.
$\therefore$ Point F_1 must fall on point F.	Two straight lines intersect in only one point.
$\therefore$ $\triangle DEF \cong \triangle D_1E_1F_1$.	For they coincide.

III. **Theorem:**® *If the sides of one triangle are equal to the sides of another the triangles are congruent.*

Given △s ABC and $A_1B_1C_1$, Figure 44, having $AB = A_1B_1$, $BC = B_1C_1$, and $CA = C_1A_1$.

To prove $\triangle ABC \cong \triangle A_1B_1C_1$.

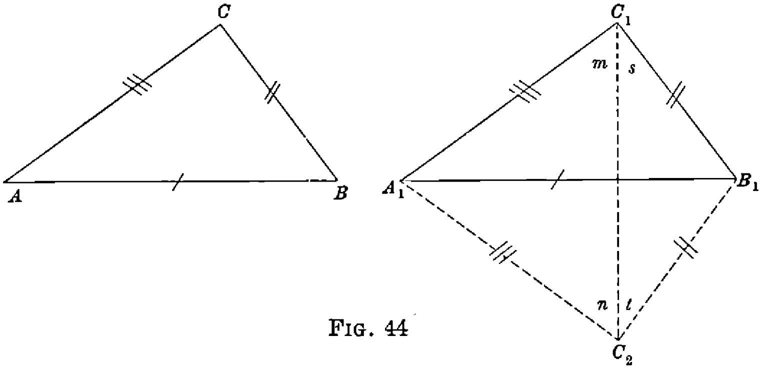


FIG. 44

STATEMENTS	AUTHORITIES
Imagine $\triangle ABC$ placed in the position $A_1B_1C_2$ adjacent to $\triangle A_1B_1C_1$ so that AB coincides with A_1B_1.	For, a figure can be moved in space without change in shape or size of its parts, and AB is given equal to A_1B_1.
Draw C_1C_2.	
Then AC or $A_1C_2 = A_1C_1$.	Given.
∴ $m = n$.	§ 23.
BC or $B_1C_2 = B_1C_1$.	Given.
∴ $s = t$.	§ 23.
∴ $m + s = n + t$.	Equals added to equals give equal sums.
∴ $\angle A_1C_2B_1$ or $\angle ACB = \angle A_1C_1B_1$.	Substituting the whole for the sum of its parts.
$\triangle ABC \cong \triangle A_1B_1C_1$.	By the theorem of Example I.

IV. **Theorem:**⊛ *Two right triangles are congruent if the hypotenuse and a side of one are equal to the hypotenuse and a side of the other.*

Given $\triangle s\ ABC$ and $A_1B_1C_1$, Figure 45, with $\angle C = 90°$ and $\angle C_1 = 90°$. $AB = A_1B_1$, $AC = A_1C_1$.

To prove $\triangle ABC \cong \triangle A_1B_1C_1$.

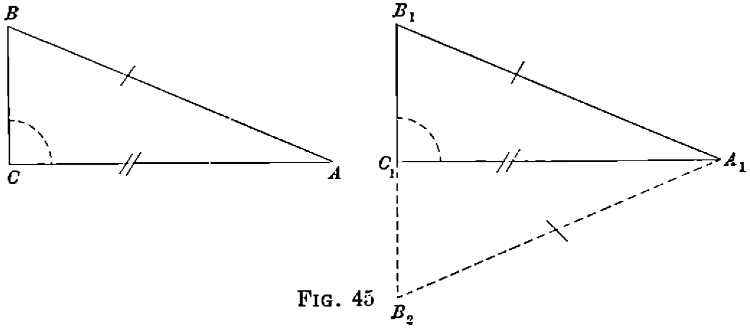


FIG. 45

STATEMENTS	AUTHORITIES
Imagine $\triangle ABC$ placed adjacent to $\triangle A_1B_1C_1$, making AC coincide with A_1C_1.	For, a figure may be moved in space without changing size and shape, and $AC = A_1C_1$.
Then $\angle B_1C_1A_1 = 90°$.	Given.
$\angle BCA$ or $\angle B_2C_1A_1 = 90°$.	Given.
$\therefore \angle B_1C_1A_1 + \angle B_2C_1A_1 = 180°$.	Addition axiom.
$\therefore$ $B_1C_1B_2$ is a straight line.	The sides of a straight angle lie in the same straight line.
AB or $A_1B_2 = A_1B_1$.	Given.
$\therefore \angle B_1 = \angle B_2$.	§ 23.
$\angle A_1C_1B_1 = \angle A_1C_1B_2$.	All right angles are equal.
$\therefore \triangle A_1B_1C_1 \cong \triangle A_1B_2C_1$.	Example II.
$\therefore \triangle A_1B_1C_1 \cong \triangle ABC$.	By substitution.

EXERCISES

1. If the angles of a triangle are equal, the triangle is equilateral. Prove.

2. A line joining the vertex of an isosceles triangle to the midpoint of the base bisects the angle at the vertex. Prove.

3. A line drawn from the vertex of an isosceles triangle perpendicular to the base, bisects the base. Prove.

4. If a line bisects an angle of a triangle and is perpendicular to the opposite side, the triangle is isosceles. Prove.

5. Given the base and one of the two equal sides of an isosceles triangle, construct the triangle.

6. Two right triangles are congruent if the two sides of the right angle of one are equal to the corresponding sides of the other. Prove.

7. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other. Prove.

8. Construct a right triangle having given the two sides of the right angle.

9. Construct a right triangle having given one of the sides of the right angle and the hypotenuse.

10. Construct a right triangle having given the hypotenuse and one acute angle.

11. Construct an isosceles triangle having given the altitude and the base angle.

12. Construct an equilateral triangle having given the altitude.

13. Prove that the bisectors of the base angles of an isosceles triangle are equal.

14. How many degrees are there in the base angles of an isosceles right triangle? Prove your answer.

15. If the vertex angle of an isosceles triangle is one-half of the base angle, find the size of each angle.

16. Make a summary of ways of proving two angles equal.

17. Make a summary of ways of proving two line segments equal.

Summary of Chapter I.

39. Geometric symbols and terms. The following symbols were used in this chapter:

1. $\angle$, $\measuredangle$	Angle, angles
2. $\triangle$, $\triangle$	Triangle, triangles
3. $\therefore$	Hence or therefore
4. $\perp$	Is perpendicular to
5. $\cong$	Is congruent to

The meaning of each of the following terms has been explained:

1. Unit of length, as inch, and centimeter
2. Geometric line
3. Angle, vertex, side
4. Acute, right, and obtuse angle
5. Bisecting segments and angles
6. Opposite angles, complementary angles, supplementary angles, adjacent angles
7. Perpendicular lines
8. Isosceles triangle, equilateral triangle
9. Congruent triangles
10. Theorem
11. Circle, arc, radius, center, diameter

40. Fundamental assumptions. The following assumptions have been used:

1. *If equals are added to equals the sums are equal.*

2. *If equals are subtracted from equals the remainders are equal.*

3. *Magnitudes equal to the same magnitude are equal to each other.*

4. *A quantity may be substituted for one equal to it.*

5. *The whole is equal to the sum of its parts.*

6. *Through two points one and only one straight line can be drawn.*

7. *Two straight lines cannot intersect in more than one point.*

8. *A geometric figure may be moved in space without changing its size or shape.*

41. Geometric principles. The following is a list of geometric principles established in Chapter I:

1. *If two lines intersect the opposite angles are equal.*
2. ⊛ *The sum of the angles of a triangle is 180°.*
3. *If two angles of one triangle are equal to two angles of another the third angles are equal.*
4. ⊛ *The base angles of an isosceles triangle are equal.*
5. *All points on the perpendicular bisector of a line segment are equidistant from the end points.*
6. ⊛ *If two sides and the included angle of one triangle are equal to two sides and the included angle of another, the triangles are congruent.*
7. ⊛ *If two angles and a side of one triangle are equal to the corresponding angles and side of another, the triangles are congruent.*
8. ⊛ *Two triangles are congruent if the sides of one are equal to the sides of the other.*
9. ⊛ *Two right triangles are congruent if the hypotenuse and one side of the right angle of one are equal respectively to the corresponding parts of the other.*

42. Geometric instruments. The following instruments have been used:

1. Ruler for measuring and drawing line segments.
2. Protractor for measuring and drawing angles.

3. Compass for drawing arcs and laying off segments.
4. Squared paper for measuring line segments.

43. Geometric constructions. The following constructions have been taught:

1. *To construct an equilateral triangle.*
2. *To construct an angle of 60°.*
3. *To bisect an angle.*
4. *To bisect a line segment.*
5. *At a point on a line to construct a perpendicular to the line.*
6. *From a point outside of a given line to construct a perpendicular to the given line.*
7. *To construct an angle equal to a given angle.*

44. Algebraic processes. The following algebraic processes were reviewed:

1. Solving linear equations in one unknown, as $8x+15=16x-9$.
2. Solving linear equations with literal coefficients, as $r=tx+s$.
3. Solving simple verbal problems by means of equations.

CHAPTER II

PARALLEL AND PERPENDICULAR LINES

Conditions Making Two Lines Parallel

45. Meaning of parallel lines. Let A, Figure 46, represent a point in a plane, and let BC be any line not passing through A. Draw AD passing through A and intersecting BC at D.

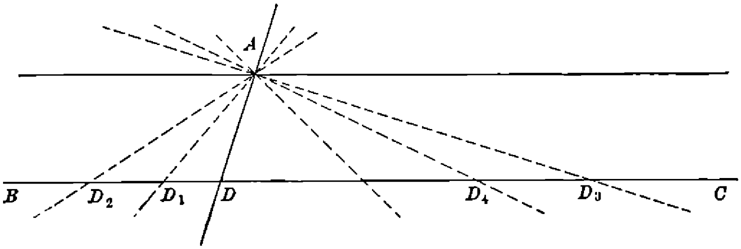


FIG. 46

Rotate AD about point A in the clockwise direction. Then point D moves along CB to the left, taking positions as D_1, D_2, etc., until it is removed infinitely far to the left.

If you continue to turn AD, the point of intersection appears at the right, moving along CB to the left and taking positions as D_3, D_4, etc. Thus somewhere during the rotation of AD the point of intersection D has passed from the left over to the right. It will be assumed that there is a position AE of AD, Figure 47, such that the point of

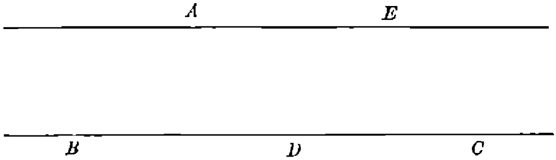


FIG. 47

intersection of AD and BC is found at the left of D immediately before and at the right of D immediately after taking this position. It will be assumed further, that in this position AE does not intersect BC at all. AE is then said to be *parallel* to BC. The preceding discussion may now be summarized in the following two statements:

1. *Two straight lines are parallel if they lie in the same plane and do not intersect, however far they are extended.*

2. *Through a given point outside of a given straight line one, and only one, line can be drawn parallel to the given line.*

EXERCISES

1. In the classroom find two lines that do not intersect, however far extended, and that are not parallel.

2. Locate parallel lines in the classroom.

46. A symbol for parallelism. The symbol $\parallel$ means "is parallel to." Thus $AB \parallel CD$ means that line AB is parallel to line CD. The symbol $\nparallel$ means "is not parallel to."

47. Axiom. An **axiom** is a statement accepted as true without proof. For example, the statement "If equal numbers are added to equal numbers the sums are equal" is an axiom.

48. Theorem: *If two lines are parallel to a third line, they are parallel to each other.*

The preceding theorem states three facts, namely that each of two given lines is parallel to a third line, and that the given lines are parallel to each other. The first two facts are regarded as "known" or "given." The last fact is to be established or to be "proved."

In demonstrating the truth of a geometric principle we shall follow the custom of restating the facts contained in it in two other ways: (1) by means of a diagram, and (2) by means of symbols. In the second case the "given" facts are separated from the facts to be "proved." Thus, the foregoing theorem is restated as follows:

Given $AB \parallel EF$, Figure 48.
$CD \parallel EF$.

To prove $AB \parallel CD$.

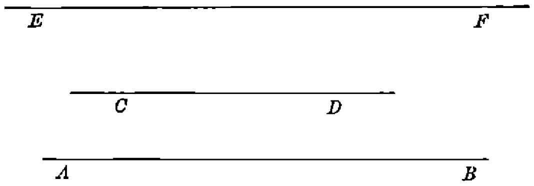


FIG. 48

After the facts are clearly stated, as in the foregoing, a method of proof is to be selected. In this case we may reason as follows: Suppose AB is not parallel to CD. Then AB must intersect CD at some point, which we may call P. Then there are two lines passing through P that are given parallel to EF. This is contradictory to what we have learned in § 45, where it was shown that through a point outside of a given line only one line can be drawn parallel to the given line. This means that if the foregoing reasoning is correct, the original supposition must have been false, for in no other way could a statement be inferred which we know to be false.

The preceding is a "round about" method of proof, known as the *indirect method.* This method is also called *Reductio ad absurdum* which means "a reduction to an absurdity."

It consists of the following four steps:

1. The negative of the fact to be proved is "assumed." Thus, in the proof of the preceding theorem, it was assumed that AB is not parallel to CD.

2. Starting with this assumption, other statements are inferred by correct reasoning. For example, if AB is not parallel to CD, it follows that AB meets CD if far enough extended.

3. Further statements are inferred until a statement is derived which is known to be false.

4. Since it is impossible to derive, by correct reasoning, a statement which is known to be *false*, it follows that the original assumption is false. Therefore the statement to be proved is true.

For the sake of clearness we shall now arrange the proof in two columns, one of which contains all the statements that make up the proof, while the other contains the evidence necessary to show that each statement is true.

Proof:

STATEMENTS	AUTHORITIES
Assume $AB \nparallel CD$.	
Then AB meets CD, if far enough extended, in some point which may be denoted by P.	If two lines lying in the same plane are not parallel they intersect if far enough extended.
Then $CP \parallel EF$,	Given.
and $AP \parallel EF$.	Given.
The last two statements cannot both be true.	For, through a given point outside of a given line, one and only one line can be drawn parallel to the given line.
∴ The assumption that $AB \nparallel CD$ is wrong.	
∴ $AB \parallel CD$.	

The following is a complete arrangement of the proof of the preceding theorem:

Theorem: *If two lines are parallel to a third line they are parallel to each other.*

Given $AB \parallel EF$, Figure 49.
$CD \parallel EF$.

To prove $AB \parallel CD$.

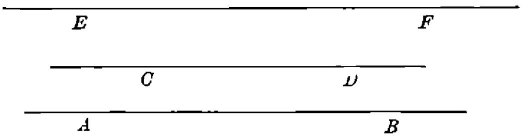


FIG. 49

Proof:

STATEMENTS	AUTHORITIES
Assume $AB \nparallel CD$. Then AB meets CD, if far enough extended, at some point, as P.	For, if two lines lying in the same plane are not parallel, they intersect if far enough extended.
Then $CP \parallel EF$,	Given.
and $AP \parallel EF$.	Given.
The preceding two statements cannot be true.	Through a given point outside of a given line one and only one line can be drawn parallel to the given line.
∴ The assumption that $AB \nparallel CD$ is wrong.	
∴ $AB \parallel CD$.	

49. Summary of suggestions for proving a theorem. The following suggestions are helpful in working out proofs:

1. Read the theorem carefully and note the various facts that are contained in it.

2. When you are able to state the theorem without referring to the book, separate the facts contained in the theorem into two groups, those that are *given*, or assumed, and those whose truth is to be established, i.e., those that are to be *proved*.

3. Draw a figure representing the conditions of the theorem, and at the same time, under the headings "given" and "to prove," list in brief symbolic form all of the facts of the theorem, not more and not less. Similar care must be taken to illustrate in the figure only the facts stated in the theorem and no others.

4. Begin to reason out the truth of the theorem, proceeding from one inference to another, always stating but one fact at a time, until finally the fact to be proved is reached. (When the indirect method is used continue the process of reasoning until a contradiction is obtained.) No statement is complete unless a reason or authority is given. Any given fact, definition, axiom, or theorem previously established is accepted as satisfactory authority. The various statements in the proof should be arranged in the best logical order. It will add greatly to clearness if you use a form similar to the one shown in § 48.

The importance of exercising great care in a geometric proof is illustrated by two well-known puzzles of geometry, given on pages 45 and 46. Read carefully and try to find the fallacies in the proofs.

I. **Theorem:** *Every triangle is isosceles.*

Given any triangle, as ABC, Figure 50.

To prove that ABC is isosceles.

Proof: Let DE be the perpendicular bisector of AB, and let CE be the bisector of angle C, meeting DE at E.

From E draw EA and EB.

Draw EG perpendicular to AC, and EF perpendicular to CB.

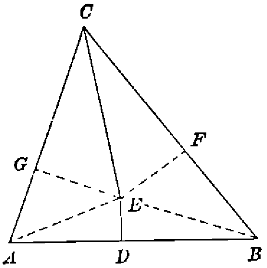


FIG. 50

Then $\triangle ADE \cong \triangle BDE$ (§ 18).

Hence, $AE = BE$.

(Since corresponding sides of congruent triangles are equal.)

$\triangle CEG \cong \triangle CEF$ (§ 24).

Hence, $EG = EF$ and $CG = CF$.

Therefore $\triangle AEG \cong \triangle BEF$.

(Hypotenuse and a side, p. 34.)

Hence, $GA = FB$.

Since $CG = CF$,

it follows that $CG + GA = CF + FB$,

or $CA = CB$.

Therefore the triangle ABC, although known not to be isosceles, would seem to have been proved to be isosceles.

II. *To show geometrically that 64 = 65.*

Draw two right triangles like the one shown in Figure 51, having the sides, including the right angle, equal to 3 and 8, respectively. Draw two quadrilaterals like the one shown in Figure 51, having one pair of opposite

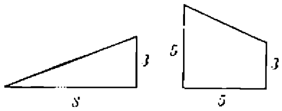


FIG. 51

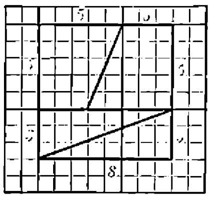


FIG. 52

sides parallel and equal to 3 and 5, respectively, and the third side perpendicular to the parallel sides and equal to 5.

By placing the triangles and quadrilaterals as in Figure 52 a square is obtained.

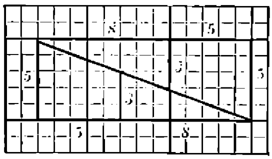


FIG. 53

The area of this square is equal to $8 \times 8 = 64$.

Now place the four figures in the position shown in Figure 53 and a rectangle is formed.

The area of this rectangle is equal to $13 \times 5 = 65$.

Hence, $64 = 65$.

50. Theorem:® *From a point outside of a given line, one and only one line can be drawn perpendicular to the given line.*

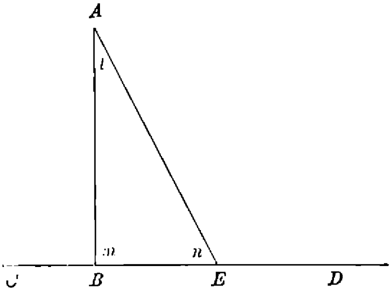


FIG. 54

Given $AB \perp CD$, Figure 54.

To prove AB the only perpendicular from A to CD.

Proof: Assume that AB is not the only perpendicular to CD.

Draw $AE \perp CD$.

In $\triangle BAE$ denote the angles by m, n, and t.

Then $m+n+t=180$.	The sum of the angles of a triangle is 180°.
$AB \perp CD$.	Given.
$\therefore \quad m=90$.	The angle formed by two perpendicular lines is 90°.
Similarly, $n=90$.	
$\therefore \quad m+n=180$.	If equal numbers are added to equal numbers the sums are equal.

This contradicts the statement above that $m+n+t=180$.

$\therefore$ The assumption that AB is not the only perpendicular to CD is wrong, and AB is the only perpendicular to CD.

51. Theorem: *Two lines perpendicular to the same line are parallel.*

The theorem contains three facts, namely, the facts that each of the two lines is perpendicular to the same line, and that the two lines are parallel to each other. Note that the first two facts are given and the third is to be proved. The indirect method of proof, explained in § 48 will be used. A complete arrangement of the proof follows.

Given $AB \perp EF$, Figure 55.

$CD \perp EF$.

To prove $AB \parallel CD$.

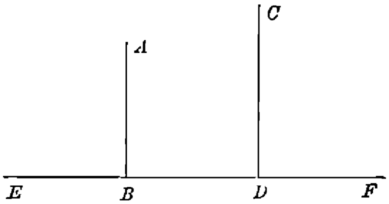


FIG. 55

Proof (Indirect method):

Assume $AB \nparallel CD$.

Then AB meets CD, if far enough extended, at some point, as G.

$GB \perp EF$.	Given.
$GD \perp EF$.	Given.
This is impossible.	From a point outside of a given line only one line can be drawn perpendicular to the given line.
∴ The assumption that $AB \nparallel CD$ is wrong, and $AB \parallel CD$.	

EXERCISES

1. Why are lines drawn along the edge of a T square that is sliding along the edge of a drawing board parallel?

2. Show that the opposite edges of a sheet of notebook paper are parallel.

52. What is meant by a transversal. The line *AB*, Figure 56, is a transversal of lines *CD* and *EF*. In general, a straight line cutting two or more straight lines is called a **transversal.**

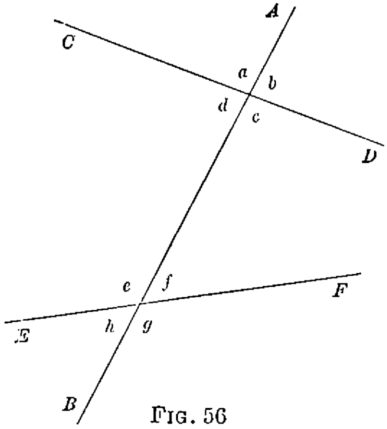


FIG. 56

53. Angle pairs formed by two lines and a transversal. If two lines, as *CD* and *EF*, Figure 56, are cut by a transversal, 8 angles are formed. They may be grouped in pairs as follows:

1. *Opposite angles:* *a*, *c*; *b*, *d*; *e*, *g*; *h*, *f*.

2. *Supplementary adjacent angles:* *a*, *b*; *a*, *d*; *d*, *c*; *c*, *b*; *e*, *f*; *f*, *g*; *g*, *h*; *h*, *e*.

3. *Interior angles:*

a) *Alternate interior angles:* *d*, *f*; *c*, *e*.

b) *Interior angles on the same side* (of the transversal): *d*, *e*; *c*, *f*.

4. *Exterior angles:*

a) *Alternate exterior angles:* *a*, *g*; *b*, *h*.

b) *Exterior angles on the same side:* *a*, *h*; *b*, *g*.

5. *Corresponding angles:* *a*, *e*; *d*, *h*; *b*, *f*; *c*, *g*.

EXERCISES

1. By means of equations state the relations between the opposite angles, Figure 56.

2. State the relations between the adjacent angles, Figure 56.

3. If angle e is 3 times as large as angle f, find the values of e and f.

4. If angle e is 3 times as large as c, if f and e are supplementary, and if the difference between c and f is 20°, find e, c, and f.

5. Eliminate x from the following equations, (1) by subtraction:

$$3x=8y.$$
$$3x+5y=17.$$

6. Eliminate x from the following equations:

$$x+y=12.$$
$$3x-5y=2.$$

54. Conditions which make two lines parallel. It has been shown above that two lines are parallel:

1. if they do not meet however far extended.

2. if they are parallel to the same line.

3. if they are perpendicular to the same line.

In the following pages further conditions will be worked out that may be used when two lines are to be proved parallel.

55. Theorem: ⊛ *If two lines are cut by a transversal making a pair of alternate interior angles equal, the lines are parallel.*

Given lines AB and CD, Figure 57, intersected by EF. $a=b$.

To prove $AB \parallel CD$.

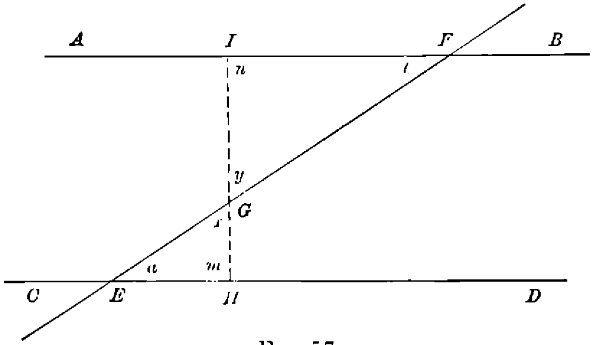


FIG. 57

Proof: Select any point on EF, such as G.

Draw $GH \perp CD$.

Extend HG until it meets AB at I.

$a=b$.	Given.
$x=y$.	If two straight lines intersect, the opposite angles are equal.
$m=n$.	If two angles of one triangle are equal to two angles of another, the third angles are equal.
$m=90$.	An angle whose sides are perpendicular to each other is a right angle.

$n = 90.$	Things equal to the same thing are equal to each other.
$AB \perp IH.$	Two lines are perpendicular to each other if they form a right angle.
$AB \parallel CD.$	Two lines perpendicular to the same line are parallel.

EXERCISES

1. Let $a = e$, Figure 58.
Prove that $b = f$.

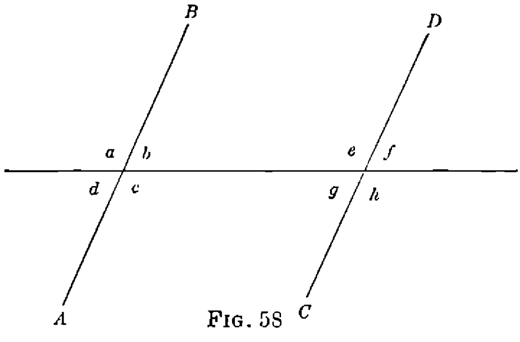


FIG. 58

Suggestion: Select relations which are known to hold for angles a, e, b, and f.

Thus, show that

$$a + b = 180.$$
$$e + f = 180.$$

It follows that

$a + b = e + f.$	Why?
$a = e.$	Given.

Show, by subtracting the last equation from the preceding, that

$$b=f, \text{ which was to be proved.}$$

Exercise 1 shows that if two corresponding angles are equal, the other corresponding angles are also equal.

2. Using the method of Exercise **1**, prove the following:

If $c=h$, Figure 58, then $c=e$.

In words, this means that if two lines are cut by a transversal, making the corresponding angles equal, the alternate interior angles are equal.

3. If $b+e=180$, prove that $b=g$.

4. If $c+g=180$, prove that $c=e$.

5. Show that it follows, from Exercises 3 and 4, that if two lines are cut by a transversal, making the interior angles on the same side supplementary, the alternate interior angles are equal.

6. Using Exercises 1–5 and the theorem of § 55, show the following to be true:

If two lines are cut by a transversal they are parallel: (1) *if the corresponding angles are equal;* (2) *if the interior angles on the same side are supplementary.*

7. Make a summary of all theorems so far mentioned in this chapter that state conditions which make two lines parallel.

8. Let $a=3x+2$; $e=5x-28$. Determine the values of x, a, and e so that lines AB and CD, Figure 58, may be parallel.

9. Let $b=7x+5$ and $g=11x-15$. Determine the values of x, b, and g so that AB may be parallel to CD.

10. Let $c=3x-5$ and $g=6x+50$. Determine x, c, and g so that AB may be parallel to CD.

Relations between Angles Formed by Parallel Lines and a Transversal

56. Converse of a theorem. You know that when a quadrilateral is a square it is also a rectangle. Is it true that a quadrilateral which is a rectangle is always a square? You have learned that if two triangles are congruent, the angles of one are equal to the corresponding angles of the other. Is it always true that if the angles of one triangle are equal to the corresponding angles of another, the triangles are congruent? Illustrate your answer by means of a drawing.

The preceding examples show that if in a geometric theorem the given facts and the facts to be proved are interchanged, the resulting statement is not always true.

By interchanging the given facts of a theorem with the facts to be proved another theorem is formed. Either theorem is called the **converse** of the other.

The foregoing examples show that the converse of a theorem is not always true, and that therefore a converse theorem cannot be accepted without proof.

EXERCISES

1. State the converse of the following: If a triangle is equilateral, it is also isosceles. Is the converse true?

2. State the converse of the theorem in § 55.

3. State a theorem the converse of which is true.

4. State the converse of the theorem: *all right angles are equal.* Is the converse true?

57. Symbols of inequality. The symbol $<$ means "is less than," and the symbol $>$ means "is greater than."

58. Theorem: ⊛ *If two parallel lines are cut by a transversal the alternate interior angles are equal* (converse of § 55).

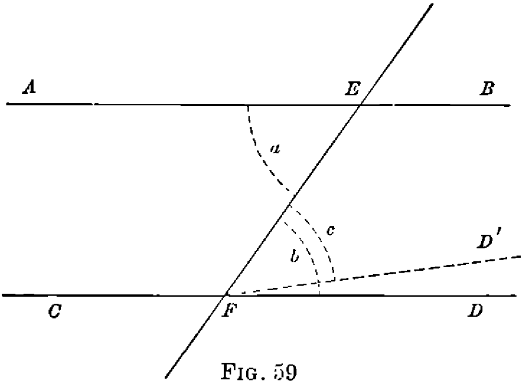


FIG. 59

Given $AB \parallel CD$ and the transversal EF, Figure 59.

To prove $a = b$.

Proof: Assume that $a \neq b$.

Then $a > b$ or $a < b$. In either case it is possible to lay off the smaller angle on the larger.

Consider $a < b$, and at F, on EF as one side, draw angle $c =$ angle a.	
Then FD', the other side of angle c, is parallel to EB.	Two lines cut by a transversal making the alternate interior angles equal are parallel.
But $FD \parallel EB$.	Given.
∴ Two lines through F are parallel to EB.	
This is impossible.	Through a point only one straight line can be drawn parallel to a given line.
∴ The assumption is wrong, and a is not less than b.	
Similarly, show that a is not greater than b.	
∴ $a = b$.	

EXERCISES

1. Draw a figure like Figure 58. If $c=e$, prove $d=g$; $c+g=180$; $b+e=180$. State this exercise in the form of a theorem.

2. If $b=g$, prove $d=g$; $a=e$; $b+e=180$; $c+g=180$.

3. Using § 58 and Exercises 1 and 2, show the following to be true:

If two parallel lines are cut by a transversal, the corresponding angles are equal and the interior angles on the same side are supplementary.

4. Show that* *if one of two parallel lines is perpendicular to a third line, the other is also.*

Suggestion: Use the theorem of § 58.

5. A road is to be laid out cutting two given parallel roads so that one of the interior angles is 5 times as large as the other interior angle on the same side. Find the number of degrees in each angle.

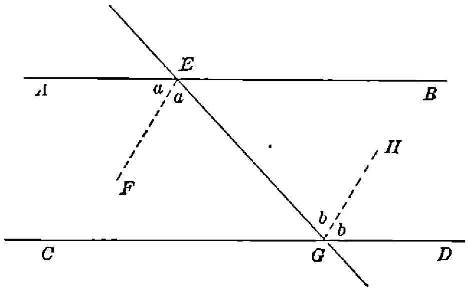


FIG. 60

6. Let $AB \parallel CD$, Figure 60. Let EF bisect $\angle AEG$, and GH bisect $\angle EGD$. Prove that $EF \parallel HG$.

Suggestion: Show that $2a=2b$.

$$a=b.$$
$$EF \parallel GH.$$

7. *If two angles have their sides respectively parallel, they are either equal or supplementary.* Prove.

Suggestion:

Case 1. Extend the sides of the angles until they intersect, Figure 61.

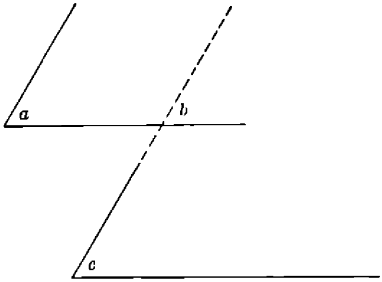


FIG. 61

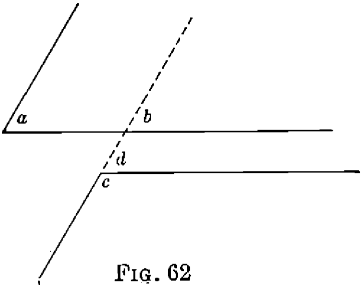


FIG. 62

Show that $a=b$, $b=c$.

$\therefore \qquad a=c.$ Why?

Case 2. As before, show that $a=b$, Figure 62.

Show that $b=d$.

$d+c=180.$ Why?

$a+c=180.$ Why?

8. *Through a given point, A,* Figure 63, *construct a line parallel to a given line, BC.*

Suggestion: Draw AD intersecting BC.

On AD, at A, construct $b=a$ (§ 33).

Draw AE.

This is the required line.

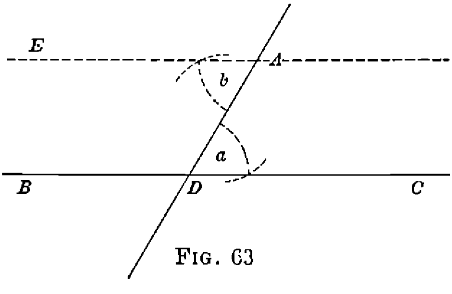


FIG. 63

9. If two parallel lines are cut by a transversal, prove that the bisectors of the interior angles on the same side are perpendicular to each other.

59. Exterior angle of a polygon. If a side of a polygon is extended, Figure 64, the angle formed by the extension

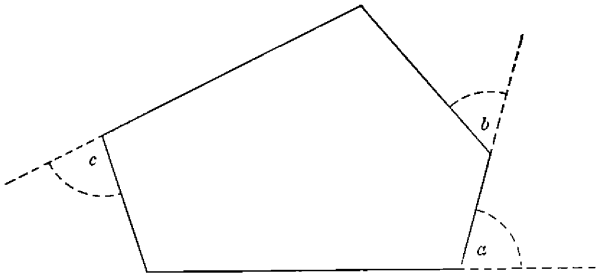


FIG. 64

and the consecutive side is called an **exterior angle** of the polygon. Thus a, b, and c are exterior angles.

EXERCISES

1. Make a summary of ways of proving two angles equal.

2. Make a summary of ways of proving two line segments equal.

3. Make a summary of ways of proving two lines perpendicular to each other.

4. Make a summary of ways of proving two lines parallel to each other.

5. Draw a polygon of 4 or more sides. At each vertex extend one of the sides. Measure each exterior angle and find the sum of the measures. State your result in the form of a theorem.

6. Show that *the sum of the exterior angles of a polygon is 360° if one exterior angle is taken at each vertex.*

Suggestion: Through a point O within the polygon, Figure 65, draw lines parallel to the extensions of the sides.

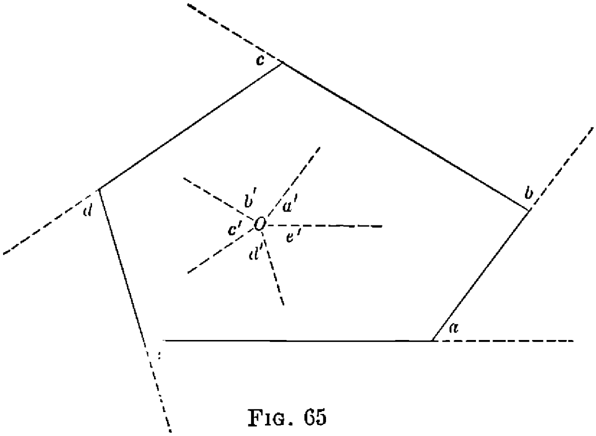


FIG. 65

Show that $a = a'$,

$b = b'$,

$c = c'$, etc.

$\therefore \quad a+b+c+ \dots = a'+b'+c'+ \dots$

Since $a'+b'+c'+d'+ \dots = 360$,

it follows that $a+b+c+d+ \dots = 360$. Why?

Note that the sum of the exterior angles of a polygon is 360°, whatever may be the number of sides of the polygon. The sum is said to be *independent* of the number of sides.

7. Draw a polygon having four sides and divide it into triangles by drawing diagonals from one vertex to each of the others. Similarly, draw polygons having five, six, or more sides and divide each into triangles by drawing diagonals from one vertex. Then complete the following table:

Number of Sides of Polygon	Number of Triangles	Sum of Interior Angles
4 . . .	2 or 4–2	(4–2) 180°
5 . .	3 or 5–2	(5–2) 180°
6 . .	4 or 6–2	(6–2) 180°
7 .		
8 .		
9 .		
n		

State a theorem expressing the sum of the interior angles of a polygon having n sides.

8. Prove that *the sum of the interior angles of a polygon of n sides is* $(n-2)$ *straight angles.*

Given polygon $ABCD$ ...., Figure 66, having n sides.

To prove $a+b+c+ \ldots . = (n-2)\ 180$.

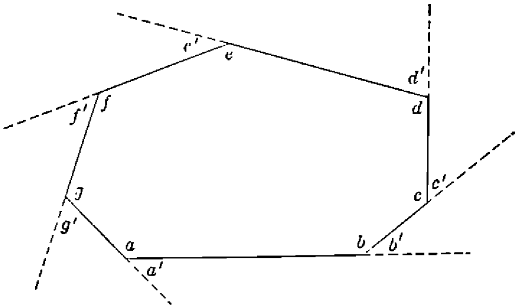


FIG. 66

Proof: Extend one side at each vertex forming the exterior angles $a', b', c', \ldots$	
Then $a+a'=180$. $b+b'=180$. $c+c'=180$, etc.,	For they are supplementary adjacent angles.
$\therefore\ a+b+c+\ldots\ldots$ $+a'+b'+c'+\ldots . = n\times 180 = 180n$.	If equal numbers are added to equal numbers the sums are equal.
$a'+b'+c'+\ldots . = 360$.	Exercise 6.
$\therefore\ a+b+c+\ldots . = 180n-360$. $= 180(n-2)$, $a+b+c+\ldots . = (n-2)\ 180$.	If equal numbers are subtracted from equal numbers the differences are equal.

Thus it is seen that the sum of the interior angles of a polygon *depends* upon the number of sides.

9. Prove that *an exterior angle of a triangle is equal to the sum of the two remote interior angles.*

Suggestion: Draw BE parallel to AC, Figure 67.

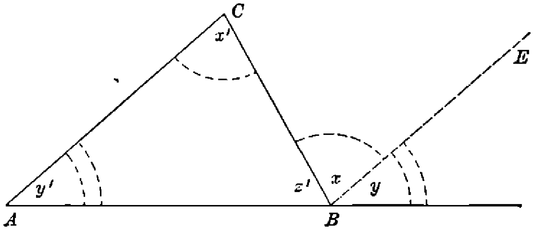


FIG. 67

Prove that $x = x'$.
and $y = y'$.
Complete the proof.

10. Using Exercise 9, show that an exterior angle of a triangle is greater than either of the two remote interior angles.

11. Using Figure 67, prove that **the sum of the angles of a triangle is 180°.*

Suggestions: $x + y + z' = 180$.
$\therefore \quad x' + y' + z' = 180$. Why?

12. Prove that the bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base, Figure 68.

13. If two straight lines are cut by a transversal making the sum of the interior angles on the same side not equal to 180°, the lines are not parallel. Prove.

Suggestion: Use the indirect method of proof.

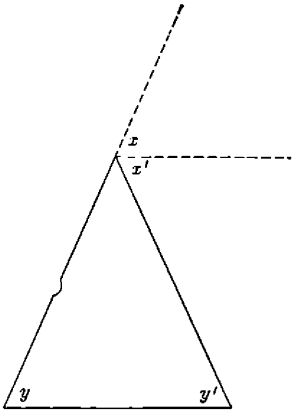


FIG. 68

14. Using Figure 69, prove that two lines perpendicular to two intersecting lines are not parallel.

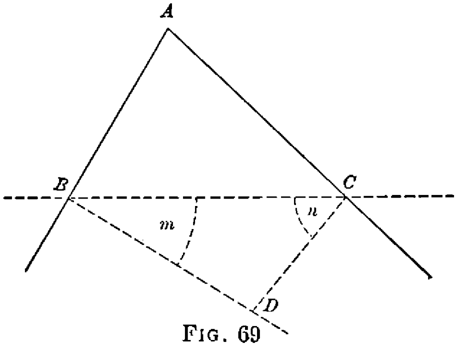


FIG. 69

Suggestions: Draw BC.

$\angle ABD = 90°$.

$\therefore \quad m < 90.$

Similarly, $n < 90.$

$\therefore \quad m + n < 180$, etc.

(If unequal numbers are added to unequal numbers of the same order, the sums are unequal in the same order.)

15. The sum of the angles of a polygon is 180°. Find the number of sides, using the equation $s = (n-2)\ 180$.

16. If the sum of the interior angles of a polygon is twice the sum of the exterior angles, what is the number of sides of the poylgon?

17. If two angles of a quadrilateral are supplementary, show that the other two are supplementary.

18. A polygon that is equilateral and equiangular is called *regular*. Prove that an interior angle of a regular polygon is $\left(\frac{n-2}{n}\right)180°$.

19. How many sides has a polygon the sum of whose angles is 720°? 18 straight angles? 36 right angles?

60. Supplementary exercises. Prove the following:

1. If a line bisects one of two opposite angles, it bisects the other also.

2. The bisectors of two adjacent supplementary angles are perpendicular to each other.

3. Divide an angle into four equal parts using only compass and straight edge.

4. If two angles have their sides perpendicular they are either equal or supplementary.

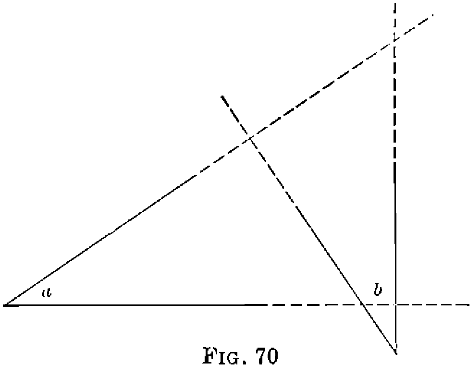


FIG. 70

Suggestions: In Figure 70 use the principle that if two angles of one triangle are equal to two angles of another, the third angles are equal.

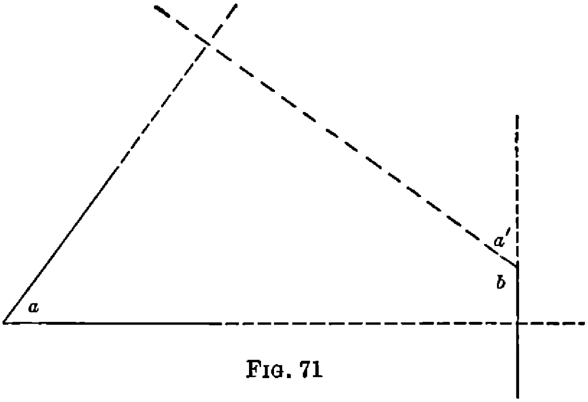


FIG. 71

In Figure 71, show first that $a=a'$.

Then $$a'+b=180, \text{ etc.}$$

5. If the perpendiculars from the midpoint of one side of a triangle to the other two sides are equal, the triangle is isosceles.

6. If a line bisects an angle of a triangle and is perpendicular to the opposite side, the triangle is isosceles.

7. The line segments drawn from the vertices of the base angles of an isosceles triangle to the midpoints of the opposite sides are equal.

8. A line parallel to the base of an isosceles triangle forms an isosceles triangle with the parts cut off from the equal sides of the given triangle.

Summary of Chapter II

61. Geometric symbols. The following symbols were introduced in the study of this chapter:

Symbols	Meaning
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$>$	is greater than
$<$	is less than

62. Geometric terms. The meaning of each of the following terms has been taught:

parallel lines	alternate interior angles
axiom	interior angles on the same side
transversal	indirect method of proof
diagonal	converse of a theorem
corresponding angles	exterior angle of a polygon

63. Geometric principles and constructions. The following is a list of the principles established in chapter ii: Many of these principles can be expressed as algebraic equations. Hence, they have been used to develop skill in solving equations.

1. **Through a given point outside of a given line, one and only one line can be drawn parallel to the given line.*

2. *If two lines are parallel to a third line they are parallel to each other.*

3. **From a point outside of a given line, one and only one line can be drawn perpendicular to the given line.*

4. *Two lines perpendicular to the same line are parallel.*

5. *If two lines are cut by a transversal they are parallel:*
   (1) ⊛*If the alternate interior angles are equal.*
   (2) *If the corresponding angles are equal.*
   (3) *If the interior angles on the same side of the transversal are supplementary.*

6. *If two parallel lines are cut by a transversal:*
   (1) ⊛*The alternate interior angles are equal.*
   (2) *The corresponding angles are equal.*
   (3) *The interior angles on the same side of the transversal are supplementary.*

7. **If one of two parallel lines is perpendicular to a third line the other is also.*

8. *If two angles have their sides parallel they are either equal or supplementary.*

9. *The sum of the exterior angles of a polygon is 360°.*

10. *The sum of the interior angles of a polygon having n sides is $(n-2)$ straight angles.*

11. *An exterior angle of a triangle is equal to the sum of the two remote interior angles.*

12. ⊛*The sum of the angles of a triangle is 180°.*

The following construction has been taught:

13. *Through a given point construct a line parallel to a given line.*

64. Algebraic skills. The chapter has given practice in solving verbal problems, in eliminating literal numbers from the equations, and in solving linear equations.

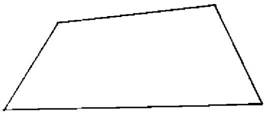
CHAPTER III

QUADRILATERALS

Properties of Parallelograms

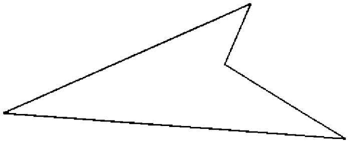
65. Quadrilaterals. A four-sided polygon is a **quadrilateral.**

Quadrilaterals are either *convex,* Figure 72, or *concave,* Figure 73. In a convex polygon no side when extended



Convex Quadrilateral

FIG. 72



Concave Quadrilateral

FIG. 73

passes through the interior of the polygon. The polygons to be studied in this course are convex.

66. Parallelogram. A quadrilateral whose opposite sides are parallel is a **parallelogram.** The symbol for parallelogram is ▱.

EXERCISES

1. Point out parallelograms in the classroom.

2. Draw a quadrilateral no two sides of which are parallel.

67. Diagonal of a parallelogram. A line joining two vertices of a parallelogram that do not lie on the same side is a **diagonal.**

68. Theorem: ⊛*A diagonal of a parallelogram divides it into two congruent triangles.*

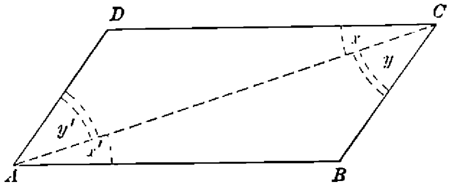


FIG. 74

Given ▱ $ABCD$, Figure 74, i.e., $AB \parallel DC$ and $AD \parallel BC$. The diagonal AC.

To prove

$\triangle ADC \cong \triangle ABC$.

Proof: $AB \parallel DC$.	Given.
$\therefore$ $x = x'$.	Alternate interior angles formed by two parallels and a transversal are equal.
$AD \parallel BC$.	Given.
$y = y'$.	Why?
$AC \equiv AC$.	
$\triangle ADC \cong \triangle ABC$.	Two triangles having two angles and a side of one equal to two angles and a side of the other are congruent.

EXERCISES

1. Using § 68, prove that *the opposite sides of a parallelogram are equal.*

2. Using § 68, prove that *the opposite angles of a parallelogram are equal.*

3. Using Exercise 3, § 58, prove that *the consecutive angles of a parallelogram are supplementary.*

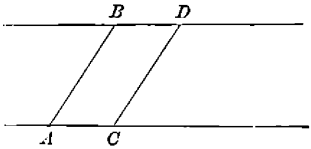


FIG. 75

4. Prove that parallel line segments, Figure 75, intercepted by parallel lines are eoual, i.e., prove that $AB = CD$.

69. Theorem: *The diagonals of a parallelogram bisect each other.*

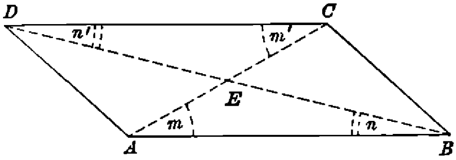


FIG. 76

Given ▱ $ABCD$, Figure 76, with the diagonals AC and DB intersecting at E.

To prove $AE = EC$.
$DE = EB$.

Proof:	$DC \parallel AB$.	Given.
∴ and	$m = m'$ $n = n'$.	If two parallel lines are cut by a transversal, the alternate interior angles are equal.
	$DC = AB$.	The opposite sides of a parallelogram are equal.
	$\triangle AEB \cong \triangle DEC$.	Two triangles are congruent if two angles and a side of one are equal to the corresponding parts of the other.
∴ and	$AE = EC$ $DE = EB$.	Corresponding sides of congruent triangles are equal.

EXERCISE

Prove the theorem of § 69 by using the triangles AED and BEC.

Conditions Under Which a Quadrilateral Is a Parallelogram

70. Theorem: ⊛ *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

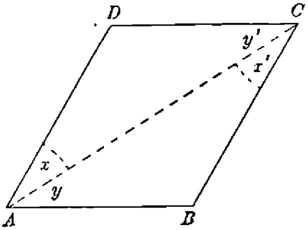


FIG. 77

Given the quadrilateral $ABCD$, Figure 77. $AB = DC$; $AD = BC$.

To prove $ABCD$ a parallelogram.

Proof: Draw AC.	
$AB = DC$.	Given.
$AD = BC$.	Given.
$AC \equiv AC$.	
$\triangle ABC \cong \triangle ADC$.	Two triangles are congruent if three sides of one are equal to three sides of the other.
$x = x'$.	Corresponding angles of congruent triangles are equal.
$AD \parallel BC$.	Two lines cut by a transversal are parallel if the alternate interior angles are equal.
Similarly, prove $AB \parallel DC$.	
$\therefore$ $ABCD$ is a parallelogram.	If the opposite sides of a quadrilateral are parallel, the quadrilateral is a parallelogram.

EXERCISES

1. Construct a parallelogram, Figure 78, using the principle of § 70.

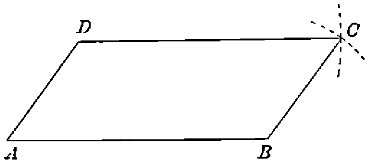


FIG. 78

2. Two opposite sides of a parallelogram are denoted by $3x+5$ and $32x-24$. Determine x and the lengths of the sides.

3. The shelf A, Figure 79, when moved about in space always remains in horizontal position. Explain this fact by means of one of the properties of parallelograms.

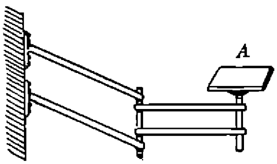


FIG. 79

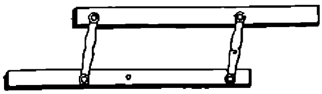


FIG. 80

4. A *parallel ruler*, Figure 80, is used to draw parallel lines. On what principle of parallelograms is the construction of the parallel ruler based?

5. Prove that the bisectors of the opposite angles of a parallelogram are parallel.

6. If one of the angles of a parallelogram is a right angle, prove that all angles are right angles.

7. Two opposite angles of a parallelogram are $8x-6$ and $5x+12$. Find x and the angles.

8. The opposite sides of a parallelogram are $4x+8$ inches and $6x-2$ inches.

Find the length of the sides.

71. Theorem: ⊛*If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.*

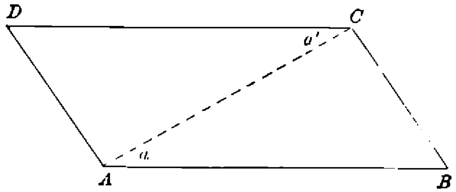


FIG. 81

Given the quadrilateral $ABCD$, Figure 81. $AB = DC$; $AB \parallel DC$.

To prove $ABCD$ a parallelogram.

Proof: Draw the diagonal AC.	
$AB \parallel DC$.	Given.
$a = a'$.	If two parallel lines are cut by a transversal, the alternate interior angles are equal.
$AB = DC$.	Given.
$AC \equiv AC$.	
$\triangle ACD \cong \triangle ACB$.	Two triangles are congruent if two sides and the included angle of one are equal to the corresponding parts of the other.
$AD = BC$.	Corresponding sides of congruent triangles are equal.
$\therefore$ $ABCD$ is a parallelogram.	If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.

EXERCISES

1. Prove that a line joining the midpoints of two opposite sides of a parallelogram divides it into two parallelograms.

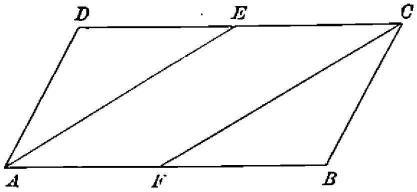


FIG. 82

2. Points E and F, Figure 82, are the midpoints of two opposite sides of the parallelogram $ABCD$. Prove that $AFCE$ is a parallelogram.

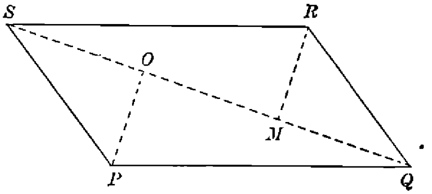


FIG. 83

3. In parallelogram $PQRS$, Figure 83, $RM \perp SQ$ and $PO \perp SQ$. Prove $RM = PO$.

4. Two opposite sides of a parallelogram are $8y-4(4y-5)$ and $5(4y+3)-16$. Find y and the sides.

5. Prove that lines OP and RM, Figure 83, are parallel to each other.

6. Prove the theorem of § 71, if it is given that AD and BC are equal and parallel.

7. Prove that the quadrilateral formed by joining the midpoints of the sides of a parallelogram is a parallelogram.

72. Theorem: *If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.*

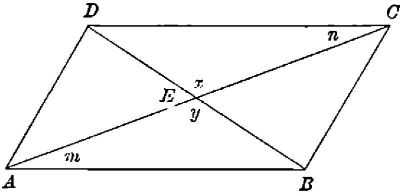


FIG. 84

Given the quadrilateral $ABCD$, Figure 84. $AE = EC$; $DE = EB$.

To prove $ABCD$ a parallelogram.

Proof: $AE = EC$.	Given.
$DE = EB$.	Given.
$x = y$.	Opposite angles are equal.
$\triangle AEB \cong \triangle DEC$.	Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.
$DC = AB$.	Corresponding sides of congruent triangles are equal.
$m = n$.	Corresponding angles of congruent triangles are equal.
$AB \parallel DC$.	If two lines are cut by a transversal making the alternate interior angles equal the lines are parallel.
$\therefore$ $ABCD$ is a parallelogram.	If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

EXERCISES

1. Prove that a line segment, PQ, Figure 85, passing through

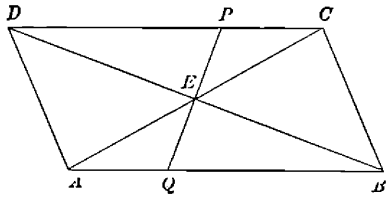


FIG. 85

the point of intersection E of the diagonals of a parallelogram and intercepted by two sides is bisected at E.

Suggestion: Use the method of congruent triangles.

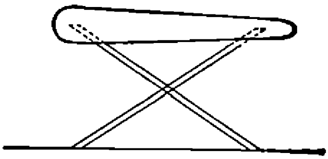


FIG. 86

2. Explain why the ironing board, Figure 86, adjusted to any height is always parallel to the floor.

3. The opposite sides of a parallelogram are $\frac{3x-17}{x+3}+\frac{5x-4}{14}$ and $\frac{3x-1}{7}-\frac{2x+11}{28}$. Determine the value of x.

4. If the diagonals of a quadrilateral are perpendicular to each other and bisect each other the quadrilateral is an equilateral parallelogram.

5. If the diagonals of a quadrilateral are equal and bisect each other, the quadrilateral is an equiangular parallelogram.

73. Theorem: *If the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.*

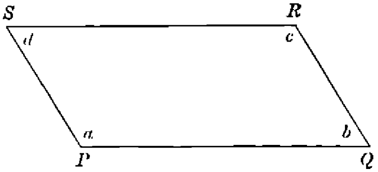


FIG. 87

Given the quadrilateral $PQRS$, Figure 87, with $a=c$, $b=d$.

To prove $PQRS$ a parallelogram.

Proof: $a+b+c+d=360$.	The sum of the angles of a quadrilateral is 360°.
$a=c$.	Given.
$b=d$.	Given.
$2a+2b=360$.	By substitution, and by combining similar terms.
$a+b=180$.	Halves of equal numbers are equal.
$PS \parallel QR$.	If two lines are cut by a transversal making the interior angles on the same side supplementary, the lines are parallel.
Similarly, prove $PQ \parallel SR$.	
$\therefore$ $PQRS$ is a parallelogram.	If the opposite sides of a quadrilateral are parallel, the quadrilateral is a parallelogram.

EXERCISES

1. If $ABCD$, Figure 88, is a parallelogram, $PQ \perp AB$, and $RS \perp AB$, prove that $PQ = RS$.

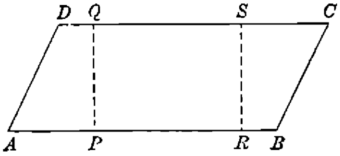


FIG. 88

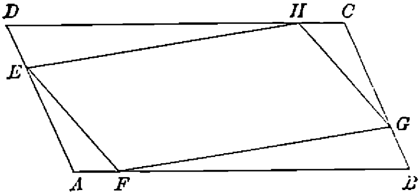


FIG. 89

2. In the parallelogram, Figure 89, $DE = AF = BG = CH$. Prove $EFGH$ is a parallelogram.

3. Make a summary of the properties of a parallelogram which are discussed in chapter iii.

4. Make a summary of the conditions under which a quadrilateral is a parallelogram.

5. Determine the value of y, if $y+30$ and $17(y-2)$ denote the number of degrees in the opposite angles of a parallelogram.

6. Two opposite angles of a parallelogram are denoted by

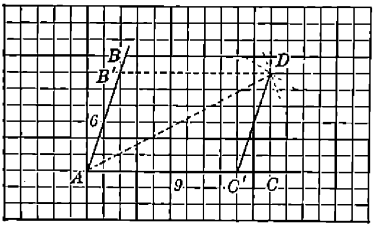


FIG. 90

$5x-5$ and $29-3y$. One of the remaining angles is $182-4y-2x$. Find x, y, and the number of degrees in each angle.

7. A boat, Figure 90, is being rowed across a stream in the direction AB, 30° east of north at a rate of 6 miles an hour, while the current moves it east in the direction AC at the rate of 9 miles an hour. Find the direction in which the boat is actually moving.

Solution: On squared paper lay off 9 units on AC, and 6 units on AB. Thus, $AC' = 9$ and $AB' = 6$.

Construct with ruler and compass the parallelogram $AC'DB'$. Draw the diagonal AD. Then the *direction* of the diagonal AD is the direction in which the boat is actually moving, and the *length* of AD is the number of miles the boat has passed over in one hour.

Since the sides AC' and AB' represent the velocity of the current and the rate of rowing, respectively, $AC'DB'$ is called the *parallelogram of velocities*. AC' and AB' are the *component velocities*. The diagonal AD is the *resultant velocity*.

8. A boat is being rowed southward with a force which would carry it 5 miles an hour in still water, and the wind is driving it southwestward with a force which carries it 3 miles an hour. Construct the parallelogram of forces and the resultant. Then state the distance and direction of the boat from the starting-point after one hour.

9. Two forces of 26 pounds and 54 pounds are exerted upon an object at an angle of 80° to each other. Find the resultant.

10. Two opposite sides of a parallelogram are respectively $7x$ and $4y+19$ inches long. The other sides are $4x$ and $3y+8$ inches. Find x and y.

Rectangles

74. What is meant by a rectangle. If the adjacent sides of a parallelogram intersect at right angles, Figure 91, the parallelogram is called a **rectangle.**

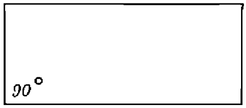


FIG. 91

EXERCISES

1. Prove that the diagonals of a rectangle are equal.

2. State and prove the converse of Exercise 1.

3. Make a summary of the properties of a parallelogram which are true for the rectangle.

4. Prove that the bisectors of the angles of a parallelogram form a rectangle.

Squares

75. Meaning of a square. A **square** is a rectangle whose sides are equal. Since a square is equilateral and equiangular, it is called a *regular* quadrilateral.

EXERCISES

1. Prove that the diagonals of a square are equal.

2. Prove that the diagonals of a square bisect the angles of the square.

3. Prove that the diagonals of a square intersect at right angles.

4. If the diagonals of a quadrilateral are equal, bisect each other, and are perpendicular to each other, the quadrilateral is a square. Prove.

Trapezoids

76. What is meant by a trapezoid. A quadrilateral which has one, and only one, pair of parallel sides is a **trapezoid,** Figure 92.

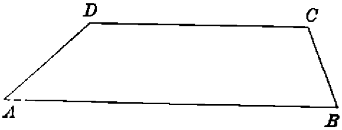


FIG. 92

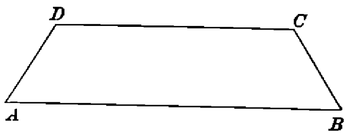


FIG. 93

If the two non-parallel sides are equal, the trapezoid is **isosceles,** Figure 93.

EXERCISES

1. If a trapezoid is isosceles, the base angles are equal. Prove.

Suggestions: Draw $DE \parallel CB$, Figure 94.

Show that $AD = BC = ED$.

$\therefore \quad a = b$. For the base angles of an isosceles triangle are equal.

Show that $\quad b = c$.

$\therefore \quad a = c$.

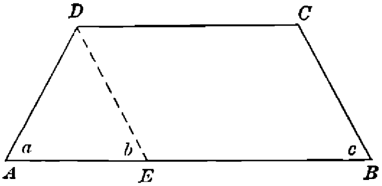


FIG. 94

2. If the base angles of a trapezoid are equal, prove that the trapezoid is isosceles.

3. Prove that the diagonals of an isosceles trapezoid are equal.

77. Supplementary exercises. Prove the following:

1. If a parallelogram is equilateral but not equiangular, it is a **rhombus,** Figure 95. Prove that the diagonals of a rhombus bisect each other.

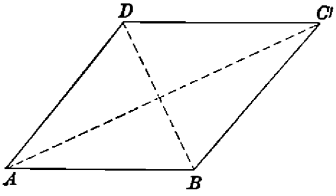


FIG. 95

2. Prove that the diagonals of a rhombus are perpendicular to each other.

3. Prove that the diagonals of a rhombus bisect the angles.

4. Prove that the altitudes of a rhombus are equal, i.e., that the perpendicular distances between AB and DC, Fig. 95, and BC and AD are equal.

5. With ruler and compass construct a rhombus having one of the sides equal to 8 centimeters and the acute angles equal to 45°.

6. Given the diagonals of a rhombus, construct the rhombus.

7. Prove that the sum of the sides of a quadrilateral is greater than the sum of the diagonals and less than twice the sum.

Suggestions: $AB+BC>AC$, Figure 96, since the shortest distance between two points is measured along the straight line joining them. For the same reason $AE+EB>AB$.

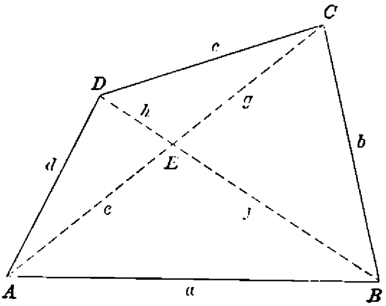


FIG. 96

Proof: $a+b>AC$.	Why?
$c+d>AC$.	
$a+d>BD$.	
$b+c>BD$.	
$\therefore 2a+2b+2c+2d>2AC+2BD$.	If unequals are added to unequals of the same order the sums are unequal in the same order.
$\therefore a+b+c+d>AC+BD$.	
$a<e+f$.	
$b<f+g$.	
$c<g+h$.	
$d<h+e$.	
$a+b+c+d<2e+2g+2f+2h$.	Why?
$a+b+c+d<2(e+g+f+h)$,	
or $a+b+c+d<2(AC+BD)$.	

8. If from a point within a triangle line segments are drawn to the end points of a side, their sum is less than the sum of the other two sides. Prove.

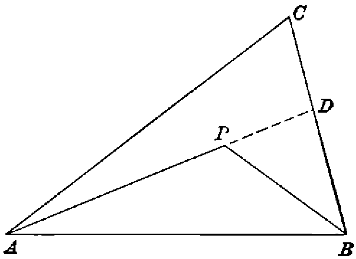


FIG. 97

Suggestions: Extend AP, Figure 97, until it intersects BC as at D.

Then $PB < PD + DB$.	The shortest distance between two points is the straight line joining the points.
$AP + PB < AP + PD + DB$	Adding equals to unequals gives unequal sums.
and $AP + PB < AD + DB$.	Substituting the whole for the sum of the parts.
$AD < AC + CD$.	Why?
$\therefore AP + PB + AD < AD + DB + AC + CD$.	By adding unequals to unequals.
$AP + PB < DB + AC + CD$.	Why?
$AP + PB < AC + CB$.	Why?

9. Prove that if each of two points on a given line is equally distant from two points, the given line is the perpendicular bisector of the segment joining the given points.

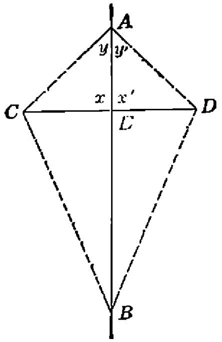


FIG. 98

Given the line AB, Figure 98, and the points C and D such that

$$AC = AD \text{ and } BC = BD.$$

To prove $x = x'$ and $CE = ED$.

Proof:

STATEMENTS	REASONS
$AC = AD$, $BC = BD$.	By hypothesis.
$AB \equiv AB$.	Common to both triangles ACB and ADB.
$\therefore \triangle ACB \cong \triangle ADB$.	Why?
Hence, $y = y'$.	Corresponding parts of congruent triangles are equal.
$AE \equiv AE$.	Common to both triangles.
$AC = AD$.	By hypothesis.
$\therefore \triangle ACE \cong \triangle ADE$.	Why?
Hence, $x = x'$, and $CE = ED$.	Corresponding parts of congruent triangles are equal.

10. The sum of the perpendiculars from a point on the base of an isosceles triangle to the two equal sides is equal to the altitude to one of these sides.

11. The sum of the perpendiculars from a point within an equilateral triangle to the three sides is equal to the altitude.

Make the following constructions:

12. Given a side and the diagonal of a rectangle, construct the rectangle.

13. Given a side and an angle of a rhombus, construct the rhombus.

14. Given the diagonal of a square, construct the square.

15. Given the diagonals of a rhombus, construct the rhombus.

Solve the following systems of equations:

16. $9r+s=5.$
$6r+5s=12.$

17. $5x+2y=4.$
$3x-5y=21.$

Summary of Chapter III

78. A topical organization of the chapter. The facts and principles taught are stated below.

A) Terms and geometric concepts:

convex quadrilateral	square
concave quadrilateral	rhombus
parallelogram	trapezoid
rectangle	isosceles trapezoid

B) Properties of parallelograms.

If a quadrilateral is a parallelogram:

1. *A diagonal divides it into congruent triangles.*⊛
2. *The opposite sides are equal.*
3. *The opposite angles are equal.*

4. *The consecutive angles are supplementary.*
5. *The diagonals bisect each other.*

C) Conditions under which a quadrilateral is a parallelogram.

A quadrilateral is a parallelogram:

1. *If the opposite sides are equal.*®
2. *If two sides are equal and parallel.*®
3. *If the diagonals bisect each other.*
4. *If the opposite angles are equal.*

D) Special cases of parallelograms. Important properties of the following figures were studied:

rectangle, square, trapezoid, and isosceles trapezoid.

E) Geometric constructions:

1. Construct a parallelogram having given the lengths of two adjacent sides and the included angle.

2. Construct a rhombus having given the length of the sides and the acute angle.

F) Algebraic abilities:

1. To solve linear equations, such as

$$3x+2=5x-28.$$

2. To eliminate an unknown number by substitution and by comparison. Thus, if $a+b=c$ and $a=2b$, eliminate a. Or, if $a=b$ and $b=c$, eliminate b.

3. To solve linear equations in two unknowns.

CHAPTER IV

RELATIONS BETWEEN LINE SEGMENTS FORMED BY PARALLEL LINES INTERSECTING TRANSVERSALS.

Proportions

79. What is meant by the ratio of two segments. *To measure* a line segment means to find how many times it contains a *unit segment.* **Unit segments** are as a rule such well-defined standards of length as the inch, foot, yard, and centimeter. Thus, the measure of AB, Figure 99, is 5 when the centimeter is used as the unit of measure.

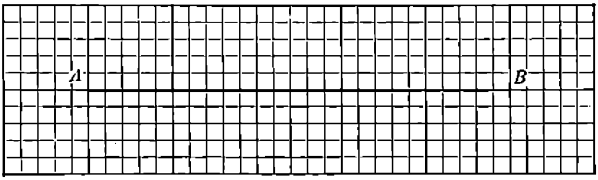


FIG. 99

Unknown measures are denoted by small letters, called **literal numbers.** Thus, the length of AB, Figure 99, may be denoted by a. The number which a stands for may be determined by measuring AB.

By the **ratio of two segments** is meant the ratio of their numerical measures, both being measured with the same unit.

EXERCISES

1. The ratio of two segments is $\frac{3}{4}$. If one segment is 27 inches, find the length of the other.

2. Find the value of x if $\frac{24}{x}=\frac{3}{7}$; $\frac{18}{5}=\frac{7}{x}$; $\frac{2\frac{1}{2}}{3}=\frac{6}{x}$.

80. A geometric method of finding a unit common to two segments. The ratio of two line segments may be found by means of the compass, as follows:

Let AB and CD, Figure 100, be two segments whose ratio is to be found.

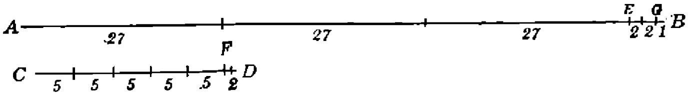


FIG. 100

Lay off the smaller segment, CD, on the larger, AB, as often as possible, leaving a remainder, EB, which is less than CD.

Lay off EB on CD, leaving a remainder FD, which is less than EB.

Lay off FD on EB, leaving a remainder $GB < FD$.

Lay off GB on FD and continue the process until there is no remainder. The last remainder is a common unit of measure of AB and CD.

Assume that GB is the last remainder.

Using GB as unit, show that $AB = 86$, and $CD = 27$.

Therefore $\frac{AB}{CD} = \frac{86.}{27.}$

81. Commensurable magnitudes. In § 80 it was assumed that a unit was contained exactly in both AB and CD. Two magnitudes which have a common unit of measure are said to be **commensurable.**

82. Incommensurable magnitudes. Not all magnitudes have a common unit of measure. Magnitudes not having a common unit of measure are said to be **incommensurable.**

EXAMPLE

The side and diagonal of a square are incommensurable segments. This may be seen as follows:

Since $AB<AC$, Figure 101, AB may be laid off on AC, leaving a remainder B_1C.

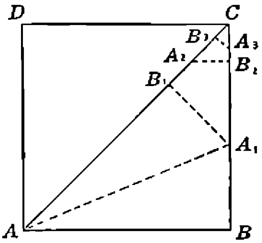


Fig. 101

Draw $B_1A_1 \perp AC$ at B_1.

$$AB_1 = AB.$$
$$AA_1 \equiv AA_1.$$
$$\angle B_1 = \angle B.$$
$$\therefore \triangle AB_1A_1 \cong \triangle ABA_1.$$
$$\therefore B_1A_1 = A_1B.$$

Prove $B_1C = B_1A_1 = A_1B$.

Since $B_1C < A_1C$, it follows that B_1C may be laid off on A_1C leaving a remainder, as B_2C.

Thus, B_1C may be laid off on BC twice, leaving a remainder, B_2C.

In the same way it may be shown that B_2C may be laid off on B_1C twice, leaving a remainder, B_3C; that B_3C may be laid off twice on B_2C, leaving a remainder, etc.

In each case the process is a repetition of the preceding case, only with smaller segments. Since in each case there is a remainder, the process may be kept up indefinitely.

Hence, *no common unit of AB and AC can be found.*

It will be seen later that the ratio $\frac{AC}{AB} = \sqrt{2}$, which is an *irrational* number, i.e., a number which cannot be expressed exactly in terms of integers, or of fractions whose terms are integers.

Since all measurement is approximate, the distinction between commensurable and incommensurable segments does not occur in practical work. When measurement is required, engineers and draftsmen select a unit sufficiently small to allow them to neglect, without noticeable effect upon the final result, the remainder, which can always be made less than the unit used.

83. Theorem:⊛ *If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.*

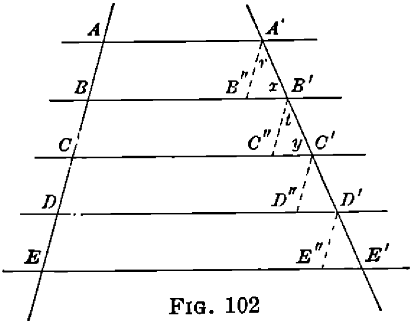


FIG. 102

Given $AA' \parallel BB' \parallel CC'$, etc., Figure 102.

$$AB = BC = CD, \text{ etc.}$$

To prove $A'B' = B'C' = C'D'$, etc.

Proof:

Draw $A'B'' \parallel AB$, $B'C'' \parallel BC$, $C'D'' \parallel CD$, etc.	
Since $B'B'' \parallel C'C''$,	Given.
$x = y$.	If two parallels are cut by a transversal, the corresponding angles are equal.
Show that $A'B'' \parallel B'C''$.	Two lines parallel to the same line are parallel to each other.
$r = t$.	

Show that $A'B'' = AB = BC = B'C''$.	
$\therefore \quad \triangle B''A'B' \cong \triangle C''B'C'$.	Two triangles are congruent if a side and two angles of one are equal to the corresponding parts of the other.
$A'B' = B'C'$.	Corresponding sides of congruent triangles are equal.
Similarly, $B'C' = C'D'$, etc.	

EXERCISES

1. *If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side and is equal to one-half of the second.* Prove.

Suggestions:

I. Draw $GCH \parallel AB$, Figure 103.

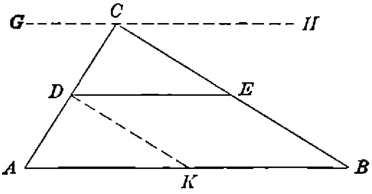


FIG. 103

$CD = DA$.	Given.
$CE = EB$.	If three parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.
II. Draw $DK \parallel CB$.	
Then $AK = KB$.	Why?
$\therefore \quad KB = \frac{1}{2}AB$.	Since it is one of the two equal parts of AB.
$DE = KB$.	Parallels intercepted by parallels are equal.
$DE = \frac{1}{2}AB$.	By substitution.

2. If a line bisects one of the non-parallel sides of a trapezoid and is parallel to the base, it bisects the other of the non-parallel sides and is equal to one-half the sum of the bases. Prove.

Suggestion:

Since $DC \parallel EF \parallel AB$, Figure 104, and since $DE = EA$, show that $CF = FB$.

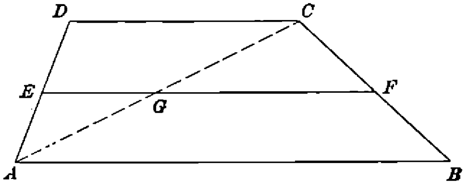


FIG. 104

Draw AC.

Show that $GF = \frac{1}{2}AB$

and that $EG = \frac{1}{2}DC$.

$\therefore \quad EG + GF = EF = \frac{1}{2}(DC + AB)$.

3. *Divide a given line segment into a given number of equal parts.*

Given line segment AB, Figure 105.

Required to divide AB into equal parts.

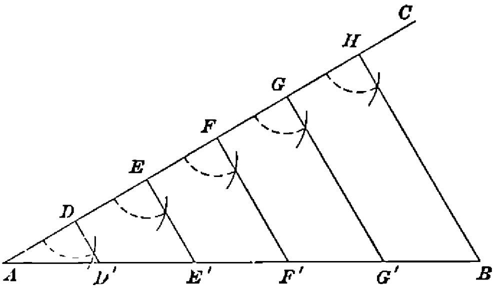


FIG. 105

Construction: Draw AC.

Beginning at A, lay off on AC equal lengths, such as AD, DE, EF, FG, and GH.

Draw HB.

Construct $GG' \parallel HB$, $FF' \parallel HB$, $EE' \parallel HB$, and $DD' \parallel HB$.

Then AD', $D'E'$, $E'F'$, $F'G'$, and $G'B$ are the required equal parts of AB.

Proof: $DD' \parallel EE' \parallel FF' \parallel GG' \parallel HB$. By construction.

$AD = DE = EF = FG = GH$. By construction.

$AD' = D'E' = E'F' = F'G' = G'B$. Why?

4. Prove that the midpoint D of the hypotenuse of a right triangle, Figure 106, is equidistant from the vertices.

Suggestions:

Draw $DE \parallel AB$.

Prove $\triangle CDE \cong \triangle BDE$.

$\therefore \quad DB = DC$.

Show that $DA = DB$.

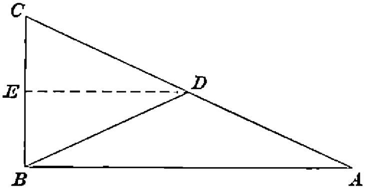


FIG. 106

5. If $CD = 3$ centimeters, Figure 103, $DA = 3$ centimeters, $CE = 4$ centimeters, and $EB = x - 2$ centimeters, find x and the length of EB.

6. If $CD = DA$, $CE = \frac{3x-2}{4} - \frac{4x-15}{5}$, $EB = \frac{7x+5}{10} - 4$, find x and the length of CE and EB.

84. Median of a trapezoid. The line joining the midpoints of the non-parallel sides of a trapezoid as EF, Figure 104, is the **median** of the trapezoid.

85. Proportional segments. Equations of two equal ratios, as $\frac{4}{6} = \frac{2}{3}$, $\frac{1}{5} = \frac{3}{15}$ are *proportions.* Any four segments AB, CD, EF, and GH are *proportional* if $\frac{AB}{CD} = \frac{EF}{GH}$. Thus four segments are **proportional** if the ratio of two is equal to the ratio of the other two.

EXERCISES

1. The lengths of a flagpole and its shadow are proportional to the lengths of a pole and its shadow. If a flagpole casts a shadow 26 feet long, and if at the same time a 6-foot pole casts a shadow $5\frac{1}{4}$ feet long, find the height of the pole.

2. Solve for x: $\quad \frac{x}{x-16} = \frac{5}{3}$; $\frac{x}{8} = \frac{2x+3}{5}$; $\frac{2}{2x+1} = \frac{7}{x}$.

86. Uses of proportional line segments. Figure 107 represents a pair of **proportional compasses.** It is used to make scale drawings of given figures. The instrument consists of two intersecting lines AA' and BB', that can be clamped together at the point of intersection, O, by means of a screw. By making $OB' = \frac{1}{2}OB$ and $OA' = \frac{1}{2}OA$, and by opening the compass so that AB equals a given line segment, we obtain $A'B'$ equals $\frac{1}{2}AB$. You will see that this fact follows from the principle of proportional line segments given in Exercise 5, § 87.

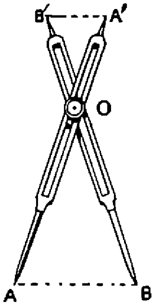


FIG. 107

The **pantograph,** Figure 108, is used to draw figures to definite scales, and to enlarge or to reduce maps, drawings, designs, etc. The instrument consists of four pointed bars, fastened together in such a way as to make BB_1A_2A a parallelogram. According to the principles of proportional line segments, if $\frac{OB}{OB_1}$ is made equal to $\frac{B_1A_2}{B_1A_1}$, points O, A, and A_1 must fall in a straight line and make $\frac{OA}{OA_1} = \frac{OB}{OB_1}$. Keeping point O fixed and making point A describe figure (a) makes the pencil point A_1 describe figure (b). The two figures are similar in shape. In fact, figure (b) is really figure (a) *magnified* to the scale OB_1 to OB.

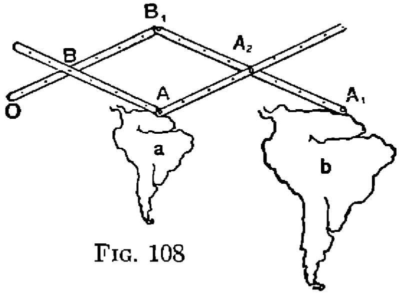


FIG. 108

The **diagonal scale,** Figure 109, is another instrument whose construction is based upon principles of propor-

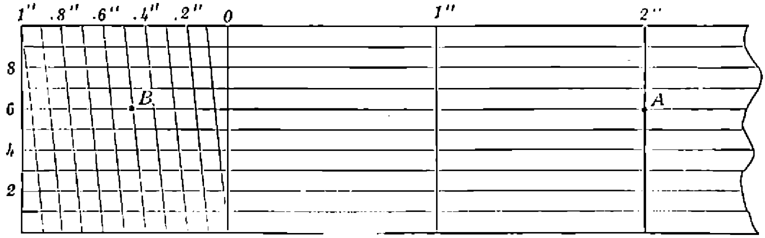


FIG. 109

tional line segments. By means of it lengths may be measured to hundredths of an inch.

Figure 110 represents part of Figure 109, enlarged.

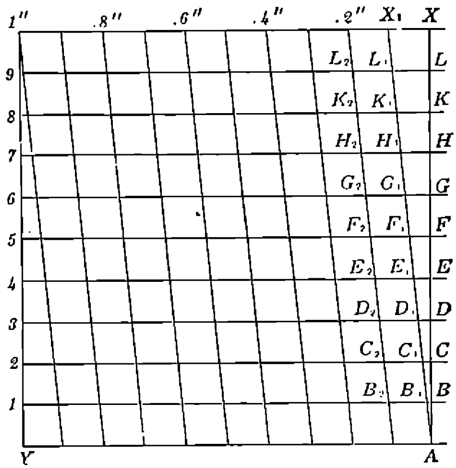


FIG. 110

Note that YA and the side opposite YA are divided into 10 equal parts.

Similarly, AX and the side opposite AX are divided into equal parts. The points of division are then joined by straight lines as shown in the diagram.

By § 87, $\frac{BB_1}{XX_1}=\frac{AB}{AX}=\frac{1}{10}.$

Hence, $BB_1=\frac{1}{10}XX_1.$

Similarly, $CC_1=\frac{2}{10}XX_1,$

$DD_1=\frac{3}{10}XX_1$, etc.

Since $XX_1=\frac{1}{10}AY=\left(\frac{1}{10}\right)''$,

you have $BB_1=.01''$, $CC_1=.02''$, $DD_1=.03''$, etc.

Likewise, $BB_2=.1''+.01''=.11''.$

$CC_2=.1''+.02''=.12''.$

$DD_2=.1''+.03''=.13''$, etc.

EXERCISES

1. Draw three parallel lines. Draw two transversals.

Measure the segments intercepted on one of the transversals, and find their ratio. Measure the segments intercepted on the other transversal and find their ratio.

2. Compare the ratio of the segments on one transversal in Exercise 1 with the ratio of the corresponding segments on the other.

3. State the result of Exercise 2 in the form of a theorem.

4. Compare your statement with the theorem given in § 87.

5. Show that the theorem given in § 83 is a special case of that of § 87.

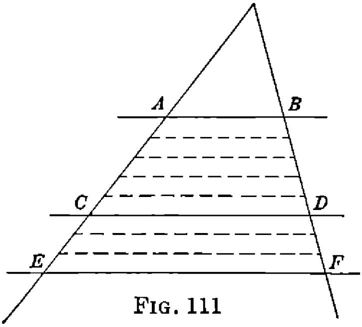


FIG. 111

87. Theorem: ⊛*If two, or more, parallel lines are cut by two transversals, the ratio of two segments on one transversal is equal to the ratio of the corresponding segments on the other.*

Given $AB \parallel CD \parallel EF$, Figure 111, and the transversals AE and BF.

To prove $\frac{AC}{CE}=\frac{BD}{DF},$

$$\frac{AE}{AC}=\frac{BF}{BD},$$

and $\frac{AE}{CE}=\frac{BF}{DF}.$

Proof: Imagine a common unit of measure applied to AC and CE (§ 80).

Suppose that this unit is contained m times in AC and in CE n times (in Figure 111 $m=5$ and $n=3$).

Then $\frac{AC}{CE}=\frac{m}{n}.$ (The ratio of two segments is the ratio of their measures.)

Through each of the points of division on AE draw a line parallel to EF.

These parallels divide BF into equal parts (§ 83), such that to every part on AE there corresponds one and only one part on BF.

Using one of the equal parts of BF as a unit, the numerical measures of BD and DF are, respectively, m and n.

$$\frac{BD}{DF}=\frac{m}{n}. \qquad \text{Why?}$$

$$\frac{AC}{CE}=\frac{BD}{DF}. \qquad \text{Why?}$$

Similarly, show that $\frac{AE}{AC}=\frac{BF}{BD}$, and that $\frac{AE}{CE}=\frac{BF}{DF}$.

Note that this proof applies only to the case when AC and CE are commensurable. The incommensurable case is discussed in § 88.

EXERCISES

1. Prove the foregoing theorem, using Figure 112.

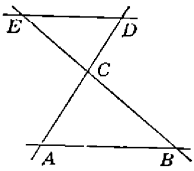


FIG. 112

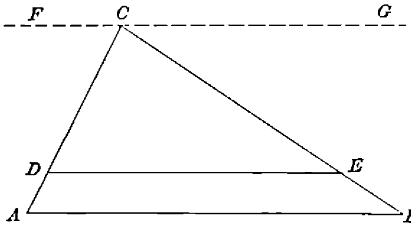


FIG. 113

2. Prove that *if a line is parallel to one side of a triangle it divides the other two proportionally.*

Suggestion: Draw $FG \parallel AB$, Figure 113, and apply § 87.

3. *Construct a segment forming a proportion with three given segments (fourth proportional).*

Given segments a, b, c, Figure 114.

Required to construct a segment x, such that $\frac{a}{b}=\frac{c}{x}$.

Construction: Draw lines AB and AC.

Lay off a, b, and c as shown in the diagram.

Draw DF.

Draw $EG \parallel DF$.

Then FG is the required fourth proportional.

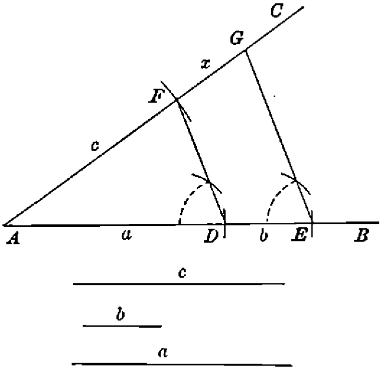


FIG. 114

Prove that $\frac{a}{b}=\frac{c}{x}$.

4. Construct a segment proportional to two given segments.

Suggestion: Let a and b be two given segments. Construct segment x so that $\frac{a}{b}=\frac{b}{x}$ (third proportional).

5. Prove that if two given lines are cut by two parallel lines the segments of the parallel lines are proportional to the corresponding segments of the given lines.

Thus, in Fig. 115, $\frac{CD}{CA}=\frac{DE}{AB}$ and $\frac{CE}{CB}=\frac{DE}{AB}$.

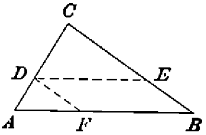


FIG. 115

Suggestion: Draw $DF \parallel CB$, Figure 115. Show that $BF = ED$

$$\text{Then } \frac{CD}{CA}=\frac{BF}{BA}.$$

$$\text{Since } \frac{BF}{BA}=\frac{DE}{BA},$$

$$\text{it follows that } \frac{CD}{CA}=\frac{DE}{BA}.$$

88.† The incommensurable case of the theorem in § 87. The theorem has been proved for the commensurable case. The following is a discussion of the incommensurable proof. The method of proof is indirect.

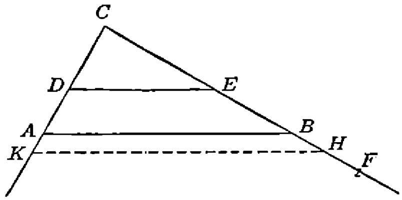


FIG. 116

Assume $\frac{DA}{CD} \neq \frac{EB}{CE}$, Figure 116.

Then either $\frac{DA}{CD} > \frac{EB}{CE}$, or $\frac{DA}{CD} < \frac{EB}{CE}$.

Let $\frac{DA}{CD} > \frac{EB}{CE}$.

Select a point F on the extension of EB, making EF long enough to make $\frac{DA}{CD} = \frac{EF}{CE}$. (1)

Determine a point H between B and F, making CE and EH *commensurable.*

Draw $HK \parallel BA$.

Then, $\frac{DK}{CD} = \frac{EH}{CE}$.

Since $DA < DK$,

it follows that $\frac{DA}{CD} < \frac{EH}{CE}$. (2)

Comparing (2) and (1), you have $\frac{EF}{CE} < \frac{EH}{CE}$.

$EF < EH$.

This is impossible, and the assumption that $\frac{DA}{CD} > \frac{EB}{CE}$ is wrong.

† This section may be omitted or assigned to special pupils.

Similarly, you may prove that $\frac{DA}{CD}$ is not *less* than $\frac{EB}{CE}$.

Hence, $$\frac{DA}{CD}=\frac{EB}{CE}.$$

89. Division of a segment in a given ratio. To divide a segment AB, Figure 117, *internally* in the ratio $\frac{m}{n}$ means to find a point, P, on AB so that $\frac{AP}{PB}=\frac{m}{n}$.

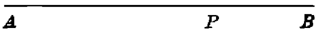


FIG. 117

EXERCISES

1. *To divide a segment internally in the ratio* $\frac{m}{n}$.

Let AB, Figure 118, be the given segment.

Draw a line AC through A and lay off $AD=m$ and $DE=n$.

Draw EB. Through D draw $DF \parallel EB$.

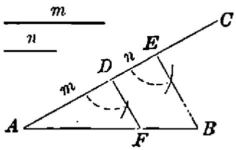


FIG. 118

Then F is the point which divides AB internally in the ratio $\frac{m}{n}$.

Test the correctness of the construction by measuring the segments and comparing the ratios. Give proof.

2. *To divide a given segment AB into segments proportional to several given segments.*

Given segment AB, Figure 119, and segments x, y, and z.

Required to divide AB into segments proportional to x, y, and z.

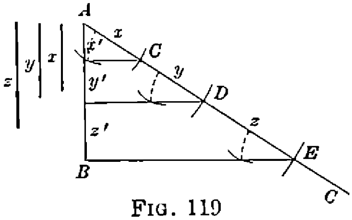


FIG. 119

Construction: Draw a line, as AC.

On AC lay off x, y, and z successively.

Join B to the last point of division, E.

Draw parallels to BE through the points of division C and D. Then x', y', and z' are the required parts of AB.

Show that $\frac{x'}{y'}=\frac{x}{y}$,

and that $\frac{y'}{z'}=\frac{y}{z}$.

90. Theorem: ⊛*Two lines that cut two given intersecting lines and make the corresponding segments of the given lines proportional are parallel* (converse of § 87).

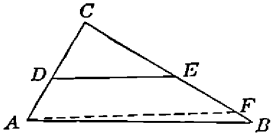


FIG. 120

Given $\frac{CD}{DA}=\frac{CE}{EB}$, Figure 120

To prove $AB \parallel DE$.

Proof (indirect method):

Suppose AB not parallel to DE.

Draw $AF \parallel DE$.

Then, $\frac{CD}{DA}=\frac{CE}{EF}$.	A line parallel to one side of a triangle divides the other two sides proportionally.
But $\frac{CD}{DA}=\frac{CE}{EB}$.	Given.
$\frac{CE}{EF}=\frac{CE}{EB}$.	By substituting $\frac{CE}{EF}$ for $\frac{CD}{DA}$.
$CE \cdot EB = CE \cdot EF$.	Multiplying both sides of the equation by $EF \cdot EB$.

$EB = EF$.	Dividing both sides by CE.
This is impossible.	For the whole segment cannot be equal to a part.

Therefore the assumption that AB is not parallel to DE is wrong, and $AB \parallel DE$.

EXERCISES

Prove the following:

1. If $\frac{CA}{CD} = \frac{CB}{CE}$ (Figure 120), then $DE \parallel AB$.

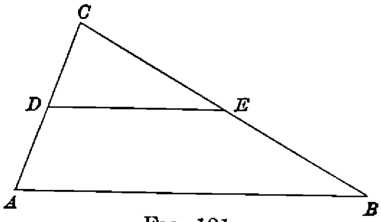


FIG. 121

2. *The line joining the midpoints of two sides of a triangle is parallel to the third side.*

Suggestion: Since $CD = DA$, Figure 121, and $CE = EB$,

it follows that $\frac{CD}{DA} = 1$.

and that $\frac{CE}{EB} = 1$. Why?

$$\frac{CD}{DA} = \frac{CE}{EB}.$$

$$\therefore \quad DE \parallel AB \ (\S\ 90).$$

3. Construct a triangle having given the midpoints of the sides.

4. Prove that the segments joining the midpoints of the sides of a quadrilateral form a parallelogram, Figure 122.

Suggestion: Draw the diagonal AC.

Prove that $EH \parallel FG$, and that $EH = FG$.

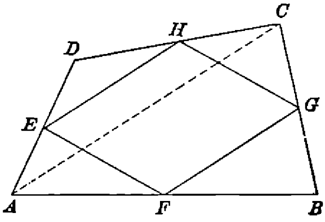


FIG. 122

5. Prove that the segments joining the midpoints of the sides of a given triangle divide it into four congruent triangles.

6. Prove that the line joining the midpoints of the non-parallel sides (median) of a trapezoid is parallel to the bases.

Suggestion: Use the indirect method of proof, as in § 90.

7. Show that the segments connecting the midpoints of the opposite sides of a quadrilateral bisect each other.

8. Draw a triangle and bisect one of the angles. Find the ratio of the segments into which the bisector divides the side opposite the bisected angle. Find the ratio of the other two sides. Compare the two ratios.

91. Theorem: ⊛ *The bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.*

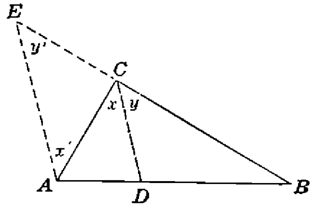


FIG. 123

Given $\triangle ABC$, $x=y$, Figure 123.

To prove: $\frac{AD}{DB}=\frac{AC}{CB}$.

Proof: Extend BC.	
Draw $AE \parallel DC$.	
Then $\frac{AD}{DB}=\frac{EC}{CB}$.	A line parallel to one side of a triangle divides the other two sides proportionally.
$x=x'$.	Why?
$y=y'$.	Why?
$x=y$.	Why?
$x'=y'$.	Why?
$EC=CA$.	If two angles of a triangle are equal, the sides opposite them are equal.
$\frac{AD}{DB}=\frac{AC}{CB}$.	By substituting AC for EC in $\frac{AD}{DB}=\frac{EC}{CB}$.

EXERCISES

1. Prove that a line passing through the vertex of a triangle and dividing the opposite side into segments proportional to the other two sides bisects the angle included between those sides (converse of § 91).

To prove this exercise, retrace the steps in the proof of § 91.

2. If in Figure 123, $AC=8$, $CB=10$, $AB=9$. Find the lengths of AD and DB.

3. If in Figure 123, $AC=5$, $CB=4$, $DB=3$. Find the lengths of AD and AB.

4. If $AC=8$, $CB=16$, and $AB=12$, Figure 123, find AD and DB.

5. If $AC=2$, $CB=3$, $AD=x-2$ and $DB=x+1$, find x and AB.

6. If $AC=a$, $CB=b$, $AD=x-m$, and $DB=x-n$, find x.

7. The sides of a triangle are 15 inches, 30 inches, and 18 inches. Find the segments of the 18-inch side made by the bisector of the opposite angle.

92. External division of a segment. A point, P, on a segment, AB, divides AB into the segments AP and PB, Figure 124. Considering the direction AB as **positive**, and the opposite direction BA as **negative**, then

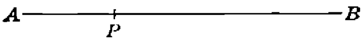


FIG. 124

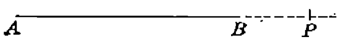


FIG. 125

$$(+AP)+(+PB)=(+AB).$$

If P is on the extension of AB, Figure 125, then AP is positive, and PB is negative. Nevertheless the statement $(AP)+(PB)=(+AB)$ still holds good. Because of this equation, AP and PB are called parts of AB, and AB is said to be divided **externally** by P. Thus, in *external* as in *internal* division of AB the two parts are measured from A to P, and from P to B.

EXERCISES

1. Prove that ®*the bisector of an exterior angle of a triangle divides the opposite side externally into segments that are proportional to the other sides.*

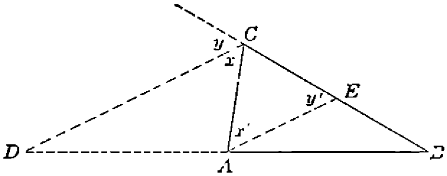


FIG. 126

Given $\triangle ABC, x=y$, Figure 126.

To prove $\dfrac{AD}{DB}=\dfrac{AC}{CB}.$

The proof is practically the same as in § 91.

2. If a segment is divided internally and externally in *the same ratio* it is said to be divided **harmonically.** Prove that the bisector of an interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side harmonically.

3. *Divide a segment externally in the ratio* $\dfrac{m}{n}$.

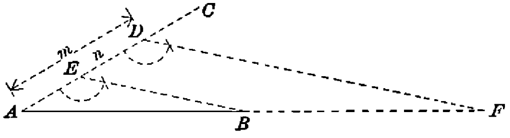


FIG. 127

Draw $AD=m$, Figure 127, $DE=n$. Join E to B and draw $DF \parallel EB$.

Then $\dfrac{AF}{FB}=\dfrac{m}{n}$. Prove.

4. A segment AB, 18 inches long, is divided internally, or externally, at a point, P. What is the ratio $\dfrac{AP}{PB}$ for $AP=2$? 3? 6? 9? 20? 30?

93. Supplementary exercises. Prove the following exercises:

1. If four numbers, or line segments, a, b, c, and d, are in proportion, *i.e.*, if $\frac{a}{b}=\frac{c}{d}$, prove that $ad=bc$.

If the first and last terms in a proportion, as a and d, are called **extremes,** and the second and third, as b and c, are called **means,** the exercise above may be stated as follows: *In a proportion the product of the means is equal to the product of the extremes.*

2. Using the preceding theorem as a test of proportionality, tell which of the following statements are proportions:

$$\frac{15}{9}=\frac{10}{6}; \qquad \frac{8}{15}=\frac{4}{7}; \qquad \frac{6}{18}=\frac{7}{21}; \qquad \frac{4}{7}=\frac{12}{20}.$$

3. The statements below are different arrangements of the four factors in the equation $8 \cdot 7=14 \cdot 4$. Some of them are equations, others only appear to be equations. Apply the *test of proportionality* and point out which statements are proportions.

$a)\ \frac{8}{14}=\frac{4}{7}.$ $c)\ \frac{4}{7}=\frac{8}{14}.$ $e)\ \frac{8}{7}=\frac{14}{4}.$ $g)\ \frac{8}{4}=\frac{7}{14}.$

$b)\ \frac{7}{4}=\frac{14}{8}.$ $d)\ \frac{14}{8}=\frac{7}{4}.$ $f)\ \frac{7}{14}=\frac{8}{4}.$ $h)\ \frac{8}{14}=\frac{7}{4}.$

4. Exercise 3 shows that proportions are formed from the numbers 4, 7, 8, and 14 only when they are taken in a certain order. From what place in the equation $8 \cdot 7=14 \cdot 4$ must the **means** be taken to form a proportion? The **extremes?**

5. Write four proportions from $3 \cdot 28=4 \cdot 21$. Apply in each case the test of proportionality.

6. Write four proportions from $a \cdot 12b = 3a \cdot 4b$, and test.

7. Exercises 5 and 6 illustrate the following theorem: *If the product of two factors is equal to the product of two others, proportions may be formed by taking as means the factors of either product, and as extremes the factors of the other product.* Prove this theorem by dividing both members of the equation $ad = bc$ by bd.

8. Let $ad = bc$. Prove that the following statements are proportions:

$$\frac{a}{c} = \frac{b}{d}. \qquad \frac{b}{a} = \frac{d}{c}. \qquad \frac{c}{a} = \frac{d}{b}. \qquad \frac{c}{d} = \frac{a}{b}.$$

Form proportions from:

9. $5a - 10b = 4x^2 - 3xy$.

10. $a^2 - b^2 = c^2 - d^2$.

11. $p^4 - 16 = a^2 - 64$.

12. $ax + ay + az = br + bs + bt$.

13. $16a^2 - 2axy = ax + ay - az$.

14. $(x+y)^2 = m^2 - 2mx + x^2$.

15. $16a^2 - 25b^2 = 36 - 25y^2$.

16. $6x^2 + 13x + 2 = a^2 + 2a + 1$.

17. $x^2 - 5x + 6 = y^2 + 3y - 28$.

18. Form a continued proportion, as $\frac{7}{12} = \frac{21}{36} = \frac{63}{108}$.

Find the sum of the numerators and the sum of the denominators. Find the ratio of the sums, and compare it with the given ratios.

19. In a proportion, as $\frac{a}{b}=\frac{c}{d}$, the numerators a and c are sometimes called **antecedents**, and the denominators, b and d, are called **consequents.**

Prove that if *two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Thus, from $\frac{2}{3}=\frac{4}{6}=\frac{8}{12}= \ldots$, it follows, according to this theorem, that $\frac{2+4+8\ \ldots}{3+6+12\ \ldots}=\frac{2}{3}$.

Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}= \ldots$

To prove that $\frac{a+c+e+g+\ \ldots}{b+d+f+h+\ \ldots}=\frac{a}{b}=\frac{c}{d}= \ldots$

Proof:

$\frac{a}{b}=\frac{a}{b}$	Why?
$\frac{c}{d}=\frac{a}{b}$	Why?
$\frac{e}{f}=\frac{a}{b}$	Why?
$\frac{g}{h}=\frac{a}{b}$, etc.	Why?
$ab=ab$	Why?
$cb=ad$	Why?
$eb=af$	Why?
$gb=ah$, etc.	Why?

$(a+c+e+g\ \ldots)b=a(b+d+f+h\ \ldots)$. Why?

$\frac{a+c+e+g\ \ldots}{b+d+f+h\ \ldots}=\frac{a}{b}=\frac{c}{d}$, etc. Why?

20. Find the fourth proportional to 2, 3, and 5.

21. Find the third proportional to 3 and 6.

22. Form a proportion from $a^2-b^2=ab$.

23. Form a proportion from $a^2-4=2a^2-10a$.

24. Apply the test of proportionality to the following statements:

a) $\frac{4}{7}=\frac{12}{21}$.   *d*) $\frac{4+7}{4}=\frac{12+21}{12}$.

b) $\frac{4}{12}=\frac{7}{21}$.   *e*) $\frac{4+7}{7}=\frac{12+21}{21}$.

c) $\frac{7}{4}=\frac{21}{12}$.   *f*) $\frac{7-4}{4}=\frac{21-12}{12}$

25. What change in the position of the terms of proportion *a*, Exercise 24, will transform it into proportion *b*? Equation *b* is said to be obtained from *a* by *alternation*.

26. What change will transform proportion *a* into *c*? Equation *c* is said to be obtained from *a* by *inversion*.

27. What change will transform proportion *a* into *d*? Equation *d* is said to be obtained from *a* by *addition*.

28. What change will transform proportion *a* into *e*?

29. What change will transform proportion *c* into *f*? Equation *f* is said to be obtained from *c* by *subtraction*.

30. Apply alternation to the proportion $\frac{15}{27}=\frac{10}{18}$.

31. Apply alternation to $\frac{a}{b}=\frac{c}{d}$.

32. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$ and $\frac{d}{b}=\frac{c}{a}$.

33. Show that if $\frac{a}{b}=\frac{a'}{b'}$ and $\frac{b}{c}=\frac{b'}{c'}$, etc.,

then $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$, etc.

34. Apply inversion to $\frac{a}{b}=\frac{c}{d}$.

35. Prove that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{b}{a}=\frac{d}{c}$.

36. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+b}{b}=\frac{c+d}{d}$.

Suggestions for proof:

Multiply $\frac{a+b}{b}=\frac{c+d}{d}$ by bd.

Then $(a+b)d=(c+d)b$.

$\therefore$ $ad+bd=cb+bd$. Why?

$ad=cb$. Why?

$\frac{a}{b}=\frac{c}{d}$. Why?

Since the last statement was given, the proof is obtained by reversing the steps in the preceding discussion:

Proof:

$\frac{a}{b}=\frac{c}{d}$ Given.

$ad=cb$ Why?

$ad+bd=cb+bd$ Why?

$(a+b)d=(c+d)b$ Why?

$\frac{a+b}{b}=\frac{c+d}{d}$ Why?

Notice the method used in obtaining this proof. Starting from the conclusion, we draw a number of inferences until a known fact is deduced, e.g., the hypothesis. This is called the *analysis.*

The steps being reversible, we start from this known fact and get the conclusion by reversing the steps.

This last part is the *proof.*

37. Similarly, prove that $\frac{a-b}{b}=\frac{c-d}{d}$.

Summary of Chapter IV

94. A topical outline of the chapter. The following facts and principles have been taught:

A) New terms:

measure of a line segment	fourth proportional
ratio of two segments	third proportional
commensurable segments	alternation
incommensurable segments	inversion
median of a trapezoid	addition
proportional segments	subtraction

B) Geometric principles:

1. ⊛ *If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.*

2. *If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side and is equal to one-half of the second.*

3. ⊛ *If three, or more, parallel lines are cut by two transversals, the ratio of two segments on one transversal is equal to the ratio of the corresponding segments on the other.*

4. ⊛ *If a line is parallel to one side of a triangle it divides the other two sides proportionally.*

5. ⊛ *Two lines that cut two given intersecting lines and make the corresponding segments of the given lines proportional are parallel.*

6. *The line joining the midpoints of two sides of a triangle is parallel to the third side.*

7. ⊛ *The bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.*

8. ⊛ *The bisector of an exterior angle of a triangle divides the opposite side externally into segments that are proportional to the other two sides.*

C) Geometric constructions:

1. *Divide a given segment into a given number of equal parts.*

2. *Construct the fourth proportional of three given segments.*

3. *Divide a segment in a given ratio:* (*a*) *internally*; (*b*) *externally.*

D) Algebraic processes taught:

1. Solving problems leading to simple equations.

2. Solving equations in the form of proportions, such as $\frac{x}{x-1}=\frac{4}{3}$.

3. Forming proportions from the factors of such equations as $a^2-b^2=c^2-d^2$.

CHAPTER V

LOCI. CONCURRENT LINES

Loci

95. Meaning of locus. When a point moves it traces a path whose shape is determined by the conditions under which the point moves. Thus, a stone falling from rest moves along a *straight line*, and a particle projected obliquely into space moves along a curve which is practically a *parabola*, Figure 128.

FIG. 128

In geometry you have learned that the location of all points in a plane at a given distance from a fixed point is a *circle;* that the place of all points in a plane at equal distances from two fixed points is a *straight line*, the perpendicular bisector of the segment joining the given points.

The place of all points satisfying some specified condition and *not containing other points* is called the **locus** of the points. *Locus** is a Latin word, meaning "place."

96. Determination of a locus. To *determine* the locus of a point mark a number of positions of the point. From these points it will be possible to obtain a notion of the locus.

Thus, marking several positions of the pedal of a bicycle on a wall beside a walk suggests the locus of the pedal.

*The plural of *locus* is *loci*.

EXERCISES

1. A circle C, Figure 129, is rolled without sliding along the edge of a ruler AB. Find the locus of a point P on the circle.

Suggestion: Cut a circle from cardboard and roll it carefully along the ruler. By pricking through with a pin, mark a number of positions of P. Draw a smooth curve through the points thus obtained. The locus of P is called a *cycloid*.

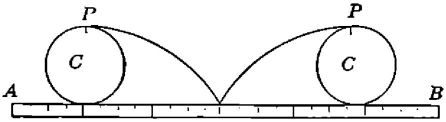


FIG. 129

2. Draw two perpendicular lines, Figure 130. On a piece of tracing paper draw a segment AB and mark a point P on AB. Move AB so that B slides along OY, and A along OX, and mark a number of positions of P. Draw the locus of P. The locus will be a quarter of an *ellipse*.

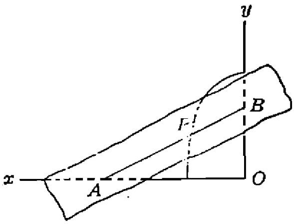


FIG. 130

3. What is the locus of points in a plane having a given distance from a given line?

Suggestion: Mark several points at the given distance from the given line. Their position will suggest the locus.

4. What is the locus of points in a plane at equal distances from two given parallel lines?

5. What is the locus of points in space having a given distance from a given point?

6. What is the locus of points in space equally distant from two given points?

7. What is the locus of points in space equally distant from two parallel lines?

8. What is the locus of points in space having a given distance from a given line?

9. Points A and B are 16 centimeters apart. By means of loci determine points 10 centimeters from A and 8 centimeters from B.

10. A straight river passes through a city. Determine points that are 8 miles from the center of the city and 5 miles from the middle of the river.

97. Proof for a locus. The locus of points satisfying given conditions must contain *all* points satisfying these conditions and *no other* points, i.e., a proof must show that:

I. *Every point on the locus must satisfy the given conditions.*

II. (*a*) *Every point satisfying the conditions must lie on the locus, or*

(*b*) *Any point not on the locus must not satisfy the conditions.*

98. Theorem:® *The perpendicular bisector of the segment joining two given points is the locus of points in a plane which are equidistant from the two given points.*

Proof: I. Show that every point on the perpendicular bisector is equidistant from the two points.

II. Let $PA = PB$, Figure 131. Let PC be a line drawn from P to the midpoint, C, of AB.

Prove that $x = y$.

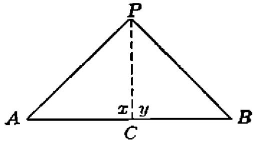


FIG. 131

PC is the perpendicular bisector of AB.

III. ∴ The perpendicular bisector of AB is the locus of points equidistant from A and B.

99. Theorem:® *The bisector of an angle is the locus of points in a plane which are within the angle and equidistant from the sides.*

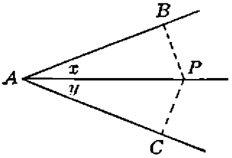


FIG. 132

Proof: I. Show that every point on the bisector is equidistant from the sides.

II. Let $PB \perp AB$, Figure 132, $PC \perp AC$, and $BP = PC$.

Show that $x = y$.

III. ∴ The bisector of angle BAC is the required locus.

EXERCISES

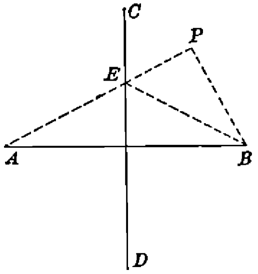


FIG. 133

1. Prove that any point not on the perpendicular bisector of a line segment is unequally distant from the end points.

Suggestions:

Prove that $PB < PE + EB$, Figure 133.

$PE + EB = PE + EA = PA$.

∴ $PB < PA$.

2. By the method of § 96, find the locus of the midpoints of the radii of a circle.

3. A circle is rolled around a second fixed circle, always touching it. Find the locus of the center of the moving circle.

4. On one side of a triangle locate a point equidistant from the other two sides.

5. On a given line locate a point equidistant from two given points.

6. Find the points equidistant from two given points and a fixed distance from a given line.

7. Prove that any point not on the bisector of an angle is unequally distant from the sides of the angle.

Suggestions: $PD < PG$, Figure 134, for the shortest distance from a point to a line is the perpendicular from the point to the line.

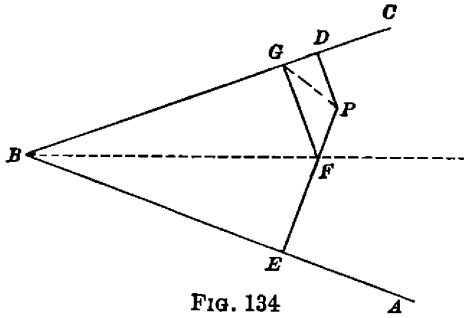


FIG. 134

$$PG < PF + FG.$$
$$PF + FG = PE.$$

$\therefore$ $PD < PE$. For, if three magnitudes are such that the first is less than the second, and the second less than the third, the first is less than the third.

8. Locate a point equidistant from two given points and equidistant from two parallel lines.

9. Locate the points equidistant from two intersecting lines, and at a given distance from a given point.

Concurrent Lines

100. Meaning of concurrent lines. If three or more lines pass through the same point they are **concurrent** lines.

EXERCISES

1. Draw an equilateral triangle and bisect each angle. If your construction is well made, the bisectors will meet in a point.

2. Draw a right triangle and construct the bisectors of the angles. Are they concurrent?

101. Theorem: *The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the sides of the triangle.*

Given $\triangle ABC$, Figure 135, with AD, BE, and CF, the bisectors of $\angle s\ A$, B, and C, respectively.

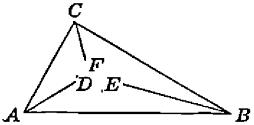


FIG. 135

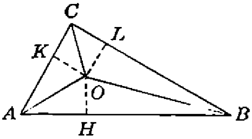


FIG. 136

To prove that AD, BE, and CF are concurrent in a point equidistant from AB, BC, and CA.

Proof: Show that AD and BE intersect, as at O, Figure 136. Exercise 13, § 59.

Draw	$OH \perp AB$, $OK \perp AC$, $OL \perp BC$.	
Then	$OH = OK$.	Why?
	$OH = OL$.	Why?
	$OK = OL$.	Why?
	CF must pass through O.	§ 99.

102. Incenter. The point of intersection of the bisectors of the interior angles of a triangle is called the **incenter** of the triangle. It is the center of the circle that may be drawn touching the three sides of the triangle.

EXERCISES

1. Construct a circle touching the three sides of a triangle. The circle is said to be *inscribed* in the triangle.

2. Show that the bisectors of one interior angle, as A, Figure 137, and of the exterior angles at B and C are concurrent.

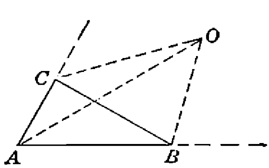


FIG. 137

3. Do the bisectors of the three interior angles of a triangle always meet within the triangle?

4. When do the bisectors of the angles of a triangle coincide with the altitudes?

103. Excenter. The point of intersection of the bisectors of two exterior angles of a triangle and the third interior angle is called an **excenter** of the triangle.

EXERCISES

1. How many excenters are there?

2. Draw a triangle. Construct four circles just touching the three sides or their extensions.

104. Tangent. A straight line having only one point in common with a circle is a **tangent.**

EXERCISES

1. Draw a circle. Draw a radius and erect a perpendicular to the radius at the outer extremity. If your work is carefully done the perpendicular will be tangent to the circle.

2. Find the center of a given circle.

Solution: Draw a chord.

Since the center is equidistant from all points on the circle it must lie on the perpendicular bisector of the chord. Therefore draw the perpendicular bisector of the chord and extend it to the circle.

Bisect the diameter. The midpoint is the required center.

105. Theorem: *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle.*

Given $\triangle ABC$, Figure 138, and DE, FG, and HK, the perpendicular bisectors of AB, BC, and CA, respectively.

To prove that DE, FG, and HK are concurrent in a point equidistant from A, B, and C.

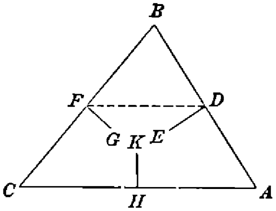


FIG. 138

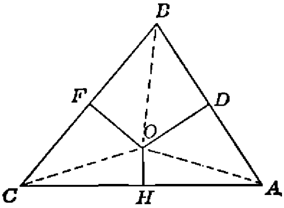


FIG. 139

Proof:

Draw FD, Figure 138.

Since $EDB = 90°$ and $GFB = 90°$,

$EDB + GFB = 180°$.

But $EDF + GFD < EDB + GFB$.

$\therefore$ $EDF + GFD < 180°$.

$\therefore$ DE intersects FG (see Exercise 13, § 59).

Show that $OC = OB$ and $OB = OA$ Fig. 139.

$\therefore$ $OC = OA$.

$\therefore$ HK passes through O.

(The perpendicular bisector of a segment is the locus of a point equidistant from the end points.)

106. Circumscribed circle. A circle passing through the vertices of a polygon is said to be **circumscribed** about the polygon.

EXERCISES

1. Show that the point O, Figure 139, is the center of the circumscribed circle of triangle ABC.

2. Draw the circle circumscribed about a triangle.

3. Draw a circle passing through three points not in the same straight line.

107. Circumcenter. The point of intersection of the perpendicular bisectors of the sides of a triangle is called **circumcenter.**

108. Theorem: *The three altitudes of a triangle are concurrent.*

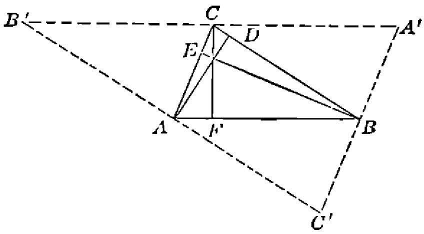


FIG. 140

Given $\triangle ABC$, Figure 140, with $AD \perp BC$, $BE \perp AC$, and $CF \perp AB$.

To prove that AD, BE, and CF are concurrent.

Proof: Draw $B'C' \perp AD$, $C'A' \perp BE$, and $A'B' \perp CF$, forming $\triangle A'B'C'$.

Then, $AB \parallel A'B'$,

$BC \parallel B'C'$,

and $CA \parallel C'A'$. Why?

Show that $B'C = AB = CA'$.

Hence, CF is the perpendicular bisector of $A'B'$.

Similarly, show that AD is the perpendicular bisector of $B'C'$ and that BE is the perpendicular bisector of $C'A'$.

$\therefore$ AD, BE, and CF are concurrent. Why?

109. Orthocenter. The point of intersection of the three altitudes of a triangle is called the **orthocenter** of the triangle.

EXERCISES

1. Construct the altitudes of an equilateral triangle and use the preceding theorem as a check of the accuracy of your construction.

2. Draw an obtuse triangle. Construct the altitudes and show that the extensions of the altitudes pass through the same point.

3. Draw a right triangle and find the point in which the altitudes intersect.

4. Prove that the three altitudes of an equilateral triangle divide the triangle into six congruent right triangles.

110. Center of gravity of a triangle. From cardboard cut a triangle. Draw lines from the vertices to the midpoints of the opposite sides (medians). If the construction is made carefully, they will meet in a point. If the triangle is supported by placing a pin under the point of intersection, the triangle will balance. For this reason the point of intersection of the three medians is called the center of gravity of the triangle. In § 111 it will be proved that the three medians meet in a point.

EXERCISES

1. Draw a triangle. Construct the orthocenter, circumcenter, and center of gravity. As a test of accuracy of your construction use the fact that the three points lie on the same straight line.

2. Prove that the medians drawn to the equal sides of an isosceles triangle are equal.

111. Theorem: *The medians of a triangle are concurrent in a point which lies two-thirds the distance from the vertex to the midpoint of the opposite side.*

Given $\triangle ABC$, Figure 141, with the medians AE, BF, and CD.

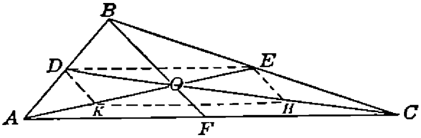


FIG. 141

To prove that AE, BF, and CD are concurrent at some point, O, such that

$$AO = \tfrac{2}{3}AE,$$
$$BO = \tfrac{2}{3}BF,$$
$$CO = \tfrac{2}{3}CD.$$

Proof:

Show that $\angle EAC + \angle DCA < 180°$.

$\therefore$ AE must intersect CD at some point, as O.

Draw KH joining K, the midpoint of AO to H, the midpoint of OC.

Draw DE, DK, and EH.

Then, $DE \parallel AC$ and $DE = \tfrac{1}{2}AC$ (§ 90, Exercise 2, and § 83, Exercise 1).

Similarly, $KH \parallel AC$ and $KH = \tfrac{1}{2}AC$.

$\therefore$ *KHED is a parallelogram* (§ 71).

$\therefore$ $EO = OK = KA$,

and $DO = OH = HC$.

$\therefore$ $AO = \tfrac{2}{3}AE$ and $CO = \tfrac{2}{3}CD$.

Similarly, we may show that CD and BF meet in a point which is two-thirds the distance from C to D and from B to F, i.e., at O.

EXERCISES

1. Prove that if two medians of a triangle are equal the triangle is isosceles.

2. Prove that the perpendiculars drawn from two vertices of a triangle to the median of the side between them are equal.

Summary of Chapter V

112. Geometric terms. The following terms have been used:

locus	tangent to a circle
concurrent lines	circumscribed circle
incenter	inscribed circle
excenter	median of a triangle
circumcenter	altitude of a triangle
orthocenter	perpendicular bisector of
center of gravity	the side of a triangle

113. Loci. It has been shown: 1. How to determine the locus of a point moving according to given conditions. 2. How to give a proof for a locus.

114. Theorems. The following theorems have been proved:

1.⊛ *The perpendicular bisector of the segment joining two given points is the locus of the points in a plane which are equidistant from the two given points.*

2.⊛ *The bisector of an angle is the locus of points within the angle and equidistant from the sides.*

3. *The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.*

4. *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle.*

5. *The altitudes of a triangle are concurrent.*

6. *The medians of a triangle are concurrent in a point two-thirds the distance from a vertex to the midpoint of the opposite side.*

CHAPTER VI

SIMILARITY

Similar Triangles

115. What is meant by similar polygons? The following construction shows how to draw a triangle which is of the same shape as a given triangle.

Construction: Let $\triangle ABC$, Figure 142, be the given triangle.

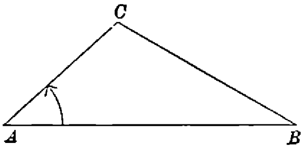


FIG. 142

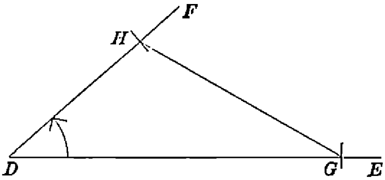


FIG. 143

Draw a line of indefinite length, as DE, Figure 143.

Draw DF, making with DE an angle equal to $\angle A$.

On DE lay off any convenient length, as DG.

Construct the fourth proportional (Exercise 3, § 87) to AB, DG, and AC, and lay it off on DF.

Draw HG.

Then $\triangle DGH$ is the required triangle.

If your construction is well done, the two triangles should be *of the same shape*. For this reason they are called *similar* triangles.

Apply the following tests of accuracy to your construction: (1) Measure all angles. The angles of $\triangle ABC$ should be equal to the corresponding angles of $\triangle DGH$.

(2) Measure all sides of both triangles and find the ratios: $\frac{AB}{DG}$, $\frac{BC}{GH}$, and $\frac{AC}{DH}$. These ratios should be equal.

The preceding construction illustrates the following two properties of similar triangles: (1) the corresponding angles are equal, and (2) the ratios of the corresponding sides are equal.

The same properties are possessed by similar polygons. For this reason **similar polygons** are defined as *polygons having the corresponding sides proportional and the corresponding angles equal.* Hence, the statement:

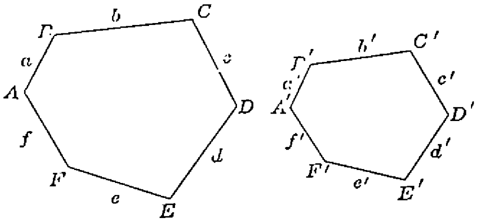


FIG. 144

polygon $ABCDEF \backsim A'B'C'D'E'F'$, Figure 144, may be expressed symbolically by the two following statements:

$$\begin{cases} 1. & \angle A = \angle A',\ \angle B = \angle B',\ \angle C = \angle C', \text{ etc.} \\ 2. & \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'} = \frac{f}{f'} \end{cases}$$

116. Conditions sufficient to make triangles congruent. Previously you have seen the importance of congruent triangles in proving theorems and solving problems. The definition of *congruent* triangles contains **six** conditions, viz.:

1. The equality of the corresponding angles,

$$\angle A = \angle A',\ \angle B = \angle B',\ \angle C = \angle C'.$$

2. The equality of the corresponding sides,

$$a = a',\ b = b',\ c = c'.$$

However, it was shown that we do not need to establish all of these conditions to prove two triangles congruent and that the following conditions are **sufficient:**

1. Two sides and the angle included between them in one triangle equal, respectively, to the corresponding parts of the other triangle.

2. Two angles and a side of one triangle equal to the corresponding parts of the other.

3. Three sides equal, respectively.

Thus, the problem of proving two triangles congruent is greatly simplified.

117. Conditions sufficient to make two triangles similar. The definition of *similar* triangles contains **five** conditions, viz.:

1. The equality of the corresponding angles, or

$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C'.$$

2. The proportionality of the corresponding sides, or $\frac{a}{a'} = \frac{b}{b'}$, $\frac{b}{b'} = \frac{c}{c'}$, from which it follows that $\frac{c}{c'} = \frac{a}{a'}$.

As in the case of congruent triangles, it is not necessary to show that all five of these conditions are satisfied to make two triangles similar. It will be shown that any one of the following three conditions is **necessary and sufficient:**

1. The equality of two pairs of corresponding angles.

2. The proportionality of two pairs of corresponding sides, and the equality of the included angle.

3. The proportionality of the corresponding sides.

It will be seen later that many problems may be solved by means of similar triangles.

118. Theorem: *A line parallel to one side of a triangle forms with the other two sides a triangle similar to the given triangle.*

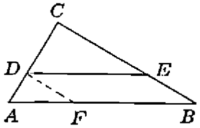


FIG. 145

Given $\triangle ABC$, and $DE \parallel AB$, Figure 145.

To prove $\triangle DEC \backsim \triangle ABC$.

Proof: Prove that the angles of $\triangle DEC$ are respectively equal to the angles of $\triangle ABC$.

Since	$DE \parallel AB$,	
	$\frac{CD}{CA}=\frac{CE}{CB}$.	Why?
Draw	$DF \parallel CB$.	
Then	$\frac{FB}{AB}=\frac{DC}{AC}$.	Why?
	$DE=FB$.	Parallels intercepted by parallels are equal.
Substituting for FB its equal, DE, you have	$\frac{DE}{AB}=\frac{DC}{AC}$.	
Hence,	$\frac{CD}{CA}=\frac{CE}{CB}=\frac{DE}{AB}$.	Why?
	$\triangle ABC \backsim \triangle DEC$.	§ 115.

119. Theorem:[*] *Two triangles are similar if two angles of one are respectively equal to two angles of the other.*

Given $\triangle\triangle ABC$ and $A'B'C'$, with $A = A'$ and $C = C'$, Figure 146.

To prove that $\triangle ABC \backsim \triangle A'B'C'$.

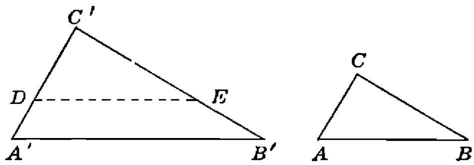


Fig. 146

Proof:

On $C'A'$ lay off	$C'D = CA$.	
Draw	$DE \parallel A'B'$.	
Then	$\triangle DEC' \backsim \triangle A'B'C'$	§ 118.
	$\angle D = \angle A'$.	Why?
	$\angle A' = \angle A$.	Given.
	$\angle D = \angle A$.	Why?
	$\triangle DEC' \cong \triangle ABC$.	Why?
	$\triangle A'B'C' \backsim \triangle ABC$.	By substitution.

EXERCISES

To find the height of a chimney.

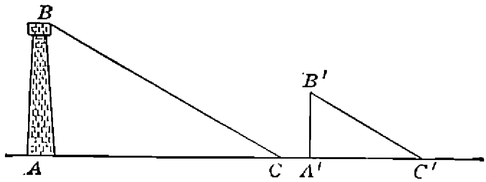


Fig. 147

Let AC, Figure 147, represent the shadow of the chimney AB, and $A'C'$ the shadow of a vertical stick $A'B'$.

Assuming rays of sunlight to be parallel, show that $\angle C = \angle C'$.

Since triangles ABC and A′B′C′ have two angles equal respectively, they can be shown to be similar (§ 119).

Hence, $\frac{AB}{A'B'}=\frac{AC}{A'C'}$ Why?

and $AB=\frac{AC\cdot A'B'}{A'C'}$. Why?

Using this equation as a formula, find the height of a chimney whose shadow is 108 feet, if at the same time the shadow of a 4-foot vertical stick is 9 feet long.

2. To determine the distance across a river.

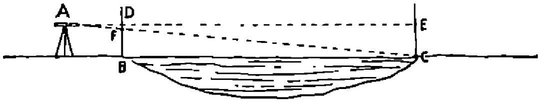


FIG. 148

Sighting across the river with telescope A, Figure 148, place in the line of sight vertical rods, as at B and C. Take readings of rods at E and D. Depress the telescope sighting at C and take the reading at F. From the readings compute the length of DF and EC.

$EC \parallel DF$.

∴ Triangles AFD and ACE are similar.

Hence, $\frac{AE}{AD}=\frac{EC}{DF}$.

$AE=\frac{AD\cdot EC}{DF}$, which

expresses AE in terms of the known lengths AD, EC, and DF.

The length, ED, may be found by subtracting AD from AE.

Prove the following:

3. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

4. Two isosceles triangles are similar if the vertex angle of one is equal to the vertex angle of the other.

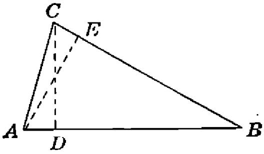


FIG. 149

5. If in triangle ABC, Figure 149, AE and CD are two altitudes, it follows that $\frac{AE}{CD}=\frac{AB}{BC}$. Hence, two altitudes of a triangle are said to be *inversely* proportional to the corresponding bases.

6. Using § 119, construct upon a given line segment as a side a triangle similar to a given triangle. Prove your construction.

7. All equilateral triangles are similar.

8. Corresponding altitudes of similar triangles are proportional to corresponding sides.

Suggestion: Prove $\triangle DBC \sim \triangle D'B'C'$, Figure 150.

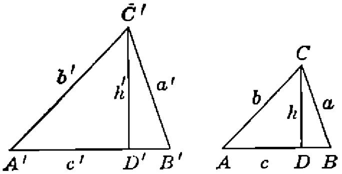


FIG. 150

9. The bisectors of corresponding angles of two similar triangles are proportional to corresponding sides.

10. The triangle formed by joining the midpoints of the sides of a given triangle is similar to the given triangle.

11. The shadow of a chimney is 30 yards long. At the same time the shadow of a vertical pole 6 feet high is 4.5 feet long. Find the height of the chimney.

12. If two triangles are similar to a third triangle they are similar to each other.

13. Two triangles are similar if the corresponding sides are parallel, or perpendicular.

For, if the sides of two angles are respectively parallel the angles are either equal or supplementary.

Thus, (1) $A = A'$, or (2) $A + A' = 2$ right angles.
(3) $B = B'$, or (4) $B + B' = 2$ right angles.
(5) $C = C'$, or (6) $C + C' = 2$ right angles.

Show that the three equations (2), (4), and (6) cannot all be true at the same time.

Show that two of the equations (2), (4), and (6) cannot both be true at the same time.

Hence, at least two of the equations (1), (3), and (5) must be true and the triangles are mutually equiangular.

14. Let ABC be a given angle. From a point P on BA draw PP' perpendicular to BC, and from point Q on BC draw QQ' perpendicular to BA. Prove that $\triangle BPP' \backsim \triangle BQQ'$.

15. Draw a triangle ABC and extend two sides through the vertex C. Draw a line parallel to AB intersecting the extensions in B' and A'. Prove triangle $B'CA'$ similar to triangle ABC.

16. Using the formula of Exercise 1 find the height of a building whose shadow is 48 feet, if the shadow of a 7-foot pole is $5\frac{1}{2}$ feet.

17. Using the formula of Exercise 2 find the distance across a river if $AD = 4$ feet, $EC = 7\frac{1}{2}$ feet, and $DF = 5$ inches.

18. Are two quadrilaterals similar if the angles of one are equal to the angles of the other? Illustrate your answer by means of a drawing.

19. Two angles of a triangle are 40° and 50°, and the included side is 150 yards long. Make a scale drawing in which one unit of length represents 30 yards.

20. Prove Exercise 13 for the case of two triangles whose sides are perpendicular to each other.

120. Theorem:[®] *Two triangles are similar if the ratio of two sides of one is equal to the ratio of two sides of the other and if the angles included between these sides are equal.*

Given $\triangle ABC$ and $A'B'C'$, with $C = C'$ and $\frac{CA}{C'A'} = \frac{CB}{C'B'}$, Figure 151.

To prove that $\triangle ABC \backsim \triangle A'B'C'$.

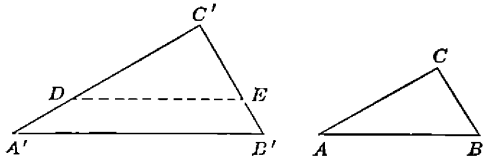


FIG. 151

Proof:

On $C'A'$ lay off $C'D = CA$.

On $C'B'$ lay off $C'E = CB$.

Then show that $\frac{C'D}{C'A'} = \frac{C'E}{C'B'}$.

$DE \parallel A'B'$. Why?

$\triangle DEC' \backsim \triangle A'B'C'$. Why?

Show that $\triangle DEC' \cong \triangle ABC$.

$\triangle ABC \backsim \triangle A'B'C'$. Why?

EXERCISES

1. Two right triangles are similar if the ratio of the sides including the right angle of one, is equal to the ratio of the corresponding sides of the other.

2. Two corresponding medians of two similar triangles are proportional to two corresponding sides.

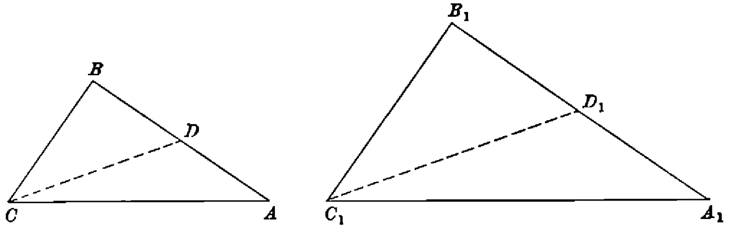


FIG. 152

Suggestion: Show that $\triangle DBC$ and $D_1B_1C_1$, Figure 152, are similar.

3. Two sides of a triangle are 16 inches and 12 inches and the included angle is 43°. In another triangle two sides are 40 inches and 30 inches and the included angle is 43°. Show that the triangles are similar.

4. To determine an inaccessible distance.

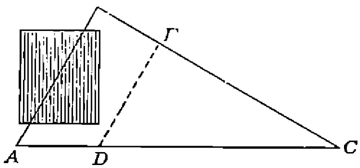


FIG. 153

Let AB, Figure 153, be the distance to be measured.

From a point, C, chosen conveniently, measure BC and AC. Mark point D on AC.

On BC determine point E so that $\frac{CD}{CA}=\frac{CE}{CB}$. Measure DE.

Show that triangles CDE and CAB are similar.

$$\text{Hence,} \quad \frac{AB}{DE}=\frac{AC}{DC}$$

$$\text{and} \quad AB=\frac{DE\cdot AC}{DC}.$$

Thus DE, AC, and DC being known, AB may be found as the quotient of the product $DE\cdot AC$ and DC.

5. Two sides of a triangle are 14 inches and 3.5 inches, and the included angle is 75°. Two sides of another triangle are 20 inches and 5 inches, and the included angle is 75°. Show that the triangles are similar.

6. Prove that two equilateral triangles are similar, by using § 120.

121. Theorem:[⊛] *Two triangles are similar if the corresponding sides are in proportion.*

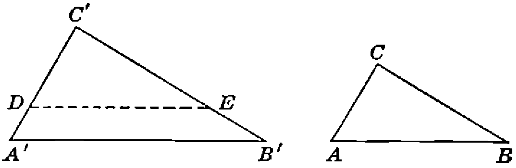


Fig. 154

Given $\triangle ABC$ and $A'B'C'$, with $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CA}{C'A'}$, Figure 154.

To prove that $\triangle ABC \backsim \triangle A'B'C'$.

On $C'A'$ lay off $C'D = CA$.

On $C'B'$ lay off $C'E = CB$.

Draw DE.

$\frac{CA}{C'A'}=\frac{CB}{C'B'}$.	Given.
$\frac{C'D}{C'A'}=\frac{C'E}{C'B'}$.	By substitution.
$\triangle DEC' \backsim \triangle A'B'C'$.	(§ 120).
$\frac{C'D}{C'A'}=\frac{DE}{A'B'}$.	Why?
$\frac{CA}{C'A'}=\frac{AB}{A'B'}$.	Given.
$\frac{C'D}{C'A'}=\frac{AB}{A'B'}$.	Why?
$\frac{DE}{A'B'}=\frac{AB}{A'B'}$.	Why?
$DE = AB$.	Why?
$\triangle DEC' \cong \triangle ABC$.	Why?
$\triangle ABC \backsim \triangle A'B'C'$.	Why?

EXERCISES

1. *The perimeters of similar triangles are to each other as any two corresponding sides.*

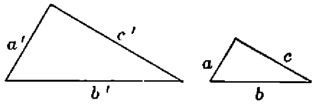


FIG. 155

Since the triangles, Figure 155, are similar,

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'} \qquad \text{Why?}$$

Let the common value of these ratios be denoted by r.

Then $$\frac{a}{a'}=r,\ \frac{b}{b'}=r,\ \frac{c}{c'}=r.$$

$$a=a'r,\ b=b'r,\ c=c'r. \qquad \text{Why?}$$

$$a+b+c=(a'+b'+c')r. \qquad \text{Why?}$$

$$\therefore\ \frac{a+b+c}{a'+b'+c'}=r=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'} \qquad \text{Why?}$$

2. Prove Exercise 1 by means of Exercise 19, § 93.

3. The lengths of the sides of a triangular piece of land are approximately 125 rods, 54 rods, and 112 rods. A drawing is made of it, the longest side of which is 3 inches. What are the lengths of the other sides of the triangle in the drawing?

4. The perimeter of a triangle is 15 centimeters, and the sides of a similar triangle are 4.5 centimeters, 6.4 centimeters, and 7.1 centimeters. Find the lengths of the sides of the first triangle.

Similar Polygons

122. Construction of similar polygons. Similar polygons may be constructed as follows:

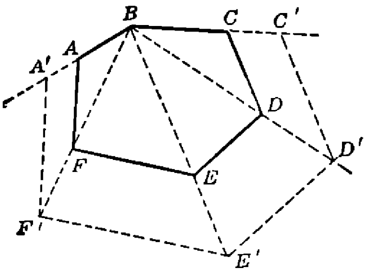


FIG. 156

Let $ABCDEF$, Figure 156, be a given polygon.

To construct a polygon similar to $ABCDEF$.

Construction: Draw diagonals from one vertex, as B, to the other vertices, and extend them. From any point on BA or its extension, as A', draw $A'F' \parallel AF$.

Draw $F'E' \parallel FE$, $E'D' \parallel ED$ and $D'C' \parallel DC$.

Then $A'B'C'D'E'F'$ is the required polygon.

Proof: Prove $\angle D = \angle D'$, $\angle E = \angle E'$, etc.

Show that $$\frac{CD}{C'D'} = \frac{BD}{BD'} = \frac{DE}{D'E'}, \text{ etc.}$$

Hence, $$\frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'}, \text{ etc.}$$ Why?

123. Homologous parts. Corresponding sides of similar polygons are **homologous** sides.

Corresponding angles, diagonals, altitudes, and medians are **homologous** angles, diagonals, altitudes, and medians.

124. Theorem:[*] *Similar polygons may be divided by homologous diagonals into triangles similar to each other and similarly placed.*

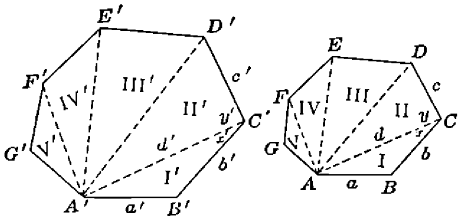


FIG. 157

Given polygon $ABCD$ etc., $\backsim$ polygon $A'B'C'D'$ etc., Figure 157, with diagonals drawn from A and A'.

To prove $\triangle \text{I} \backsim \triangle \text{I}'$, $\triangle \text{II} \backsim \triangle \text{II}'$, etc.

Proof:

$\frac{a}{a'} = \frac{b}{b'}$.	Corresponding sides of similar polygons are in proportion.
$\angle B = \angle B'$.	Corresponding angles of similar polygons are equal.
$\triangle \text{I} \backsim \triangle \text{I}'$.	§ 120.
$x = x'$.	Why?
$\angle C = \angle C'$.	Why?
$y = y'$.	Why?
$\frac{b}{b'} = \frac{d}{d'}$.	Corresponding sides of similar triangles are in proportion.
$\frac{b}{b'} = \frac{c}{c'}$.	Given.
$\frac{c}{c'} = \frac{d}{d'}$.	Why?
$\triangle \text{II} \backsim \triangle \text{II}'$, etc.	Why?

EXERCISES

1. *The perimeters of similar polygons are to each other as any two homologous sides.*

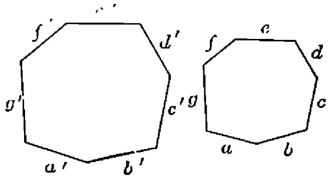


FIG. 158

Since the polygons, Figure 158, are similar,

$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$ etc.	Corresponding sides of similar polygons are in proportion.
$\therefore \quad \frac{a+b+c+\text{etc.}}{a'+b'+c'+\text{etc.}}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$ etc.	See Exercises 1, 2, § 121.

2. The line passing through the midpoint of one of the bases of a trapezoid and the point of intersection of the diagonals bisects the other base. Prove.

3. The non-parallel sides of a trapezoid of bases 18 and 60 and of altitude 6 are extended until they meet. What are the altitudes of the triangles whose bases are the upper and lower bases of the trapezoid?

4. The base of a triangle is 72 inches, and the altitude is 12 inches. Find the upper base of the trapezoid cut off by a line parallel to the base and 8 inches from it.

5. The perimeters of two similar polygons are to each other as 7 to 9. A side of the first is 2.1 inches. Find the corresponding side of the second.

6. Prove that two regular polygons having the same number of sides are similar.

Similarity in the Right Triangle

125. Theorem: *The perpendicular to the hypotenuse from the vertex of the right angle divides a right triangle into parts similar to each other and to the given triangle.*

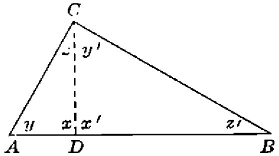


FIG. 159

Given $\triangle ABC$ with the right angle C, and $CD \perp AB$, Figure 159.

To prove

$$\triangle ADC \backsim \triangle BDC \backsim \triangle ABC.$$

Proof:	
$x = x'$.	Adjacent angles formed by two perpendicular lines are equal.
$y + z' = 90$.	Why?
$y' + z' = 90$.	Why?
$y + z' = y' + z'$.	Why?
$y = y'$.	
$\triangle ADC \backsim \triangle BDC$.	Why?

Prove that $\triangle$s ADC and ABC are mutually equiangular and therefore similar.

Similarly, prove $\triangle BDC \backsim \triangle ABC$.

126. Projection of a point. The **projection of a point** upon a given line is the **foot** of the perpendicular drawn from the point to the line. Thus, point D, Figure 160, is the projection of point A upon BC.

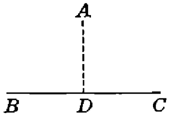


FIG. 160

127. Projection of a segment. To project a **line segment,** as AB, Figure 161, upon a line, as CD, drop perpendiculars to CD from the end points of the segment AB. Then EF is the projection of AB upon CD.

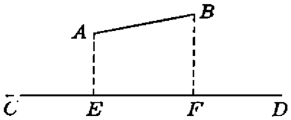


FIG. 161

In general, the **projection of a given segment** upon a line is the segment of the line whose end points are the projections of the end points of the given segment.

EXERCISES

1. In each of the following figures name the projection of AB upon CD, Figures 162–164. Give reasons for your answer.

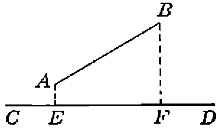


FIG. 162

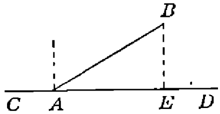


FIG. 163

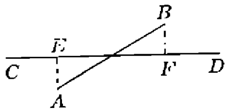


FIG. 164

Draw a figure in which the segment is equal to its projection; in which the projection is a point.

2. In triangle ABC, Figure 165, name the projection of AC upon AB; of BC upon AB.

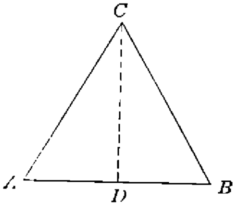


FIG. 165

3. In triangle ABC, Figure 165, project BC upon AC; AB upon BC.

4. Draw an obtuse triangle, as ABC, Figure 166. Project AB upon BC; AC upon AB; BC upon AB; AB upon AC.

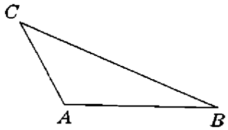


FIG. 166

128. Mean proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, b is a **mean proportional** between a and c.

EXERCISES

1. Find a mean proportional between 4 and 9.

Denoting the mean proportional by x, you have $\frac{4}{x}=\frac{x}{9}$.

$\therefore \quad x^2=4\cdot 9$ Why?

$\therefore \quad x=\pm\sqrt{4\cdot 9}$

$\therefore \quad x=\pm 2\cdot 3$ Why?

or $\quad x=+6, -6.$ Check both results.

2. Find the mean proportional between 9 and 16; 4 and 25; 8 and 50.

3. In triangle ABC, Figure 167, find the projection of the median, m, upon AB.

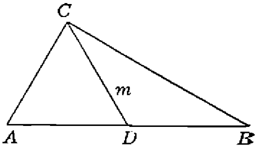


FIG. 167

129. Radical. An indicated root of a number is a **radical.** Thus $\sqrt{5}$, $\sqrt[3]{x}$, $\sqrt[4]{16}$, $\sqrt{a+b^2}$ are radicals.

130. Simplification of radicals. In computing the value of a radical it is often of advantage to change the form of the number under the radical sign. The following examples illustrate this:

$$\text{I.}\begin{cases}\sqrt{25\cdot 16}=5\cdot 4, \text{ for } (5\cdot 4)(5\cdot 4)=25\cdot 16.\\ \sqrt{36\cdot 9}=6\cdot 3, \text{ for } (6\cdot 3)(6\cdot 3)=36\cdot 9.\end{cases}$$

Thus the values of $\sqrt{25\cdot 16}$, $\sqrt{36\cdot 9}$, etc., are found by extracting the square roots of the factors separately and then multiplying the results.

In general, *the square root of a product, as ab, may be found by taking the square roots of the factors, as a and b, and* *then taking the product of these square roots.* This may be stated briefly in the form of an equation, thus,

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}.$$

This principle enables us to obtain *by inspection* the square roots of large numbers, as is shown by Examples II and III.

II. $\begin{cases}\sqrt{3136}=\sqrt{4\cdot 784}=\sqrt{4\cdot 4\cdot 196}=\sqrt{4\cdot 4\cdot 4\cdot 49}\\ \qquad =2\cdot 2\cdot 2\cdot 7=56.\\ \sqrt{4225}=\sqrt{5\cdot 845}=\sqrt{5\cdot 5\cdot 169}=5\cdot 13=65\end{cases}$

The principle previously explained may be applied to advantage even when the number under the radical sign is not a square. For example:

III. $\sqrt{50}=\sqrt{5\cdot 5\cdot 2}=5\sqrt{2}.$

Knowing the square root of 2 to be 1.414+ . . . . , it follows that $\sqrt{50}=7.070+$ . . . . .

Similarly, $\sqrt{8a^3}=\sqrt{4a^2\cdot 2a}=2a\sqrt{2a}$

and $\sqrt{108}=\sqrt{9\cdot 12}=\sqrt{9\cdot 4\cdot 3}=6\sqrt{3}.$

EXERCISES

Reduce the following radicals to the simplest form:

1. $\sqrt{576}$ **5.** $\sqrt{75}$ **9.** $\sqrt{128a^2b^2}$

2. $\sqrt{441}$ **6.** $\sqrt{27}$ **10.** $\sqrt{162x^2y^2}$

3. $\sqrt{1225}$ **7.** $\sqrt{a^5b^3}$ **11.** $\sqrt{243ab^2}$

4. $\sqrt{36a^2b^2}$ **8.** $\sqrt{20x^2y}$ **12.** $\sqrt{225a^3b^2c}$

13. Find the mean proportionals between 2 and 18; 10 and 90; 8 and 200; 20 and 180.

14. Find the mean proportionals between a^2 and b^2; c^2 and d^2.

15. Find a mean proportional between $49ab$ and $225abc^2$.

16. Show that the mean proportional between a and b is the square root of the product of a and b.

131. Theorem: *In a right triangle, the perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

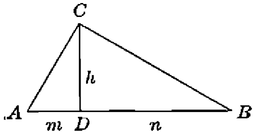


FIG. 168

That is, we are to prove $\frac{m}{h}=\frac{h}{n}$, Figure 168.

Proof:	$CD \perp AB$.	Given.
$\therefore$	$\triangle ADC \backsim \triangle BDC$.	§ 125.
	$\frac{m}{h}=\frac{h}{n}$.	In similar triangles the sides opposite equal angles are *homologous* sides and are therefore *proportional.*

132. Theorem: *In a right triangle either side of the right angle is a mean proportional between its projection upon the hypotenuse and the entire hypotenuse.*

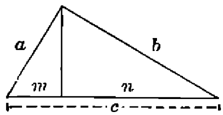


FIG. 169

You are to prove $\frac{m}{a}=\frac{a}{c}$ and $\frac{n}{b}=\frac{b}{c}$, Figure 169.

To prove this, apply the principle that homologous sides of similar triangles are in proportion.

EXERCISES

1. In a right triangle the altitude drawn to the hypotenuse divides the hypotenuse into two parts 3 inches and 12 inches long. Find the altitude and the lengths of the sides including the right angle.

2. In a right triangle the ratio of the sides of the right angle is $\frac{1}{3}$. Find the ratio of the segments in which the altitude divides the hypotenuse.

133. Problems of construction. Make the following constructions with ruler and compass:

1. *To construct a mean proportional between two segments.*

Given the segments m and n, Figure 170.

Required to construct the mean proportional between m and n.

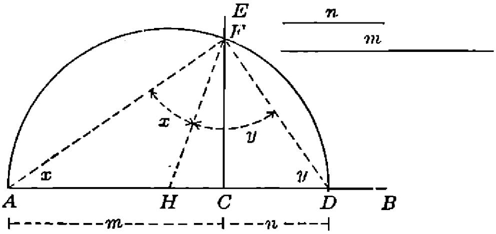


Fig. 170

Construction: On a line, as AB, lay off $AC=m$, $CD=n$.

Draw $CE \perp AB$.

Draw the circle AFD on AD as a diameter, meeting CE at F.

Then FC is the required mean proportional between m and n.

Proof: Draw AF, DF, and the median HF.

	$\angle AFH = \angle A = x.$	Base angles of an isosceles triangle are equal.
	$\angle HFD = \angle D = y.$	For the same reason.
Then	$2x+2y=180,$	The sum of the angles of triangle AFD is 180°.
	$x+y=\angle AFD=90°.$	Why?
	$\dfrac{m}{FC}=\dfrac{FC}{n}.$	§ 131.

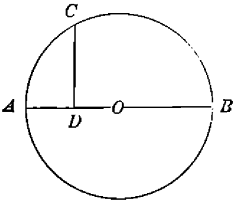


Fig. 171

2. Find the length of CD, Figure 171, if AB is a diameter, $CD \perp AB$, $AD=3$, and $AB=12$.

3. *Construct the square root of a number.*

1. To find the square root of 2, lay off on squared paper two factors of 2, as 1 and 2, Figure 172, in the same way as m and n, problem 1 (use the scale $1=2$ cm.).

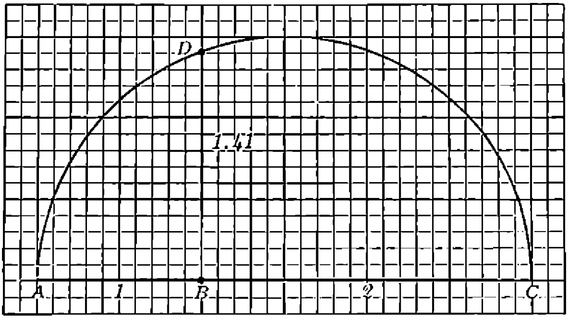


FIG. 172

The mean proportional, BD, between AB and BC, represents graphically the required square root of 2. Why?

Measure BD.

Check by extracting the square root of 2 to two decimal places.

4. Find geometrically the square root of 6; of 5; of 8.

EXERCISES

1. From a pair of opposite vertices of a rectangle draw perpendiculars to a diagonal. Show that two pairs of similar triangles are formed.

2. Prove that a side of a square is a mean proportional between the entire diagonal and one-half of the diagonal.

3. Prove that in a right triangle having two sides equal, the square of the altitude to the hypotenuse is equal to the product of the segments of the hypotenuse.

4. In triangle ABC, Figure 173, $\angle ACB$ is a right angle and $CD \perp AB$. $AD = 2$, $DB = 30$. Find the lengths of AC and CB.

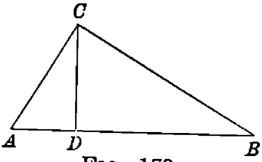


FIG. 173

5. The radius of a circle is 12.5, Figure 174. Find the projection of the chord AC upon the diameter AB passing through one of the end points of the chord.

Suggestion: Draw CB. Then $\angle ACB = 90°$.

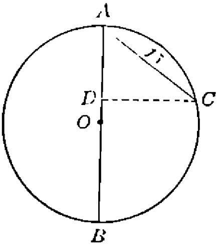


FIG. 174

6. In the right triangle, Figure 175, find c, m, n, and h, if $a = 5$ and $b = 12$.

Find b, m, n, and h, if $a = 8$, and $c = 10$.

Find a, b, c, and h if $m = 9\frac{3}{5}$ and $n = 5\frac{2}{5}$.

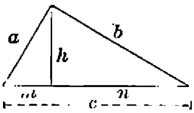


FIG. 175

7. Compute the dimensions of the section of the strongest beam that can be cut from a cylindrical log.

Let the circle, Figure 176, represent a cross-section of the log. Then the dimensions of the strongest beam are computed as follows:

Trisect the diameter AB at C and D (§ 83, Exercise 3).

Erect $CE \perp AB$ and $DF \perp AB$.

Draw the quadrilateral $AFBE$.

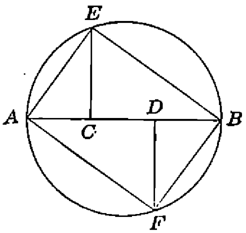


FIG. 176

This is the required section of the strongest beam.

If the diameter of the log is 15 inches, compute AE and AF.

8. If two line segments are 8 inches and 12 inches long, find the length of the third proportional.

9. Make a formula expressing AE, Figure 176, in terms of the diameter.

10. Make a formula expressing BE, Figure 176, in terms of the diameter.

Summary of Chapter VI

134. New terms. The following terms have been introduced:

similar figures	projection of line segment
homologous parts	radical
projection of point	mean proportional

135. Theorems. The following theorems have been proved:

1. *A line parallel to one side of a triangle forms with the other two sides a triangle similar to the given triangle.*

2.⊛ *Two triangles are similar: if two angles of one are equal to two angles of the other; if the ratio of two sides of one is equal to the ratio of two sides of the other and the angles included between these sides are equal; if the corresponding sides are in proportion.*

3. *The perimeters of similar polygons are to each other as any two corresponding sides.*

4. *Similar polygons may be divided by homologous diagonals into triangles similar to each other.*

5. *The perpendicular to the hypotenuse from the vertex of the right angle: divides the triangle into two triangles similar to each other and to the given triangle; is the mean proportional between the segments of the hypotenuse.*

6. *In a right triangle either side of the right angle is a mean proportional between the projection upon the hypotenuse and the entire hypotenuse.*

136. Constructions. The following constructions have been taught:

1. To construct a polygon similar to a given polygon.
2. To construct a mean proportional between two segments.
3. To construct the square root of a number.

137. Algebraic skills developed. The chapter has given practice with problems leading to proportions and calling for solutions of linear equations. It has been shown how to simplify radicals, and how to find algebraically the mean proportional between two numbers. The following principle is used in reducing radicals to the simplest form:

The square root of a product may be found by taking the square root of the factors and then taking the product of these square roots.

In symbols the principle has been stated thus:

$$\sqrt{ab}=\sqrt{a}\,\sqrt{b}$$

CHAPTER VII

THEOREM OF PYTHAGORAS. QUADRATIC EQUATION

Theorem of Pythagoras

138. Theorem of Pythagoras. The theorems of chapter vi enable you to obtain proof for one of the most important theorems of geometry, which is as follows: *The square of the hypotenuse in a right triangle is equal to the sum of the squares of the sides of the right angle.*

Given the right triangle ABC, Figure 177.

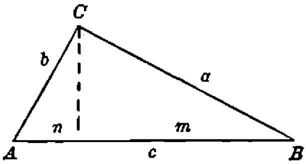


FIG. 177

To prove that $a^2+b^2=c^2$.

Proof: From C draw a perpendicular to AB.

$\frac{m}{a}=\frac{a}{c}$	§ 132.
and $\frac{n}{b}=\frac{b}{c}$.	
$\therefore\ a^2=mc.$	In a proportion the product of the means is equal to the product of the extremes.
and $b^2=nc.$	Why?
$\therefore\ a^2+b^2=(m+n)c$	Why?
or $\boldsymbol{a^2+b^2=c^2}$.	

The equation $a^2 = m \cdot c$ means that the square on BC, Figure 178, is equal to a rectangle of dimensions m and c, as $BEFD$. (Notice that the sides of rectangle $BEFD$ are m, the projection of BC on AB, and BE which is equal to c, the length of the hypotenuse AB.)

Similarly, $b^2 = n \cdot c$ means that the square on AC is equal to a rectangle as $FHAD$, having the dimensions equal to n, the projection of AC on AB, and AH, which is equal to the hypotenuse, c.

Hence, the sum of the squares on AC and BC is equal to the sum of these two rectangles, or to the square on the hypotenuse.

This illustration is practically an outline of Euclid's proof of the theorem of Pythagoras.

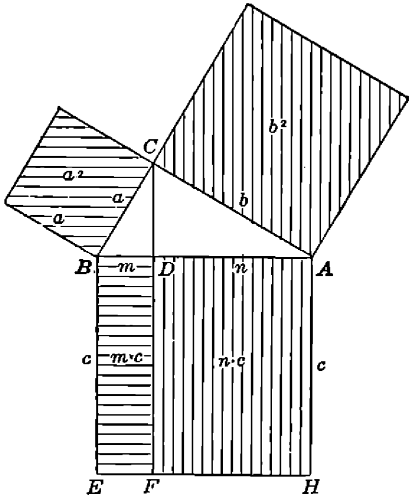


FIG. 178

139. Historical note: It is said that Pythagoras, jubilant over his great accomplishment of having found a proof of the theorem, sacrificed a hecatomb to the muses who inspired him. The invention was well worthy of this sacrifice, for it marks historically the first conception of **irrational numbers.** It is believed that Pythagoras showed the existence of irrational numbers by showing that the hypotenuse of a certain isosceles right triangle is equal to $\sqrt{2}$, Figure 179.

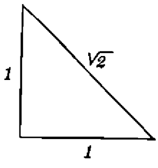


FIG. 179

His followers found much pleasure in discovering special sets of integral values of a, b, c satisfying the equation $a^2 + b^2 = c^2$, the simplest set being 3, 4, and 5. Such numbers are called *Pytha-*


FERMAT,
né en 159.. mort en 1665.

PIERRE DE FERMAT

PIERRE DE FERMAT was born near Toulouse in 1601 and died at Castres in 1665. The great mathematical historian Cantor and others have called Fermat "the greatest French mathematician of the seventeenth century," and this was a century of great French mathematicians. He was the son of a leather merchant and was educated at home. He studied law at Toulouse and in 1631 became a councilor of the Parliament of Toulouse. He is said to have performed the duties of his office with scrupulous accuracy and fidelity. He loved mathematical study, made it his avocation, spending most of his leisure on it. His disposition was modest and retiring. He published very little—only one paper—during his lifetime. Though his vocation was that of a lawyer and parliamentarian, his celebrity rests upon what he accomplished in his avocation.

Notwithstanding the fact that Fermat published very little, he exerted a great influence on the mathematicians of his age through a continual correspondence which he carried on with them. The mathematical discoveries upon which his fame rests were made known to the world through his correspondence or through the notes on his results that were found after his death, written on loose sheets of paper, or scribbled on the margins of books he had annotated while reading. A part of these notes and Fermat's marginal notes, found in his copy of Diophantus' *Arithmetic*, were published after his death by his son, Samuel. As Fermat's notes do not seem ever to have been intended for publication, it is often difficult to estimate when his discoveries were made, or whether they were really original.

For fuller information about Fermat, see Ball's *History* (pp. 293–301); Cajori's *History* (pp. 173, 179–82).

gorean numbers. The question naturally arose later whether there existed any sets of integral values of a, b, and c that would satisfy the equations $a^3+b^3=c^3$, $a^4+b^4=c^4$, etc., in general, $a^n+b^n=c^n$ for $n>2$.

The great mathematician Fermat, who lived 1601–65, states among his notes the theorem that the equation $x^n+y^n=z^n$ is not satisfied by a set of integral numbers for x, y, z, and n except for $n=2$. He also makes the statement that he has discovered a really wonderful proof for the theorem. Unfortunately, he gives not the least suggestion as to the nature of his proof.

EXERCISES

1. Prove that the diagonal of a square is equal to the product of the side by the square root of 2, Figure 180.

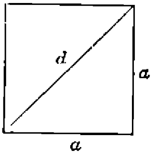


FIG. 180

2. Prove that the diagonal of a rectangle is equal to the square root of the sum of the squares of two consecutive sides, Figure 181.

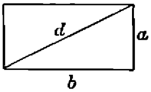


FIG. 181

3. Express the altitude of an equilateral triangle in terms of the side, Figure 182.

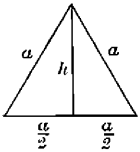


FIG. 182

4. The equal sides of an isosceles trapezoid are 20 inches. The bases are 60 and 84 inches respectively. Find the altitude and diagonal.

5. Figure 183 represents a circular window. The radius of the largest circle is 6. Find the radius, x, of the smallest window.

Suggestion: The sides of the right triangle ABC are 3, $x+3$ and $6-x$ respectively. Why?

$$\therefore \quad (x+3)^2=(6-x)^2+9.$$
$$x^2+6x+9=36-12x+x^2+9.$$
$$\therefore \quad x=2.$$

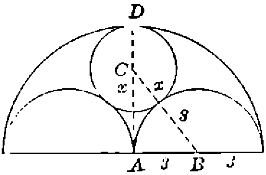


FIG. 183

6. Find the altitude of an isosceles triangle whose base is 16 feet and whose equal sides are 18 feet long.

7. Prove that the difference between the squares of two sides of a triangle is equal to the difference between the squares of their projections on the third side.

8. Measure two adjacent sides of your classroom and compute the distance between the opposite corners.

140. The generalization of the theorem of Pythagoras. In the right triangle ABC, Figure 184, imagine the angle

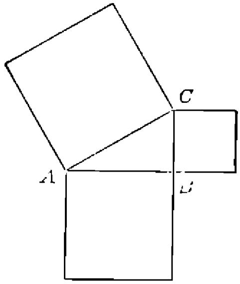


FIG. 184

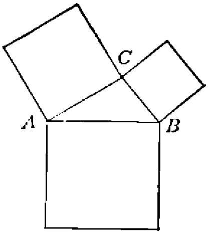


FIG. 185

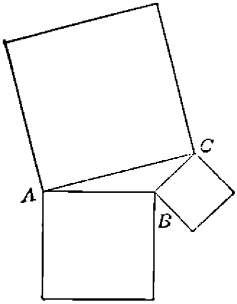


FIG. 186

ABC to decrease, leaving the lengths of the sides AB and BC unchanged. Then the squares on AB and BC are not changed in size, but as the distance between the endpoints A and C decreases, the square on AC decreases, Figure 185. Therefore in a triangle the square on the side opposite an *acute* angle is *less* than the sum of the squares on the other two sides.

In a similar way, by increasing the right angle ABC, Figure 184, as in Figure 186, we find that the square on the side opposite the *obtuse* angle B is *greater* than the sum of the squares on the other two sides.

The following two theorems will show by *how much* the square on one side of a triangle differs from the sum of the squares on the other two sides.

Although the theorem of Pythagoras was taught first it is really a special case of the relation between the three sides of a triangle, i.e., the case in which one of the angles is a right angle. The theorems in §§ 141 and 142 state the relations for obtuse and acute triangles.

141. Theorem: *In a triangle the square on the side opposite an obtuse angle is equal to the sum of the squares on the other two sides, increased by two times the product of one of them and the projection of the other upon it.*

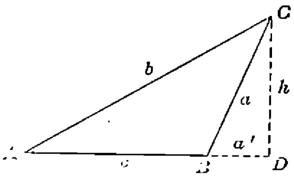


FIG. 187

Given $\triangle ABC$ with $\angle ABC$ obtuse, Figure 187.

To prove

$$b^2=a^2+c^2+2ca'.$$

Proof:

$b^2=h^2+(c+a')^2.$	By the theorem of Pythagoras.
$a^2=h^2+a'^2.$	For the same reason.
Therefore $b^2-a^2=c^2+2ca'+a'^2-a'^2.$	Why?
Hence, $b^2=a^2+c^2+2ca'.$	Why?

EXERCISES

The side opposite an obtuse angle is b, and c' is the projection of c upon a, Figure 188.

Find a' and c' when a, b, and c are respectively

1. 5, 15, 12
2. 6, 12, 8
3. 7, 11, 8

and in each case compare $2a'c$ with $2ac'$.

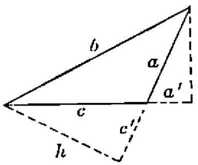


FIG. 188

142. Theorem: *In a triangle the square on the side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by two times the product of one of these two sides and the projection of the other upon it.*

Let $\angle B$ be an acute angle of triangle ABC, Figure 189.

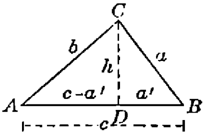


FIG. 189

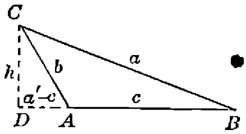


FIG. 190

Draw CD perpendicular to AB. Denote the projection of a on c by a'.

Then $\quad b^2 = h^2 + (c - a')^2. \quad$ Why?

And $\quad a^2 = h^2 + a'^2. \quad$ Why?

Subtracting, $b^2 - a^2 = (c - a')^2 - a'^2$
$= c^2 - 2ca' + a'^2 - a'^2.$

Therefore $\quad b^2 - a^2 = c^2 - 2ca'.$

Solving for b^2, $b^2 = a^2 + c^2 - 2ca'.$

This shows that the product $2ca'$ is the amount by which $a^2 + c^2$ exceeds b^2.

EXERCISES

1. Find a', Figure 189, when a, b, and c are respectively 2, 4, 5; 7, 10, 8.

2. Prove the theorem in § 142, using Figure 190.

Quadratic Equations

143. A normal form of the quadratic equation. All quadratic equations in one unknown may be arranged in the **normal form**

$$\boldsymbol{ax^2+bx+c=0,}$$

where a stands for the coefficient of x^2 when all terms in x^2 have been combined into one; b denotes the coefficient of x, and c is the constant, i.e., the term or the sum of terms not containing x.

Thus, in $5x^2+3x-4=0$, $a=5$, $b=3$, $c=-4$.

EXERCISES

Arrange each of the following equations in the normal form, $ax^2+bx+c=0$, and determine the values of the coefficients a, b, and c:

1. $x^2+4x=5$.

2. $y^2-2y=11$.

3. $c^2=4c+1$.

4. $a^2=7a-7$.

5. $8=r^2-2r$.

6. $7x+4=x^2$.

7. $3m^2=6-7m$.

8. $-12-22x=40+36x^2$.

9. $6a^2-10a=13a-21$.

10. $60-x^2=6x+12$.

11. $ax^2+bx=b+ax$.

12. $2y^2+4ay+2ab=-by$.

13. $2z^2+ab=2az+bz$.

14. $s^2+a^2=2as-2$.

144. Solution of the equation $ax^2+bx+c=0$. Since every quadratic equation may be changed to the normal form $ax^2+bx+c=0$, we may obtain a solution of *every* quadratic equation by solving $ax^2+bx+c=0$. Thus, we shall derive a formula by means of which the solution of *any* quadratic equation may readily be found.

Solution: $ax^2+bx+c=0.$

$ax^2+bx = -c.$ Subtracting c from both members.

$x^2+\frac{b}{a}x = -\frac{c}{a}.$ Dividing both members by a.

Completing the square on the left side by adding $\frac{b^2}{4a^2}$ to both sides of the equation, we have

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a},$$

or
$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{4ac}{4a^2},$$

or
$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}.$$

Whence, $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$ Changing the left member to a square of a binominal.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}},$$

or
$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Whence,
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

145. General quadratic formula. The values of x in the equation

$$ax^2+bx+c=0$$

have been found to be

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

and

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

These are the **general quadratic formulas.** They may be combined into a single formula thus,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

EXERCISES

By means of the quadratic formula solve the following equations. In these equations consider a, b, and c as *knowns* and all other letters as *unknowns:*

1. $3x^2+5x-2=0$

Here $a=3$, $b=5$, $c=-2$.

Substituting these values in the formula,

$$x=\frac{-5\pm\sqrt{25+24}}{6}=\frac{-5\pm7}{6}=\frac{1}{3}\text{ and }-2.$$

2. $2x^2+5x+2=0.$

3. $6x^2-11x+5=0.$

4. $2r^2-r-6=0.$

5. $2x^2+x=15.$

6. $r^2-9r-36=0.$

7. $t^2+15t=-44.$

8. $x^2-72=6x.$

9. $3m^2=6-7m.$

10. $6+11x=-18x^2-20.$

11. $2x^2+3x=2.$

12. $1.4x^2+5x=2.4.$

13. $3x^2-37x+112=0.$

14. $6x^2-1.4x=3.2.$

15. $14y^2+2y=28y-10y^2+5.$

16. $11R^2-10R=24-10R^2.$

17. $6p^2-13p=10p-21.$

18. $8l^2-12l+3=0.$

19. $y^2+ay+b=0.$

20. $s^2-2as+a^2+2=0.$

21. $t^2-3abt+2a^2b^2=0.$

22. $28b^2=-17by+3y^2.$

23. $12m^2-16am-3a^2=0.$

Solve the following problems:

24. The diagonal of a rectangle, Figure 191, is 17 inches. One of the sides is 7 inches longer than the other. Find the length of each side.

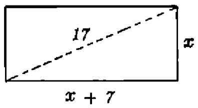


FIG. 191

25. The diagonal of a rectangle is 8 units longer than one side and 9 units longer than the other. How long is the diagonal?

26. A ladder 33 feet long leans against a house. The foot of the ladder is 14 feet from the house. How far from the ground is the point of the house touched by the top of the ladder?

27. The diagonal of a rectangle, Figure 192, is 26. The distance from the vertex to the diagonal is 12. Find the segments into which the perpendicular divides the diagonal.

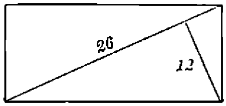


FIG. 192

146. Historical note: To solve a pure quadratic equation, such as $x^2=25$, is merely to extract a square root. A way of extracting square roots of numbers has been known since the dawn of history. Early mathematical students did what amounted to solving a pure quadratic long before they even thought about quadratic equations.

But no one could have written the tenth book of *Euclid's Elements* (300 B.C.) without a good knowledge of ways of solving quadratic equations. Since this tenth book contains most of Euclid's original work, it may safely be assumed that Euclid had this knowledge. He solved no quadratics algebraically, but he proved geometrical theorems that amounted to such solutions. Euclid was a Greek and Greek geometers did not like calculatory processes like solving quadratics, because they did not think practical numerical calculating scientific work. Plato (429–348 B.C.) had said calculating is a childish art beneath the dignity of a philosopher.

The great skill of Archimedes (287–212 B.C.) in difficult calculations, makes men think that he also must have known how to solve quadratics algebraically, but his writings contain nothing about it.

Heron of Alexandria (first century B.C.) was a scientific engineer and surveyor and he solved correctly numerous quadratic equations. In his *Geometria* he solves a problem leading to a quadratic, which, in modern symbolism, is

$$\frac{11}{14}d^2+\frac{29}{7}d=S,$$

in which S is a given number and d is the diameter of a circle. He gives correctly a rule which, in modern form, is

$$d=\frac{\sqrt{154S+841}-29}{11}.$$

Thus by Heron's time the algebraic rule had become entirely dissociated from geometry, and was known and studied for itself, without any connection with geometrical theorems of area or of lines. It had taken centuries, however, to bring about this separation from geometry.

The next important appearance of the solution of the quadratic equation is in the *Arithmetic* of Diophantus (third and fourth centuries A.D.). He distinguishes three normal forms, viz.:

1. $ax^2+bx=c$. 2. $ax^2=bx+c$. 3. $ax^2+c=bx$.

As the Greeks knew no negative numbers, the three forms had to be kept separate for treatment, and of course they could not handle the form

$$x^2+px+q=0,$$

for it requires a knowledge of both negative and complex numbers, which neither antiquity nor later times until the seventeenth century B.C. was able to comprehend.

The union of the three normal forms into one was first accomplished by the Hindus. The rule of Brahmagupta (b. 598 A.D.), which was assumed as known by his predecessor Aryabhatta (b. 476 A.D.), expressed in modern form, was

$$ax^2+bx=c, \text{ whence } x=\frac{\sqrt{ac+\left(\frac{b}{2}\right)^2}-\frac{b}{2}}{a},$$

the agreement of which with Diophantus' first form perhaps suggests a Greek origin of Hindu algebraic knowledge

A later Hindu scholar, Cridhara, introduced a slight improvement by changing the form to the following:

$$x=\frac{\sqrt{4ac+b^2}-b}{2a}.$$

The eastern Arab Alkarchi (about 1010 A.D.), who was the greatest Arabian algebraist, introduced the higher degree equations of quadratic form

$$ax^{2n}+bx^n=c;\ ax^{2n}=bx^n+c;\ \text{and } ax^{2n}+c=bx^n,$$

and solved them by reducing them to the three principal cases.

Medieval European mathematicians before Cardan (1501–76), still unable to construe the significance of negative number, continued to split up the solution of quadratics into numerous special cases, often including as many as twenty-four special cases, each with its special rule of reckoning. Finally, Cardan succeeded in gaining the correct insight into negative number, and the Italian school of thinkers attacked the imaginary. Through the work of this school it became possible to supply the lacking form,

$$x^2+px+q=0, \text{ for the cases of } p>0 \text{ and } q>0.$$

Summary of Chapter VII

147. New terms. The following terms have been introduced:

irrational numbers quadratic formula
normal form of a quadratic equation

148. Theorems. The following theorems have been proved:

1. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the sides of the right angle.*

2. *The square of the side opposite an obtuse angle of a triangle is equal to the sum of the squares of the other two sides increased by two times the product of one of them and the projection of the other on it.*

3. *The square of the side opposite an acute angle of a triangle is equal to the sum of the squares of the other two sides diminished by two times the product of one of them and the projection of the other on it.*

149. Algebraic skills developed. It has been shown that every quadratic equation in one unknown can be solved by the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

where a, b, c are the coefficients in the equation

$$ax^2+bx+c=0.$$

CHAPTER VIII

TRIGONOMETRIC RATIOS

The Meaning of Trigonometric Ratios

150. Finding angles and distances. The theorem of Pythagoras, the fact that two right triangles are similar if an acute angle of one is equal to an acute angle of the other, and the principle that the acute angles of a right triangle are complementary, enable us to work out a method for finding unknown angles and distances.

These principles are the basis of **trigonometry,** a subject which is useful not only in the study of more advanced mathematics, but also in all the exact sciences.

EXERCISES

1. Show that all right triangles having an acute angle of one equal to an acute angle of the other are similar.

2. On squared paper draw a right triangle having an angle of 30°. Measure the sides carefully and find the ratio of the side *opposite* the angle of 30° to the *hypotenuse.*

3. Prove that this ratio is the same for all right triangles having an angle of 30°.

4. In a right triangle having one angle equal to 60°, find, by careful measurement, the ratio of the side *opposite* the angle of 60° to the *hypotenuse.* Compare your result with the results obtained by other members of the class.

5. Prove that this ratio is constant for all right triangles that have an angle of 60°.

6. In a right triangle having an angle of 45°, find the ratio of the side *opposite* the angle of 45° to the *hypotenuse.*

7. Prove that this ratio is constant for all right triangles having an angle of 45°.

8. Draw with a protractor an angle of 40°, Figure 193. From points on either side of the angle, as A_1, A_2, A_3, draw perpendiculars to the other side. Measure A_1C_1 and A_1O and find their ratio.

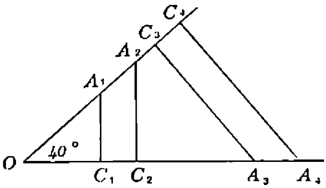


FIG. 193

9. Prove that the ratio of the side opposite the angle of 40° to the hypotenuse is the same for all triangles of Figure 193.

Exercise 9 illustrates the fact that the *ratio* of the sides, Figure 193, remains *constant* as the *lengths* of the sides *vary*.

The constant *ratio of the opposite side to the hypotenuse*, as in Figure 193, is called *the sine of 40°*.

151. Trigonometric ratios of an angle. Let angle A, Figure 194, be a given angle. From any point, as B, on either side of the angle draw a perpendicular to the other side. Thus, a right triangle is formed, as ACB.

In this triangle, the ratio of the *side opposite* the vertex of $\angle A$ to the hypotenuse is the *sine of angle* A† (written: sin A),

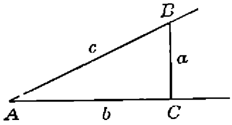


FIG. 194

$$\text{i.e., } \sin A = \frac{a}{c}.$$

† The word "sine" is a shortened form of the Latin *sinus*, which is the translation of an Arabic word meaning a "bay," or "gulf." Albert Girard (1595–1632), a Dutch mathematician, was the first to use the abbreviations "sin," and "tan" for "sine" and "tangent" (Ball, p. 235). Ball (p. 243) says the term "tangent" was introduced by Thomas Finck (1561–1646) in his *Geometriae Rotundi* of 1583. The same historian says (p. 243) the term "cosine" was first employed by E. Gunter in 1620 in his *Canon on Triangles*, and that the abbreviation "cos" for "cosine" was introduced by Oughtred in 1657. These contractions, "sin," "cos," and "tan," did not however come into general use until the great Euler reintroduced them in 1748. The word "cosine" is an abbreviation for "complementary sine."

The ratio of the side *adjacent* to the vertex of $\angle A$ to the *hypotenuse* is the *cosine* of *angle* A (written: cos A),

$$\text{i.e., } \cos A = \frac{b}{c}.$$

The ratio of the side *opposite* to the side *adjacent* is the *tangent of angle* A (written: tan A),

$$\text{i.e., } \tan A = \frac{a}{b}.$$

Stated more compactly:

$$\sin A = \text{the } ratio,\ \frac{\text{opposite side}}{\text{hypotenuse}},\ \text{or the } quotient,\ \frac{a}{c}$$

$$\cos A = \text{the } ratio,\ \frac{\text{adjacent side}}{\text{hypotenuse}},\ \text{or the } quotient,\ \frac{b}{c}$$

$$\tan A = \text{the } ratio,\ \frac{\text{opposite side}}{\text{adjacent side}},\ \text{or the } quotient,\ \frac{a}{b}$$

152. Values of the trigonometric ratios determined by means of a drawing. The values of the trigonometric ratios of a given angle may be found from a drawing of a right triangle containing the angle, as shown in the following exercises:

EXERCISES

1. Find the numerical value of sin 50°.

With a protractor construct on squared paper an angle equal to 50°, Figure 195. Draw $AB \perp CB$. Measure AB and AC.

Find the value of the ratio $\frac{AB}{AC}$.

This is the required number.

2. Find the numerical value of sin 20°; sin 35°; sin 55°; sin 70°.

3. The values of the trigonometric ratios of angles from 1°

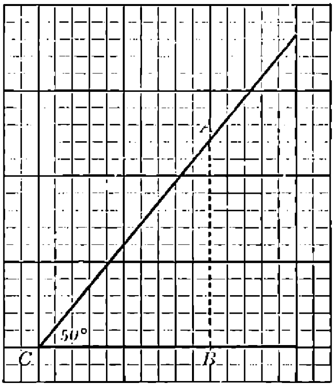


FIG. 195

TABLE OF SINES, COSINES, AND TANGENTS OF ANGLES FROM 1°–90°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	.0872	.9962	.0875	**50**	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	**55**	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	**60**	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	.3420	.9397	.3640	**65**	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	**70**	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3 0777
28	.4695	.8829	.5317	73	.9563	.2924	3 2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	**75**	.9659	.2588	3.7321
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	**80**	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	**85**	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	.7071	.7071	1.0000	**90**	1.0000	.0000	∞

to 90° are tabulated in the table on p. 165. Compare your results for Exercises 1 and 2 with the corresponding values given in the table.

153. Values of the trigonometric ratios found by means of the table. The table, p. 165, gives approximately to 4 places the values of the ratios for angles containing an integral number of degrees from 1° to 90°. This is quite sufficient for our purposes.

Where greater accuracy is required, tables are available which give the values of the trigonometric ratios of angles containing fractions of degrees.

EXERCISE

From the table find the values of the following ratios. State your results in the form of equations.

sin 2°	cos 11°	tan 20°
sin 42°	cos 63°	tan 85°

154. Trigonometric functions. Examine the table, p. 165, and notice how the values of the trigonometric ratios change as the angle changes from 1° to 90°. Since a change in the angle produces a *corresponding* change in the ratio, the trigonometric ratios are also called **trigonometric functions.**

From the table, obtain the changes of sin A as A increases from 0° to 90°.

Similarly, obtain the changes of cos A.

155. To find the values of the trigonometric functions if the value of one of them is given. Having given the *value* of a single *function* of an angle the values of the other functions and the number of degrees in the angle may be determined in various ways. If a table of trigonometric functions is available, they may be looked up in the table. The following exercises show a graphical method:

EXERCISES

1. Given the sine of an angle equal to $\frac{1}{2}$, find the values of the other functions and the number of degrees in the angle.

Solution:

Draw a right angle, A, Figure 196.

On one side of the angle lay off $AB=1$.

With B as center and radius equal to 2, draw a circle arc meeting AC at C.

Measure AC and find the values of the cosine and tangent of angle C.

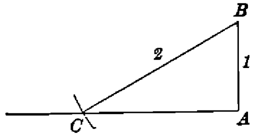


FIG. 196

With a protractor find the number of degrees in $\angle C$.

2. Find the number of degrees in an angle whose sine is $\frac{7}{8}$; .2; .75. Also find the values of the other functions.

3. Find the angle and the values of the other two functions if $\cos B=0.6$; if $\tan A=\frac{5}{3}$.

Exact Values of the Functions of 30°, 45°, and 60°

156. Exact values of the functions of 30° and 60°. Since angles of 30°, 45°, and 60° are used in a large number of problems, you should *remember* the exact values of the functions of these angles, as found in the following exercises:

EXERCISES

1. To construct a right triangle containing an angle of 30°, draw an equilateral triangle, Figure 197, and divide it into two congruent triangles by drawing the altitude to one side.

Show that the acute angles of triangle ADC are 60° and 30°.

Show that the hypotenuse is twice as long as the side opposite the 30° angle.

Hence, if AD be denoted by x, AC must be $2x$.

Show by the theorem of Pythagoras that $CD=x\sqrt{3}$.

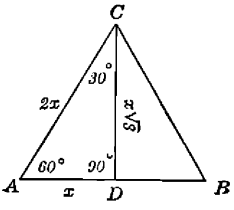


FIG. 197

2. Find the value of sin 30°, using Figure 197.
Solution:

$$\sin 30° = \frac{x}{2x} = \frac{1}{2}. \quad \text{Why?}$$

3. Find the value of sin 60°.
Solution:

$$\sin 60° = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2} = \tfrac{1}{2}\sqrt{3}.$$

4. Find the value of cos 30°.
5. Find the value of cos 60°.
6. Find the value of tan 30°.

$$\tan 30° = \frac{x}{x\sqrt{3}} = \frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

7. Find the value of tan 60°.

157. Rationalizing the denominator. In Exercise 6 the fraction $\frac{1}{\sqrt{3}}$ was changed to $\frac{1}{3}\sqrt{3}$ by multiplying numerator and denominator by $\sqrt{3}$. This does not change the **value** of the fraction, but changes the denominator to a *rational number*. This process is called **rationalizing the denominator.** The object of the rationalizing process is to obtain a form of the fraction more easily calculated arithmetically.

EXERCISES

Rationalize the denominators in the following fractions:

1. $\frac{1}{\sqrt{2}}$
2. $\frac{\sqrt{5}}{\sqrt{2}}$
3. $\frac{6}{\sqrt{a}}$
4. $\frac{12}{7\sqrt{3}}$
5. $\frac{\sqrt{6}-\sqrt{3}}{2\sqrt{3}}$
6. $\frac{\sqrt{10}-\sqrt{2}}{2\sqrt{3}}$

158. Exact values of the functions of 45°. To construct an angle of 45°, draw an **isosceles right** triangle, Figure 198.

EXERCISES

1. In the isosceles right triangle ABC, Figure 198, show that $A = C = 45°$.

2. Denoting the equal sides of triangle ABC, Fig 198, by x, show that $AC = x\sqrt{2}$.

3. Find the values of the functions of 45°, giving all results with rational denominators.

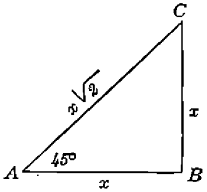


FIG. 198

159. Summary of the exact values of the functions of 30°, 45°, and 60°. The following is a simple device for memorizing these values. For the sake of symmetry, let $\frac{1}{2}$ be written in the form $\frac{1}{2}\sqrt{1}$; then

$$\sin 30° = \tfrac{1}{2}\sqrt{1},\ \sin 45° = \tfrac{1}{2}\sqrt{2},\ \sin 60° = \tfrac{1}{2}\sqrt{3}.$$

The values of the cosine are the same as above, but in reverse order, thus:

$$\cos 30° = \tfrac{1}{2}\sqrt{3},\ \cos 45° = \tfrac{1}{2}\sqrt{2},\ \cos 60° = \tfrac{1}{2}\sqrt{1}.$$

This may be conveniently arranged in the form of a table:

Function \ Angle	30°	45°	60°
Sine	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
Cosine	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$

It will be seen in § 160 that it is not necessary to memorize the values of the tangent function because they are easily computed from a simple relation existing between the trigonometric functions.

160. Relations of trigonometric functions. Important relations between the *sine*, *cosine*, and *tangent* of an angle can be shown by simple formulas.

EXERCISES

1. Prove that if A is any acute angle

$$(\sin A)^2+(\cos A)^2=1.$$

Proof:

In Figure 199 $\quad \sin A=\frac{a}{c}$. (1)

$\cos A=\frac{b}{c}$. (2)

Squaring (1) and (2),

$(\sin A)^2=\frac{a^2}{c^2}$. (3)

$(\cos A)^2=\frac{b^2}{c^2}$. (4)

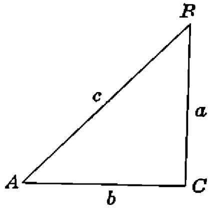


FIG. 199

Adding (3) and (4),

$(\sin A)^2+(\cos A)^2=\frac{a^2+b^2}{c^2}$. (5)

$\because \quad a^2+b^2=c^2,$

$(\sin A)^2+(\cos A)^2=1.$ (6)

$(\sin A)^2$ is usually written $\sin^2 A$. Similarly $(\cos A)^2$ and $(\tan A)^2$ are written $\cos^2 A$ and $\tan^2 A$.

Thus, $\quad \sin^2 A+\cos^2 A=1.$

2. Using the formula $\sin^2 x+\cos^2 x=1$, show that

$\sin x=\sqrt{1-\cos^2 x}$† (1)

and $\quad \cos x=\sqrt{1-\sin^2 x}.$ (2)

† We shall not use the double sign before the radical because so far we have found no meaning for a negative sine or cosine.

3. From Figure 199 show that

$$\tan A = \frac{\sin A}{\cos A},$$

and $$\tan B = \frac{\sin B}{\cos B}.$$

161. Trigonometric identities. The two fundamental relations,

$$\sin^2 A + \cos^2 A = 1 \quad (1)$$

and $$\tan A = \frac{\sin A}{\cos A}, \quad (2)$$

are true for any value of A. They are therefore called *identities*, and are sometimes written thus,

$$\sin^2 A + \cos^2 A \equiv 1;\ \tan A \equiv \frac{\sin A}{\cos A}.$$

EXERCISES

1. Find the tangent of 30°, using equation (2) above.

2. Find the tangent of 45°.

3. Find the tangent of 60°.

162. Symbol of identity. The symbol $\equiv$ is read *is*, or *is identical to.*

EXERCISES

1. In Figure 200 show that

1. $\sin A = \cos B$; 2. $\cos A = \sin B$;

i.e., that *the sine of an angle is the cosine of the complement of the angle.*

2. In Figure 200 show that

$$\tan A = \frac{1}{\tan B},$$

i.e., *the tangent of an angle equals the reciprocal of the tangent of the complement.*

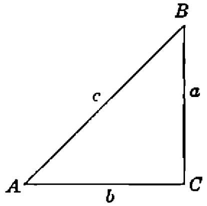


FIG. 200

Application of the Trigonometric Functions

163. Determination of a triangle. We know that all *right* triangles in which the following parts are equal, each to each, are congruent:

1. The two sides including the right angle.
2. A side and one acute angle.
3. The hypotenuse and one of the other sides.

In other words, if in a right triangle **two parts** in addition to the **right angle** are given (*at least one* being a side), the triangle is completely *determined*, and may be *constructed* from these given parts. The unknown parts may be *computed* by the methods of scale drawing, or by using the sine, cosine, and tangent of the angles, as will be seen in the following exercises.

EXERCISES

1. The rope of a flagpole is stretched out so that it touches the ground at a point 20 feet from the foot of the pole, and makes an angle of 73° with the ground. Find the height of the flagpole.

I. **Graphical solution:** With a ruler and protractor, draw the right triangle, ABC, Figure 201, to scale. By measurement, x is found to represent 66 feet, approximately.

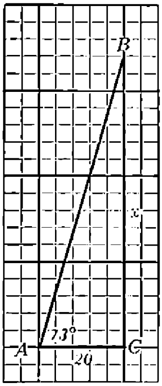


FIG. 201

II. **Trigonometric solution:** Using the tangent of $\angle A$, we have:

$\frac{x}{20} = \tan 73° = 3.2709$, from the table, p. 165.

Therefore $x = 20 \times 3.2709 = 65.418$.

The result, 65.418, is misleading, as it gives the impression that the length of BC has been determined accurately to *three* decimal places. This is impossible since the length of AC, i.e., *20*, from which 65.418 was derived by multiplication, had not been determined even to the first decimal place. Hence, the decimal .418 has no meaning and should be discarded. The length of BC is said to be 65 feet, *approximately*.

2. A balloon is anchored to the ground by a rope 260 feet long, making an angle, A, of 67° with the ground. Assuming the rope line to be straight, what is the height of the balloon?

Suggestion: Use the sine of angle A.

3. A kite-string 300 feet long, Figure 202, is fastened to a stake at A. The distance from the stake to a point C directly under the kite B is $102\frac{1}{2}$ feet. Find the height of the kite, supposing the kite-string to be straight.

Find the angle of elevation of the kite from the stake.

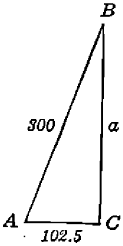


FIG. 202

I. **Graphical solution:** Draw the right triangle ABC to scale and measure a and A.

II. **Trigonometric solution:**

$$\cos A = \frac{102.5}{300} = .3.$$

From the table, p. 165, $\cos 72° = .3090$

and $\cos 73° = .2924$.

∴ The angle of elevation of the kite is about 72° or 73°.

Since $$\frac{a}{300} = \sin 72° = .9511,$$

therefore $$a = 300 \times .9511 = 285.$$

III. **Algebraic solution:** The value of a may also be obtained from the equation

$$a = \sqrt{300^2 - 102.5^2}. \quad \text{Why?}$$

4. A vertical pole 8 feet long casts on level ground a shadow 9 feet long. Find the angle of elevation of the sun.

Use the tangent ratio.

5. The angle of elevation of an aeroplane at a point A on level ground is 60°. The point C on the ground directly under the aeroplane is 300 yards from A. Find the height of the aeroplane.

6. What is the angle of elevation of the top of a hill that is $500\sqrt{3}$ feet high, at a point in the plane whose shortest distance from the top of the hill is 1,000 feet?

7. What is the angle of elevation of a road that rises one foot in a distance of 50 feet measured on the road?

8. A road makes an angle of 6° with the horizontal. How much does the road rise in a distance of 100 feet along the horizontal?

9. On a tower is a searchlight 140 feet above sea-level. The beam of light is depressed (lowered) from the horizontal, through an angle of 20°, revealing a passing boat. How far is the boat from the base of the tower?

10. A boat passes a tower on which is a searchlight 120 feet above sea-level. Find the angle through which the beam of light must be depressed from the horizontal so that it may shine directly on the boat when the boat is 400 feet from the base of the tower.

11. From the top of a cliff 150 feet high, the angle of depression of a boat is 25°. How far is the boat from the top of the cliff?

12. When an aeroplane is directly over town C, the angle of depression of town B, $2\frac{1}{4}$ miles from C, is observed to be 10°. Find the height of the aeroplane.

13. From an aeroplane, at a height of 600 feet, the angle of depression of another aeroplane, at a height of 150 feet, is 39°. How far apart are the two aeroplanes?

14. On the top of a tower stands a flagstaff. At a point A on level ground, 50 feet from the base of the tower, the angle of elevation of the top of the flagstaff is 35°. At the same point A the angle of elevation of the top of the tower is 20°. Find the length of the flagstaff.

15. Village B, Figure 203, is due north of village C. An army outpost is located at a point A, 18 miles due west of C. B bears 60° east of north from A. An aeroplane is observed to fly from C to B in a quarter of an hour. Find the average horizontal speed of the aeroplane.

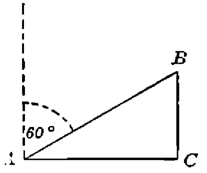


FIG. 203

16. The distance from a cannon to a straight road is 7 miles. If the range of the cannon is 10 miles, what length of the road is commanded by the cannon?

Suggestion:

Show that BAC, Figure 204, is an *isosceles* triangle, and that AH bisects BC. In the right triangle, ABH, find the length of BH.

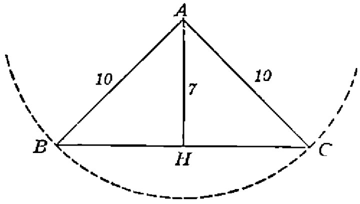


FIG. 204

17. The arms of a pair of compasses are opened to a distance of 6.25 centimeters between the points. If the arms are 11.5 centimeters long, what angle do they form?

Suggestion: In the isosceles triangle ABC, Figure 205, draw the altitude AH.

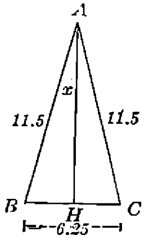


FIG. 205

18. A pair of compasses is opened to an angle of 50°. What is the distance between the points if the arms are 12.5 centimeters long?

Suggestion: Draw the altitude of the isosceles triangle.

19. A cannon with a range of 11 miles can shell a stretch of 13 miles on a straight road. How far is the cannon from the road?

20. A clock pendulum, 20 inches long, swings through an angle of 6°. Find the length of the straight line between the farthest points which the lower end reaches.

21. A clock pendulum is 25 inches long. Through what angle does the pendulum swing if the distance between the farthest points which the lower end reaches is 6 inches?

22. In triangle ABC, Figure 206, find the projection of BC on AC if $a=216$.

Find the altitude.

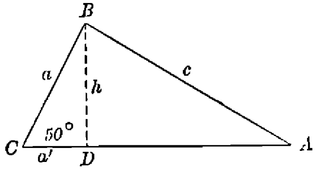


FIG. 206

23. You have seen that $c^2=a^2+b^2-2a'b$, where a' is the projection of a on c. Express a' in terms of the cosine of angle C and prove that $c^2=a^2+b^2-2ab \cos C$. This formula is known as the *law of cosines*.

24. Prove the law of cosines, using angle A.

25. Two sides and the included angle of a triangle are respectively, 8 inches, 6 inches, and 30°. By means of the law of cosines find the third side.

Summary of Chapter VIII

164. New Terms. The trigonometric ratios sine, cosine, and tangent have been defined.

165. Values of trigonometric ratios. The value of a trigonometric ratio of a given angle may be found (1) from the table, (2) graphically.

The exact values of the sine of angles of 30°, 45°, and 60° are $\frac{1}{2}$, $\frac{1}{2}\sqrt{2}$, and $\frac{1}{2}\sqrt{3}$, respectively.

The exact values of the cosine of the same angles are $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}\sqrt{2}$, and $\frac{1}{2}$, respectively.

The value of the tangent is found from the relation

$$\tan A = \frac{\sin A}{\cos A}.$$

The following fundamental trigonometric identities have been proved:

$$\sin^2 A + \cos^2 A \equiv 1$$

$$\tan A \equiv \frac{\sin A}{\cos A}.$$

If the value of one function is given, the values of the other functions may be found, (1) from the table, (2) graphically.

166. Problems. Many problems in distances, which may be solved graphically, can be solved more simply by calculating by the aid of trigonometric functions.

167. Algebraic skills developed. The irrational denominator of a fraction may be rationalized by multiplying the numerator and the denominator by the same number.

CHAPTER IX

THE CIRCLE

Properties of the Circle

168. Gothic arch. One of the uses of the circle in designs is illustrated in Figure 207. It represents the so-called equilateral **Gothic arch,** frequently found in modern architecture. Its most common use is in church windows.

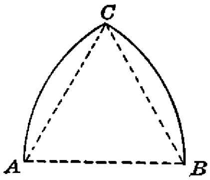


FIG. 207

ABC is an equilateral triangle and arcs AC and CB are drawn with centers at A and B and radius AB.

EXERCISES

1. In Figure 208 three Gothic arches are joined with a circle. Draw this figure with ruler and compass.

To find the center of circle O, use A and B as centers and radius equal to $\frac{3}{4}AB$. In Exercise 3, § 184, we shall learn to prove that the circles in this figure are tangent to each other in pairs.

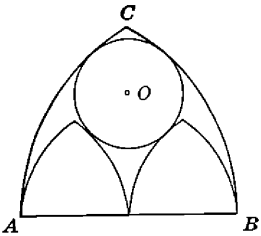


FIG. 208

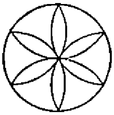


FIG. 209

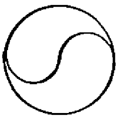


FIG. 210

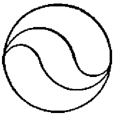


FIG. 211

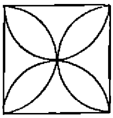


FIG. 212


FIG. 213

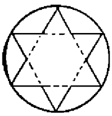


FIG. 214

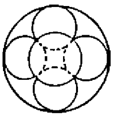


FIG. 215

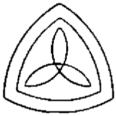


FIG. 216

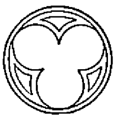


FIG. 217

2. Study the designs in Figures 209 to 217 and construct them, using ruler and compass.

3. A **circle** is a closed curved line, all points of which lie in the same plane and are equally distant from a point within, called the **center.** The segment from the center to a point on the circle is a **radius,** and the segment passing through the center and terminated by the circle is a **diameter.**

Compare the length of the radius with the distances from the center to several points taken anywhere within, upon, or outside of, the circle.

Exercise 3 shows that *a point is within, upon, or without a circle according as its distance from the center is less than, equal to, or greater than, the radius.*

169. Concentric circles. Draw several circles having the same center but unequal radii. Circles having the same center are called **concentric** circles.

EXERCISE

On notebook paper draw two circles having equal radii. If one of the circles is cut out and laid upon the other, making the centers coincide, the circles should coincide. See if you can make one of your circles coincide with the other.

If they do not coincide, what seems to cause the failure of coincidence?

In general, *two circles having equal radii are equal,* and *equal circles have equal radii.*

170. Semicircle. Major arc. Minor arc. Cut a circle from paper. Fold it along a *diameter.* How do the two parts of the circle compare as to size?

This shows that *a diameter divides a circle into two equal parts.* Each of these parts is called a **semicircle.** If a circle is divided into unequal parts, one is called the **major arc,** the other, the **minor arc.**

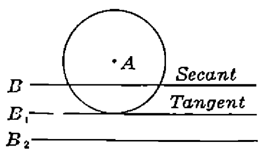


FIG. 218

171. Secant. Tangent. Draw a circle as A, Figure 218. Move the edge of a ruler, B, across the circle and notice the different positions of the edge, as B, B_1, B_2, etc. How many points may a circle and a straight line have *in common?*

A straight line intersecting a circle in two points is a **secant.**

A line touching a circle in *only one* point is a **tangent.**

172. Number of points common to two circles. By moving one circle over another, Figure 219, show that

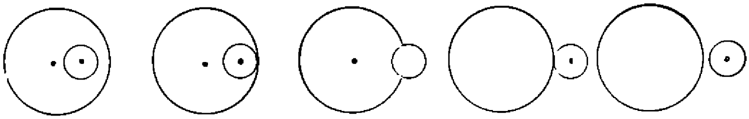


FIG. 219

two circles must intersect in *two* points, or touch in *one* point, or have *no* point in common.

173. Chord. A segment *joining* two points of a circle is a **chord,** Figure 220.

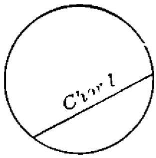


FIG. 220

174. Symbol for arc. The symbol $\frown$ means **arc.** Thus, arc AB may be written $\overset{\frown}{AB}$.

Draw two equal circles. Lay one circle upon the other, making the centers coincide. If $\overset{\frown}{AB}$ on one circle is equal to $\overset{\frown}{CD}$ on the other, they can be made to coincide.

How do the chords AB and CD compare?

In a given circle construct two equal arcs.

Relations between Central Angles, Arcs, Diameters, and Chords

175. Theorem:○ *In the same or equal circles equal central angles intercept equal arcs, and equal arcs are intercepted by equal central angles.*

For if the arcs are made to coincide, the central angles coincide, and *conversely.*

176. Subtending chord. The chord joining the end-points of an arc **subtends** (stretches under or across) the arc.

177. Theorem: ○ *In the same or equal circles equal arcs are subtended by equal chords; and conversely, equal chords subtend equal arcs.*

To prove this theorem show that $\angle C = \angle C'$, Figure 221.

Then prove
$$\triangle ABC \cong \triangle A'B'C'.$$
It follows that $\overline{AB} = \overline{A'B'}$.

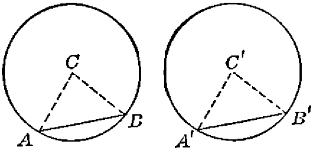


FIG. 221

To prove the converse, show that $\triangle ABC \cong \triangle A'B'C'$.

Then, $\angle C = \angle C'$ Why?

$\widehat{AB} = \widehat{A'B'}$ Why?

178. Theorem: ⊛ *A line drawn through the center of a circle perpendicular to a chord bisects the chord and the arcs subtended by the chord.*

Given $\odot O$† and CD drawn through the center O, intersecting the chord AB at E; also $CD \perp AB$, Figure 222.

To prove $AE = EB$; $\widehat{AD} = \widehat{DB}$; $\widehat{AC} = \widehat{CB}$.

Proof (method of congruent triangles):

Draw AO and OB.

Prove $\triangle AEO \cong \triangle BEO$.

Hence, $AE = EB$. Why?

and $y = y'$. Why?

Show that $\widehat{AD} = \widehat{BD}$ § 175.

and $\widehat{AC} = \widehat{CB}$.

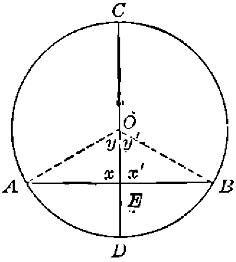


FIG. 222

† The symbol $\odot O$ means the circle whose center is O. The symbol Ⓢ means *circles*.

The theorem above is one of a group of theorems involving the following conditions:

1. A line passes through the center.
2. A line is perpendicular to a chord.
3. A chord is bisected by a line.
4. A minor arc is bisected.
5. A major arc is bisected.

By taking *as hypothesis* any two of these five conditions, and *as conclusion* one of the remaining three, we can form a number of theorems. Some of these are stated among the following exercises.

EXERCISES

1. A diameter that bisects a chord (not a diameter) is perpendicular to the chord and bisects the subtended arcs. Prove.

2. A line bisecting a chord and one of the subtended arcs passes through the center, is perpendicular to the chord, and bisects the other subtended arc. Prove.

Prove $\triangle ACE \cong \triangle BCE$, Figure 223.

Then $CD \perp AB$. Why?

Hence, CD passes through O. For the perpendicular bisector of a segment contains all points equidistant from its end points.

Show that $AD = DB$.

$\therefore \quad \overset{\frown}{AD} = \overset{\frown}{DB}$. Why?

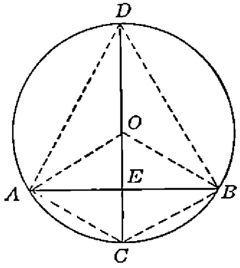


FIG. 223

3. The line segment joining the midpoints of the arcs into which a chord divides a circle is a diameter, bisects the chord, and is perpendicular to the chord. Prove.

4. A diameter bisecting an arc is the perpendicular bisector of the chord subtending the arc. Prove.

5. The perpendicular bisector of a chord passes through the center of the circle and bisects the subtended arcs. Prove.

6. A line perpendicular to a chord and bisecting one of the subtended arcs passes through the center of the circle, and bisects the chord and the other subtended arc. Prove.

7. A diameter that bisects a chord bisects the central angle between the radii drawn to the end points of the chord. Prove.

8. Bisect a given arc, using Exercise 1.

9. Given a circle, find the center, using Exercise 5.

10. Given an arc, find the center and draw the circle.

11. Draw a circle through three points not lying in the same straight line.

12. Show that the perpendicular bisectors of the sides of a polygon, all of whose vertices lie on the circle, meet in a common point.

13. Through a point within a circle draw a chord that will be bisected by the point.

14. Draw a circle that will pass through two given points and have a given radius.

15. If a circle is divided into three equal parts, and the points of division are joined by chords, an equilateral triangle is formed. Prove.

16. If the end points of a pair of perpendicular diameters of a circle are joined consecutively, what kind of polygon is formed? Prove.

17.* *Through three points not lying in a straight line one circle and only one can be drawn.*

179. Theorem:[⊛] *In the same circle, or in equal circles, equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.*

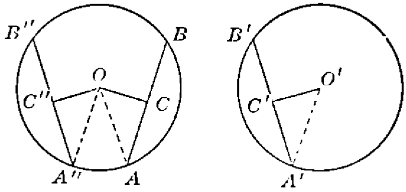


FIG. 224

Given $\odot O = \odot O'$, $AB = A'B' = A''B''$, $OC \perp AB$, $O'C' \perp A'B'$, $OC'' \perp A''B''$, Figure 224.

To prove $OC = OC'' = O'C'$.

Proof (method of congruent triangles):

Draw OA, OA'' and $O'A'$.

Prove that $AO = A'O' = A''O$.

Prove $\triangle AOC \cong \triangle A''OC'' \cong \triangle A'O'C'$.

Then, $OC = OC''$, and $OC = O'C'$.

Conversely, If $OC'' = O'C' = OC$,
prove $\triangle OAC \cong \triangle OA''C'' \cong {}'O'A'C'$.

Then $AC = A'C' = A''C''$ and $AB = A'B' = A''B''$.

EXERCISES

1. If two intersecting chords make equal angles with the line joining their common point to the center, the chords are equal. Prove.

2. In a circle the distances from the center to two equal chords are denoted by:

1. x^2+3x and $4(15-x)$.
2. $x(x+4)$ and $3(2x+5)$.
3. $x(x-3)$ and $4(3x-9)$.
4. $3x^2+4x$ and $12(1-x)$.

Find x and the distances from the center to the chords.

3. Find the locus of the midpoint of a system of chords intersecting in a point on the circumference.

Tangency

180. Theorem: ○ *The radius drawn to the point of tangency is perpendicular to the tangent; and conversely, a line perpendicular to a radius at the outer extremity is tangent to the circle.*

Given $\odot O$, AB tangent to $\odot O$ at A, Figure 225.

To prove that $OA \perp AB$.

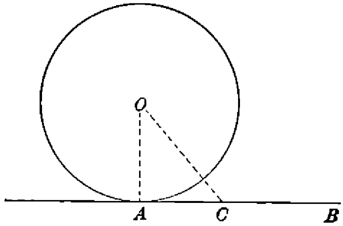


FIG. 225

Proof: Let C be any point on AB, not A.

Then C is outside of the circle. Why?

$\therefore$ $OC > OA$. Why?

Thus, OA is the shortest line that can be drawn from O to AB.

$\therefore$ $OA \perp AB$. The shortest line from a point to a given line is the perpendicular from the point to the given line.

Conversely, if $OA \perp AB$,

then $OA < OC$.

$\therefore$ C is outside of the circle.

$\therefore$ AB has but one point in common with the circle.

$\therefore$ AB is tangent to the circle.

EXERCISES

1. Show that the perpendicular to a tangent at the point of contact passes through the center of the circle.

2. Construct a tangent to a circle at a given point on the circle.

3. To a given circle draw a tangent that shall be parallel to a given line.

181. Theorem: *The arcs included between two parallel secants are equal; and, conversely, if two secants include equal arcs, and do not intersect within the circle, they are parallel.*

I. **Given** circle O and $AB \parallel CD$, cutting the circle at A and B, and at C and D, respectively, Figure 226.

To prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Proof: Draw $OE \perp AB$, and prolong it to meet CD.

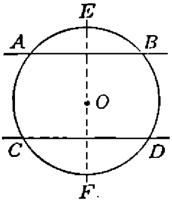


FIG. 226

Then $OE \perp CD$. Why?

$\overset{\frown}{CE} = \overset{\frown}{DE}$. Why?

$\overset{\frown}{AE} = \overset{\frown}{EB}$. Why?

$\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$. Why?

II. **Conversely, given** $\overset{\frown}{AC} = \overset{\frown}{BD}$, Figure 226, AB and CD not intersecting within the circle,

To prove $AB \parallel CD$.

Proof: Draw $OE \perp AB$ and prolong it to F.

Prove $\overset{\frown}{EC} = \overset{\frown}{ED}$. Why?

Then $EF \perp CD$. Why?

$\therefore AB \parallel CD$. Why?

III. Prove the theorem with one of the lines, as AB, tangent to the circle, as in Figure 227.

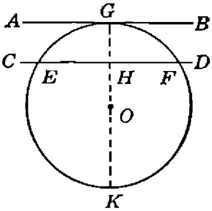


FIG. 227

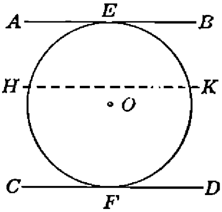


FIG. 228

IV. Prove the theorem with both parallels tangent to the circle, as in Figure 228.

Suggestion:

Draw $HK \parallel AB$ and apply Case III.

182. Theorem: *The line joining the centers of two intersecting circles bisects the common chord perpendicularly.*

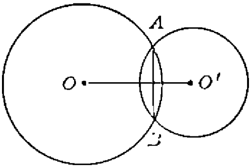


Fig. 229

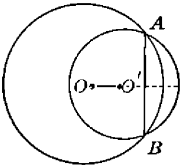


Fig. 230

Let O and O' be the intersecting circles, Figures 229, 230. Let AB be the common chord.

To prove OO' the perpendicular bisector of AB.

To prove this, draw OA, OB, $O'A$, and $O'B$.

Since $OA = OB$ and $O'A = O'B$, it follows that $OO' \perp AB$ and bisects it. Why?

183. Tangent circles. Two circles are said to be **tangent to each other** if both are tangent to the same line at the same point. This point is the **point of tangency,** or the **point of contact** of the circles.

If the tangent circles lie wholly without each other they are **tangent externally,** Figure 231.

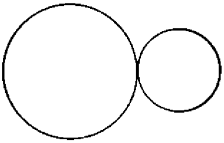


Fig. 231

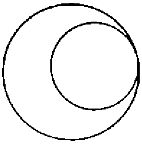


Fig. 232

If one of the tangent circles lies within the other they are **tangent internally,** Figure 232.

184. Theorem: *If two circles are tangent to each other, the centers and the point of tangency lie in a straight line.*

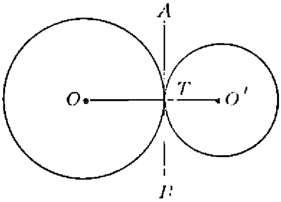


FIG. 233

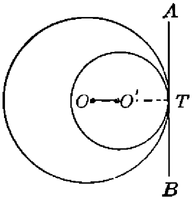


FIG. 234

I. **Let** O and O' be the centers of two circles tangent externally, T being the point of tangency, Figure 233.

To prove O, O', and T lie in a straight line.

Draw OT and $O'T$.

Prove that OTO' is a straight angle.

Then OT and $O'T$ are in a straight line. Why?

II. Prove the theorem for the case shown in Figure 234.

EXERCISES

1. Draw a circle tangent to a given circle at a given point. How many such circles can be drawn?

2. Draw a circle through a given point and tangent to a given circle.

3. If the distance between the centers of two circles is equal to the sum of their radii, the circles are tangent externally. Prove.

4. The distance between the centers of two tangent circles is $2\frac{1}{2}$ inches. The radius of one is $\frac{3}{4}$ inch. Draw the two circles.

5. The radii of three circles are 1 inch, $1\frac{1}{2}$ inches, and $\frac{3}{4}$ inch, respectively. Draw the circles tangent to each other externally.

6. Construct a circle with a given point as center and tangent to a given circle.

7. Find the locus of the midpoints of a system of parallel chords in a circle.

ARCHIMEDES

A R C H I M E D E S

ARCHIMEDES was born at Syracuse on the island of Sicily 287 B.C. and died there 212 B.C. It is said that he was related to the royal family of Syracuse. He studied mathematics under Conon at the University of Alexandria in Egypt. His great mechanical ingenuity was often called into the service of his government. He held it to be beneath the dignity of a scientist to apply his science to practical use; nevertheless he was the inventor of numerous practical devices and mechanical contrivances. Read about his detecting the fraudulent goldsmith, his invention of burning-glasses, his lever device for launching ships, and his device for pumping the water out of ships and even of inundated fields, etc. in Ball's *History of Mathematics*.

It was on the occasion of launching one of the king's large new ships that he remarked that he could move the earth if he but had a fulcrum to place his lever on.

He wrote many mathematical and scientific works, including important contributions to almost every field of science then known. He did especially valuable work in plane and solid geometry. In a book on the *Circle* he showed for the first time that the ratio of the length of a circumference to its diameter is between $3\frac{1}{7}$ and $3\frac{10}{71}$. He also worked out in another place the relations as to volume and surface of the cone, cylinder, and sphere. He regarded his discoveries on the round bodies as so important and so beautiful that he requested that the figure of a sphere inscribed in a cylinder be carved on his tombstone.

His contributions to pure and applied science were so numerous and so important that he is often referred to as "the Newton of antiquity." See whether you think the title appropriate after reading in Ball or elsewhere what both he and Newton did for science.

In his Sand Counter, Archimedes undertook to calculate and to express in numbers the number of grains of sand it would take to fill the universe. It is thought that the reason he did this was to show his scientific countrymen that there could be devised a great deal more powerful way of writing numbers than the way the Alexandrian scholars were teaching people to write them.

8. Given the radius of a circle to construct a circle tangent to two given circles.

9. With the vertices of a triangle as centers, construct three circles tangent to each other, Figure 235.

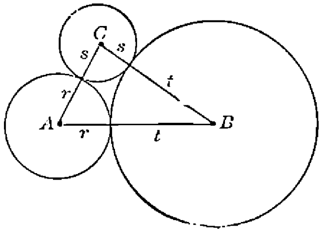


FIG. 235

Show algebraically that one of the radii is equal to half the perimeter diminished by one of the sides.

185. Historical note: The part of the theory of the circle that deals with chords, tangents, and secants is older than the time of Euclid. Most of it was probably first worked out by the Pythagoreans. It is well known that Archytas of Tarentum (430–365 B.C.), at a certain point in his construction of the problem of doubling a cube, assumed a knowledge of the theorem that the angle between a tangent and the contact-radius is a right angle. The first use of the theorem of the equality of the two tangents to a circle from an outside point of which we have knowledge is with Archimedes (287–212 B.C.). Heron (first century B.C.) is the first to give it place as an independent theorem.

The converse theorem, that the center of the circle lies on the bisector of the angle between two tangents is first met with in the seventh book of the *Synagoge* of Pappus about the end of the third century A.D. Archimedes is said to have written an entire work on the tangency of circles. The so-called *taction-problem* of Apollonius was to draw a circle which should fulfil *three* conditions, viz., go through a given point, be tangent to a given straight line, and a given circle. In the fourth book of his *Synagoge*, Pappus studied the problem to draw a circle tangent externally to three given circles, and treated another interesting problem, "through three points of a straight line to draw three other straight lines that should form an inscribed triangle within a given circle."

186. Supplementary exercises. The exercises following will give further practice in solving problems involving circles.

EXERCISES

Prove each of the following:

1. If the diameter of a circle bisects each of two chords, the chords are parallel.

2. Draw a radius *AO* of circle *O*. Through *A* draw two chords making equal angles with *AO*. Prove that the chords are equal.

3. From a given point of a circle two equal chords are drawn. Prove that they make equal angles with the radius drawn to the given point.

4. Two equal chords intersect each other within the circle. Prove that the segments of one are equal to the segments of the other.

5. The tangents from a point outside of a circle are equal.

6. Find the locus of the midpoints of equal chords of a circle.

7. Draw two central angles equal to 45° and 90°. Measure the intercepted chords. Are the chords proportional to the central angles?

8. Construct a line tangent to a circle and parallel to a given line.

9. Prove that a trapezoid inscribed in a circle is isosceles.

10. What is the locus of the centers of circles tangent to two intersecting lines?

11. The major arc cut off by a chord exceeds the minor arc by 50°. Find the number of degrees in each arc.

Summary of Chapter IX

187. New terms. The meaning of the following terms was taught:

circle	major arc
radius	minor arc
diameter	secant
center	tangent
concentric circles	chord
arc	subtending chord
semicircle	tangent circles

The following symbols were introduced: ⌒ for arc, ⊙ for circle, Ⓢ for circles.

188. Theorems. The truth of the following theorems has been shown by observation:

1. *A point is within, upon, or without, a circle according as its distance from the center is less than, equal to, or greater than, the radius.*

2. *Circles having equal radii are equal, and equal circles have equal radii.*

3. *A diameter divides a circle into equal parts.*

4. ○ *In the same or equal circles equal central angles intercept equal arcs,* and *equal arcs are intercepted by equal central angles.*

189. The following theorems have been proved:

1. ○ *In the same or equal circles equal arcs are subtended by equal chords;* and conversely, *equal chords subtend equal arcs.*

2. If any two of the following conditions are taken as hypothesis the remaining three are true:

(1) A line passes through the center.
(2) A line is perpendicular to a chord.
(3) A chord is bisected by a line.
(4) A minor arc is bisected.
(5) A major arc is bisected.

3. ⊛ *In the same or equal circles equal chords are equally distant from the center;* and conversely, *chords equally distant from the center are equal.*

4. *The arcs included between two parallel secants are equal;* and conversely, *if two secants include equal arcs, and do not intersect within the circle, they are parallel.*

5. *The line joining the centers of two intersecting circles bisects the common chord perpendicularly.*

6. *If two circles are tangent to each other, the centers and the point of tangency lie in a straight line.*

7. Two arcs are equal if one of the following conditions holds:

(1) The subtending chords are diameters.

(2) The central angles intercepting the arcs are equal.

(3) The subtending chords are equal.

(4) The arcs are intercepted by parallel chords, secants, and tangents.

8. Two chords are equal if one of the following conditions holds:

(1) The chords subtend equal central angles.

(2) The chords subtend equal arcs.

(3) The chords are equally distant from the center.

9. ○ *The radius drawn to a point of tangency is perpendicular to the tangent, and a line perpendicular to a radius at the outer extremity is tangent to the circle.*

10. * *Through three points not lying in a straight line one circle and only one can be drawn.*

CHAPTER X

MEASUREMENT OF ANGLES BY ARCS OF THE CIRCLE

190. Units of angular measure. In all preceding work you have measured angles by comparing them with such angular units as *degree, minute, second, right angle,* and *straight angle.* Thus, the measure of an angle is 45 if it contains 45 *degrees;* the measure of the same angle is $\frac{1}{2}$ if the *right angle* is used as unit; or it is $\frac{1}{4}$ if the *straight angle* is the unit of measure.

In the following it will be shown that, *if the sides of an angle touch or intersect a circle, it is possible to measure the angle in terms of the arcs intercepted*† *by the sides of the angle.*

EXERCISES

1. From cardboard cut a right angle. Move it so that the sides always pass through two fixed points, as A and B, Figure 236. This may be done by letting the sides always touch two pins stuck into the paper at A and B. Mark the position of the vertex for various positions of the angle. What is the locus of the vertex?

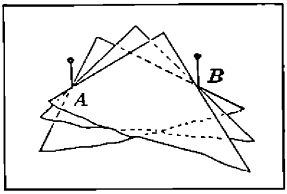


FIG. 236

2. Repeat Exercise 1 with an acute angle; with an obtuse angle.

3. Draw a semicircle. Join various points of the semicircle to the end points of the diameter, thereby forming angles whose vertices lie on the circle. With a protractor measure these angles. How do they compare in size?

† Note the difference between the words "intercept" and "intersect." The former means "to hold between" and the latter "to cut, or to cross."

191. Purpose of chapter X. If two lines intersect and also cut or touch a circle, the various positions may be illustrated as in Figures 237 to 243.

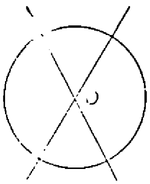


FIG. 237

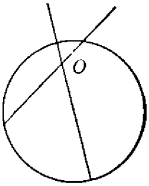


FIG. 238

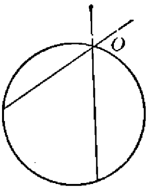


FIG. 239

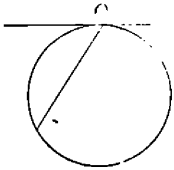


FIG. 240

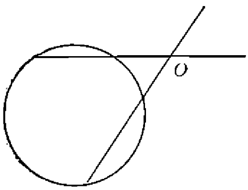


FIG. 241

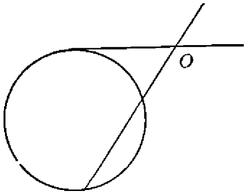


FIG. 242

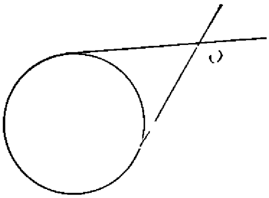


FIG. 243

In Figure 237 the lines intersect at the center of the circle, i.e., the angle is formed by *two radii.*

In Figure 238 the lines intersect within the circle, not at the center, i.e., the angle is formed by *two chords.*

Moving the intersecting lines until the vertex of the angle is on the circle, the angle becomes an *inscribed angle,* Figure 239.

Leaving one side of the angle, Figure 239, fixed, and turning the other until it is *tangent* to the circle, Figure 240 is obtained. In this figure the angle is formed by a *tangent* and a *chord.*

Figure 241 shows the lines intersecting *outside* of the circle, the angle now being formed by *two secants.*

Rotating the sides of the angle about O, Figure 241, until they became tangent to the circle, Figures 242 and 243 are obtained.

It is the purpose of chapter X to show how, in each of the Figures 237–243, the measure of the angle formed by the two intersecting lines may be expressed *in terms of the intercepted arc* or *arcs.*

192. Measure of a central angle. Let $\angle AOB$, Figure 244, be a *central angle,* and let it be divided into equal parts. If one of these is taken as a unit, the number of equal parts is the **measure** of the angle. What is the measure of $\angle AOB$?

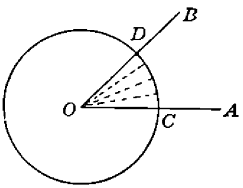


FIG. 244

Show that $\overset{\frown}{CD}$ is divided into equal parts.

If one of the equal parts of $\overset{\frown}{CD}$ is taken as the unit what is the measure of $\overset{\frown}{CD}$?

In general, if the measure of a central angle is m, the measure of the intercepted arc is also m. Why?

Briefly, we say that a central angle has the same measure as the intercepted arc, or that

A central angle is measured by the intercepted arc.

EXERCISES

1. Draw a central angle. With a protractor find the number of degrees, integral or fractional, contained in the angle. How many arc-degrees are there in the intercepted arc? What is a measure of the intercepted arc?

2. Draw a circle and mark off an arc. As in Exercise 1, find the number of arc-degrees contained in it. What is the measure of the arc?

3. Using ruler and compass only. divide a circle into 2 equal arcs, 4 equal arcs, 8 equal arcs.

4. Using ruler and compass only, construct arcs of 90°, 45°, 60°, 30°, 15°, 75°, 105°, 165°.

How may an angle of 90° be trisected? An angle of 45°?

5. Divide a circle into three arcs in the ratio 1:2:3.

Suggestion: Find algebraically the number of degrees in each. Then use the protractor to draw the arcs, as in Exercise 1.

6. A circle is divided into 4 arcs in the ratio 1:4:6:7. Find by algebra the number of degrees contained in each arc.

7.° *In the same or equal circles two central angles have the same ratio as the arcs intercepted by their sides.*

To show this, let the measures of the angles be m and n, respectively.

Show that the measures of the intercepted arcs are also m and n, respectively.

Then each ratio is $\frac{m}{n}$. Why?

8. The length of a circle is 63 inches. A central angle intercepts an arc 7 inches long. How many degrees does the angle contain?

9. In Figure 245, AB is a diameter. The number of degrees in $\angle AOC$ is denoted by x^2+4x, and in $\angle BOC$ by $3x^2+12x$. Find the values of x and the number of arc-degrees in arcs AC and CB.

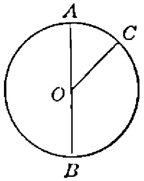


FIG. 245

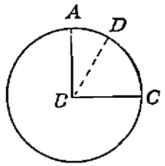


FIG. 246

10. $\angle ABC$, Figure 246, is a right angle. $\angle ABD = (2x^2-3)°$, and $\angle DBC = (10x^2-15)°$. Find the values of x and the number of degrees in arcs AD and DC.

193. Inscribed angle. An angle whose vertex is on the circle and whose sides are chords is an **inscribed angle.**

194. Measure of an inscribed angle. Draw an inscribed angle, as ABC, Figure 247. With a protractor, measure angle ABC. Find the number of arc-degrees in $\widehat{AC}$. (§ 192).

How does the measure of the inscribed angle compare with the measure of the arc?

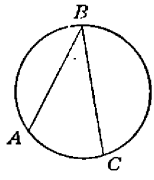


FIG. 247

195. Theorem:[⊛] *An inscribed angle is measured by one-half the arc intercepted by its sides.*

Let ABC, Figure 248, be an inscribed angle intercepting $\widehat{AC}$.

To prove that ABC is measured by $\frac{1}{2}\ \widehat{AC}$.

In proving the theorem three cases are considered:

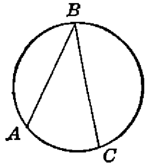


FIG. 248

Case I. The center of the circle lies *on one side* of the angle, Figure 249.

Proof: Draw the radius CD.

Denote the measures of ABC, ADC, and $\widehat{AC}$ by x, y, and x', respectively, and show that $\angle BCD = x$.

Hence, we have the relation

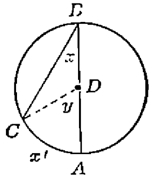


FIG. 249

$$x+x=y. \qquad \text{Why?}$$

Solving for x, $\quad x=\frac{1}{2}y.$

But, $\quad y=x'. \qquad$ § 192.

$$\therefore\ x=\tfrac{1}{2}x'. \qquad \text{By substitution.}$$

Case II. The center of the circle lies *within* the angle, Figure 250.

Proof: Draw the diameter BD.

$$x=y+z. \qquad \text{Why?}$$

$$y=\frac{y'}{2}. \qquad \text{Case I.}$$

$$z=\frac{z'}{2}. \qquad \text{Case I.}$$

$$\therefore\ x=\frac{y'+z'}{2}, \qquad \text{Why?}$$

or $\quad x=\frac{1}{2}x'. \qquad$ Why?

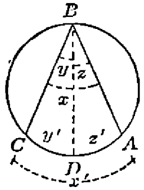


FIG. 250

Case III. The center of the circle lies *outside* of the angle, Figure 251.

Proof: Draw the diameter BD.

$$z = x + y. \quad \text{Why?}$$

$$\therefore \; x = z - y. \quad \text{Why?}$$

But $$z = \frac{z'}{2} \quad \text{Why?}$$

and $$y = \frac{y'}{2}. \quad \text{Why?}$$

$$\therefore \; x = \frac{z' - y'}{2}, \quad \text{Why?}$$

or $$x = \tfrac{1}{2}x'. \quad \text{Why?}$$

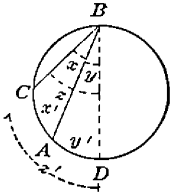


FIG. 251

196. Segment of a circle. The portion of a plane included between a chord and the arc it subtends is a **segment** of the circle. The shaded surface ABC, Figure 252, is a segment of circle O.

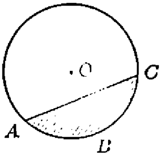


FIG. 252

EXERCISES

1. Draw a circle. With a chord cut off an arc greater than a semicircle and join various points of the arc to the end points of the chord. Compare the angles having the vertices on the arc.

2. Repeat Exercise 1, using an arc less than a semicircle.

Prove the following exercises:

3. All angles inscribed in the same segment of a circle are equal.

4. All angles inscribed in a semicircle are right angles.

5. All angles inscribed in a segment smaller than a semicircle are greater than a right angle.

6. All angles inscribed in a segment greater than a semicircle are less than a right angle.

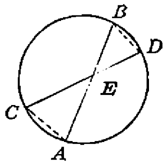


FIG. 253

7. Two chords, AB and CD, Figure 253, intersect within a circle. Show that $\triangle AEC$ and BED are mutually equiangular and therefore similar.

8. How does an inscribed angle vary as the arc increases from a short length to the length of the circle? Express your answer algebraically (Exercise 7, § 192).

9. Show how a carpenter's square may be used to test the accuracy of a semicircular groove, Figure 254.

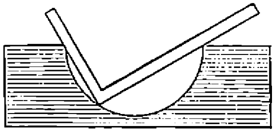


FIG. 254

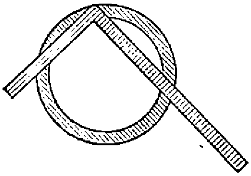


FIG. 255

10. Show how a carpenter's square may be used to find where a ring must be cut so that the two parts are equal, Figure 255.

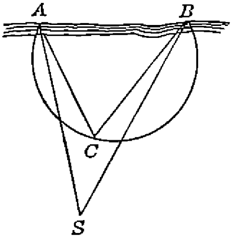


FIG. 256

11. The circle in Figure 256 represents a region of dangerous rocks to be avoided by ships passing near the coast AB. Outside the circle there is no danger. Show that the ship S is out of danger as long as angle ASB, found by observations made from the ship, is *less than* the known angle ACB.

12. An *inscribed triangle* is a triangle whose vertices lie on a circle. Two angles of an inscribed triangle are 82° and 76°. How many degrees are there in each of the three arcs subtended by the sides?

197. Theorem: *An angle formed by a tangent and a chord passing through the point of contact is measured by one-half of the intercepted arc.*

Let CD, Figure 257, be tangent to circle O, and let AB be a chord of the circle, drawn from the point of contact.

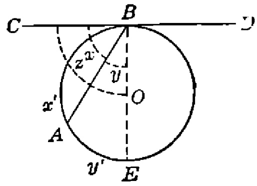


FIG. 257

To prove that $\angle ABC$ is measured by one-half of $\overset{\frown}{AB}$.

Proof: Draw the diameter BE.

Denoting the measures of $\measuredangle ABC$, EBA, and EBC by x, y, and z, respectively, and the measures of arcs BA, AE, and BAE by x', y', and z', we have the following relations:

	$z = x + y.$	Why?
	$\therefore\ x = z - y.$	Why?
But	$y = \frac{1}{2}y'.$	Why?
	Since $z = 90$ and $z' = 180$,	
	$\therefore\ z = \frac{1}{2}z'.$	
	$\therefore\ x = \frac{1}{2}z' - \frac{1}{2}y' = \frac{1}{2}(z' - y').$	Why?
	$\therefore\ x = \frac{1}{2}x'.$	Why?

EXERCISES

1. A triangle ABC, Figure 258, is inscribed in a circle and $\angle A = 57°$, $\angle B = 66°$. Tangents are drawn at A, B, and C forming the circumscribed triangle $A'B'C'$. Find the angles A', B', and C'.

2. Two angles of a circumscribed triangle $A'B'C'$ are 70° and 80°, Figure 258. Find the number of degrees in each of the three angles of the inscribed triangle ABC.

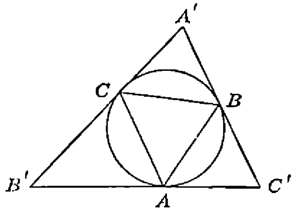


FIG. 258

198. Theorem: *If two chords intersect within a circle, either angle formed is measured by one-half the sum of the intercepted arcs.*

Draw AD, Figure 259.

Show that $x = y + z$.

$y = \frac{1}{2}y'$.

$z = \frac{1}{2}z'$.

$\therefore\ x = \frac{1}{2}(y' + z')$.

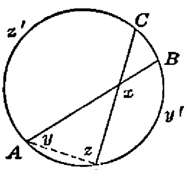


Fig. 259

199. Theorem: *If two secants meet outside of a circle the angle formed is measured by one-half the difference of the intercepted arcs.*

Draw AD, Figure 260.

Show that $y = x + z$

and $x = y - z$.

Complete the proof as in § 198.

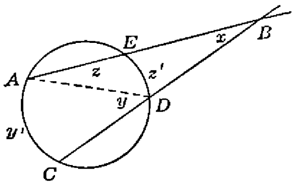


Fig. 260

200. Theorem: *The angle formed by a tangent and a secant meeting outside of a circle is measured by one-half the difference of the intercepted arcs.*

Draw CD, Figure 261.

Then $y = x + z$

and $x = y - z$.

$y = \frac{1}{2}y'$.

$z = \frac{1}{2}z'$.

$\therefore\ x = \frac{1}{2}(y' - z')$.

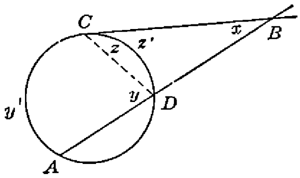


Fig. 261

201. Theorem: *The angle formed by two tangents to a circle is equal to one-half the difference of the intercepted arcs.*

Show that

$y = x + z$, Figure 262.

$x = y - z = \frac{1}{2}(y' - z')$.

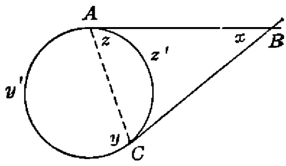


Fig. 262

EXERCISES

1. The arcs and angle being denoted as in Figure 263, find x and y by algebra.

2. Find x and y, Figure 264, the arcs and angle between the secants being as indicated in the figure.

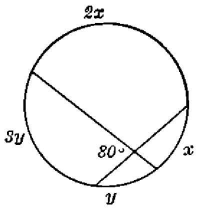


FIG. 263

3. When two tangents to a circle make an angle of 60° into what arcs do they divide the circle?

4. Into what arcs do two tangents at right angles to each other divide the circle?

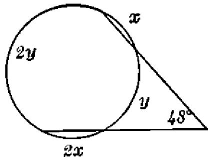


FIG. 264

5. Two tangents include two arcs of a circle, one of which is four times the other. How many degrees in the angle they form?

6. The angle between two secants intersecting outside of a circle is 76°. One of the intercepted arcs is 243°. Find the other.

7. The points of tangency of a circumscribed quadrilateral divide the circle into arcs in the ratio of 7:8:9:12. Find the angles of the quadrilateral.

8. Two tangents to a circle from an outside point form an angle of 70°. What part of the circle is the larger arc included by the points of tangency?

9. Prove that the tangents drawn from a point to a circle are equal, Figure 265.

10. The vertices of an inscribed quadrilateral divide the circle into arcs in the ratio 3:4:5:6. Find the angles of the quadrilateral.

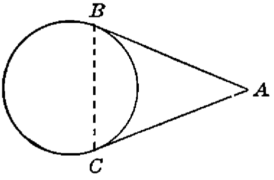


FIG. 265

202. Problems of construction. Make the following constructions:

1. *Upon a given line segment as a chord, construct a segment of a circle in which the inscribed angles are equal to a given angle.*

Given the line segment a and an angle equal to x, Figure 266.

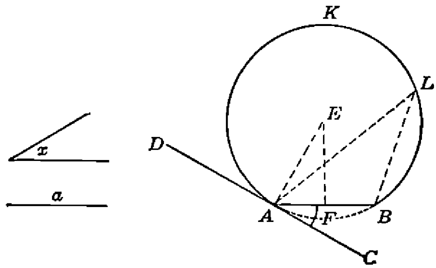


FIG. 266

To construct upon a as a chord a segment of a circle in which an angle equal to x may be inscribed.

Construction: Draw $AB = a$.

At A, on AB, construct $\angle CAB = x$.

At A draw the perpendicular to DC.

Draw the perpendicular bisector of AB.

With E, the point of intersection of the two perpendiculars, as center, and radius EA, draw a circle. This circle must pass through B. Why?

AKB is the required segment.

Proof: Let $\angle ALB$ be any angle inscribed in segment AKB.

Then	$\angle ALB = \frac{1}{2}\widehat{AB}$.	Why?
	$\angle BAC = \frac{1}{2}\widehat{AB}$.	Why?
$\therefore$	$\angle ALB = \angle BAC$.	Why?
$\therefore$	$\angle ALB$ is equal to x.	Why?

Test the accuracy of the construction with the protractor.

2. Make the construction of problem 1, using a given obtuse angle.

3. On a given line segment, construct a segment of a circle containing an inscribed angle of 60°, 30°, 120°, 45°, 135°, using ruler and compass only.

4. ○ *From a point outside of a circle to construct a tangent to the circle.*

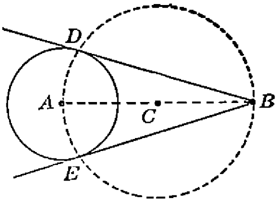


FIG. 267

Let A be the center of the given circle and B the given point outside of the circle, Figure 267.

To construct a tangent to circle A from B.

Construction: Draw AB and find the midpoint.

Draw a circle having AB as diameter, cutting circle A at D and E.

Draw BD and BE. They are the required tangents.

Proof: Draw AD and show that $\angle ADB$ is a right angle. Then BD is tangent to circle A. Why?

5. *To draw a common tangent to two circles exterior to each other.*

The number of common tangents to two circles depends upon the position of the circles. If one circle is entirely outside of the other, Figure 268, there are four common tangents, i.e., two external tangents, AB and CD, and two internal tangents, EF and GH.

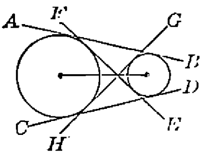


FIG. 268

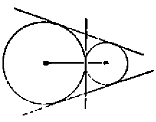


FIG. 269

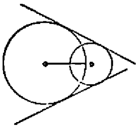


FIG. 270

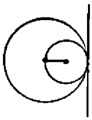


FIG. 271

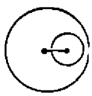


FIG. 272

If the circles are tangent to each other externally, there are two external and one internal tangent, Figure 269.

If the circles intersect, two external tangents can be drawn, Figure 270.

If the circles are tangent internally, there exists only one external tangent, Figure 271.

No common tangent exists if one circle lies entirely within the other, Figure 272.

Notice that in every case the line passing through the centers of the circles is an **axis of symmetry** of the figure.

Let A and A', Figure 273, be the centers of two circles exterior to each other.

I. *It is required to draw the common internal tangents.*

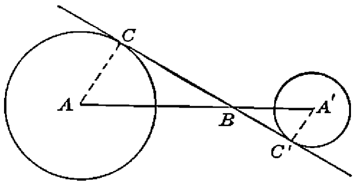


FIG. 273

Construction: Draw AA'.

Divide AA' into segments having the same ratio as the radii (§ 89), and let B be the point of division.

From B construct BC tangent to circle A (Exercise 4).

BC is one of the required internal tangents.

Proof: Draw AC. Draw $A'C' \perp CB$.

If it can be proved that $A'C'$ is equal to the radius of circle A', then BC is tangent to circle A' (§ 180).

Denote the radii of circles A and A' by R and R'.

Then $\frac{AB}{BA'} = \frac{R}{R'}$. By construction.

Prove $\triangle ABC \backsim \triangle A'BC'$

$\therefore \frac{AB}{BA'} = \frac{AC}{A'C'} = \frac{R}{A'C'}$. Why?

$\therefore \frac{R}{R'} = \frac{R}{A'C'}$. Why?

Prove that $A'C' = R'$.

$\therefore$ BC is tangent to circle A'. Why?

Show how to construct the other *common internal tangent.*

II. *To draw the external tangents.*

Construction: Draw AA', Figure 274.

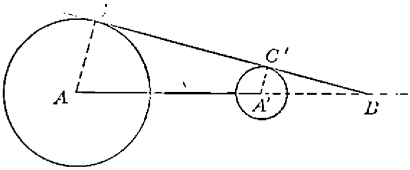


FIG. 274

Divide AA' externally in the ratio of the radii at the point B.

Draw BC tangent to circle A.

BC is one of the required external tangents.

Show how to construct the other *external tangent.*

The proof is the same as for Case I.

203. Miscellaneous exercises. A limited number of the exercises below may be worked:

1. Prove that the circles described on any two sides of a triangle as diameters intersect on the third side.

2. A circle described on one of the two equal sides of an isosceles triangle as a diameter, cuts the base at its middle point.

3. Prove that if a circle is circumscribed about an isosceles triangle, the tangents drawn through the vertices form an isosceles triangle.

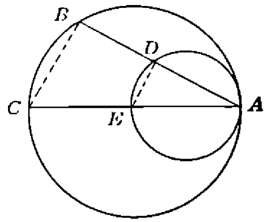


FIG. 275

4. From the point of tangency, A, Figure 275, of two circles tangent internally two chords are drawn, meeting the circles in B, C, D, and E. Prove $BC \parallel DE$.

5. (*Mathematical puzzle.*) Find the error in the proof of the following theorem: From a point not on a given line two perpendiculars may be drawn to the line.

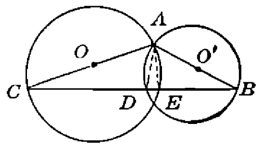


FIG. 276

In the two intersecting circles O and O', Figure 276, diameters AB and AC are drawn from A, one of the points of intersection of the circles.

Draw CB intersecting the circles in points D and E.

Draw AE and AD.

$\angle AEC$ is a right angle, being inscribed in a semicircle.

$\therefore \quad AE \perp CB.$

Similarly, $\angle ADB$ is a right angle.

$\therefore \quad AD \perp CB.$

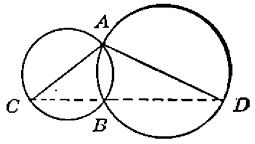


FIG. 277

6. Two circles intersect at points A and B, Figure 277. AC and AD are diameters. Prove that C, B, and D lie in the same straight line.

7. The angle between two secants is 30°, Figure 278. The number of degrees in arc DE is represented by $\frac{6x^2+29x+30}{2x+3}$, in the arc BC, by $\frac{2x^2-7x-15}{x-5}$. Find x and the number of degrees in each of the two arcs.

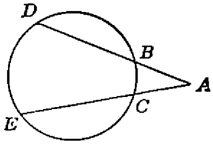


FIG. 278

Suggestion:

Reduce the fractions to lowest terms.

8. In Figure 279, $\angle AED$ is 60°, arc BC is represented by $\frac{x^2+8x+15}{x+3}$; arc AD, by $\frac{x^2+12x-45}{x+15}$. Find the number of degrees in each of the two arcs.

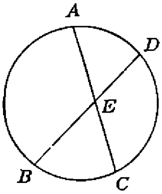


FIG. 279

9. Prove that *the sum of the three angles of a triangle is two right angles.*

Suggestions:

In Figure 280, let ABC be any triangle. Circumscribe a circle about it.

The three inscribed angles are measured by one-half the sum of the three arcs, AB, BC, and CA.

But the sum of the three arcs AB, BC, and CA is the entire circle.

∴ One-half the circle, or 180°, is the measure of the sum of the three angles of the triangle.

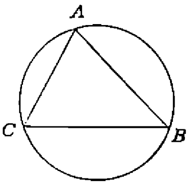


FIG. 280

10. A point moves so that the angle made by the two lines that connect it with two fixed points, C and D, is always the same. Find the *locus* of the point.

11. Prove that a parallelogram inscribed in a circle is a rectangle.

12. Two lines, Figure 281, are drawn through the point of tangency of two circles touching each other externally. If the lines meet the circles in points A, B, C, and D, prove $AB \parallel CD$.

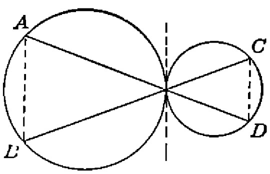


FIG. 281

13. Two circles intersect at points A and B. A variable secant through A cuts the circles in C and D. Prove that the angle CBD is constant for all positions of the secant.

14. Two circles are tangent to each other externally, and a line is drawn through the point of contact terminating in the circles. Prove that the radii to the extremities of the line are parallel.

15. Given two diagonals of a regular inscribed pentagon intersecting within it. Find the number of degrees in the angle between them.

16. In triangle ABC the altitudes BD and AE are drawn. Prove $\angle ABD = \angle AED$.

Suggestion:

Draw a semicircle on AB as diameter.

17. In laying a switch on a railway track, a "frog" is used at the intersection of two rails to allow the flanges of the wheels moving on one rail to cross the other rail. Show that the angle of the frog, a, Figure 282, made by the tangent to the curve and the straight rail DE, is equal to the central angle FOB of the arc BF.

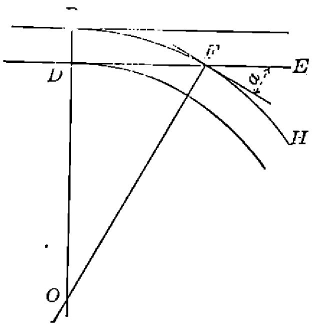


FIG. 282

18. Circular motion may be transmitted by means of a belt running over two pulleys, Figures 283, 284.

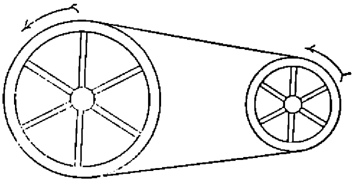


FIG. 283

Two pulleys whose radii, R and r, are 12 inches and 5 inches, respectively, are fastened to parallel shaftings and are connected by a belt, Figure 283. The distance, a, between the centers of the pulleys is 32 inches.

Find the approximate length, l, of the belt from the formula

$$l=\pi(R+r)+2a.$$

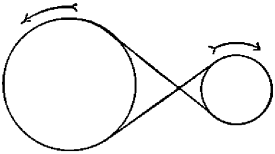


FIG. 284

In Figure 284 the pulleys are connected by a crossed belt. Find the approximate length of the belt by means of the formula

$$l=2\sqrt{(R+r)^2+a^2}+\pi(R+r).$$

Notice that the pulleys, as connected in Figure 284, turn in *opposite* directions.

19. A lunar eclipse occurs when the moon passes through the earth's shadow. If the moon is within the dark part of the shadow, Figure 285, the eclipse is said to be *total*. This part is

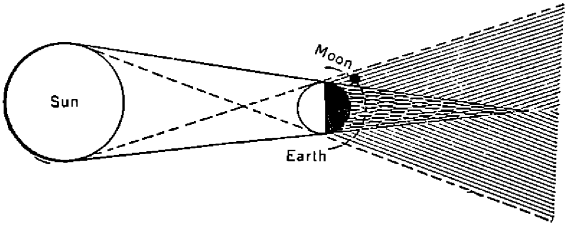


FIG. 285

included between the earth and the two *external tangents* common to the earth and the sun. If the moon is in the half-light region which is determined by the common internal tangents, the eclipse is said to be *partial*.

Find the length of the earth's shadow, taking the distance from earth to sun as 93,000,000 miles, the diameter of the sun as 866,500 miles, and the diameter of the earth as 8,000 miles.

20. Three circles, Figure 286, touch each other at A, B, and C. Lines AB and AC meet the third circle at E and D. Prove that E, O, and D lie in the same straight line.

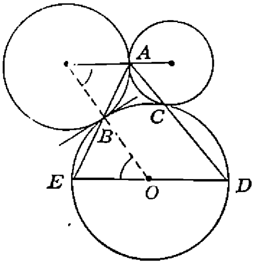


FIG. 286

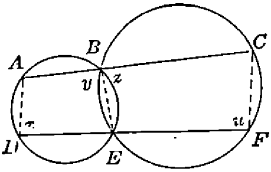


FIG. 287

21. In Figure 287, AC and DF are drawn through the points of intersection of two circles. Prove that $AD \parallel CF$.

Prove $\qquad x+y=180,\ u+z=180.$

$\therefore \qquad x=z,\ y=u.$

$\therefore \quad x+u=180.$

22. Prove that the common external tangent AB, Figure 288, to two circles that are tangent externally, is a mean proportional between the diameters of the circles.

Prove that $AE=EB=EF$ is a mean proportional between CF and FD.

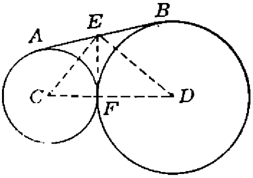


FIG. 288

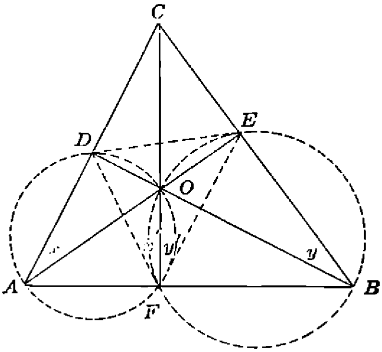


FIG. 289

23. Prove that a line from the center of a circle to the point of intersection of two tangents bisects the angle between the tangents.

24. Triangle DEF, Figure 289, is formed by joining the feet of the altitudes of $\triangle ABC$. Prove that the altitudes bisect the angles of $\triangle DEF$.

Suggestions:

Show that $x=y$, both being complements of $\angle ACB$.
Draw circles on AO and BO as diameters.
Show that $x'=x$ and $y'=y$.
$\therefore \quad x'=y'$.

Summary of Chapter X

204. New concepts. The chapter has taught the meaning of the following terms:

intercepted arc	circumscribed polygon
inscribed angle	measurement of angles by means of arcs
segment of a circle	
inscribed polygon	tangent common to two circles

205. Theorems established by observing. The following theorems were shown to be true:

1. *A central angle is measured by the intercepted arc.*

2. °*In the same or equal circles two central angles have the same ratio as the intercepted arcs.*

206. Theorems proved and constructions. The following theorems were proved:

1. ⊛*An inscribed angle is measured by one-half the arc intercepted by the sides.*

2. *An angle formed by a tangent and a chord passing through the point of contact is measured by one-half of the intercepted arc.*

3. *If two chords intersect within a circle either angle formed is measured by one-half the sum of the intercepted arcs.*

4. *If two secants meet outside of a circle the angle formed is measured by one-half the difference of the intercepted arcs.*

5. *The angle formed by a tangent and a secant meeting outside of a circle is measured by one-half the difference of the intercepted arcs.*

6. *The angle formed by two tangents to a circle is equal to one-half the difference of the intercepted arcs.*

The following constructions were taught:

1. *Upon a given line segment as a chord, construct a segment of a circle in which the inscribed angles are equal to a given angle.*

2. ○*From a point outside a circle to construct a tangent to a circle.*

3. *To draw the common external and internal tangents to two circles exterior to each other.*

207. Algebraic skills. The chapter has given practice in solving equations in one and two unknowns, and in evaluating and solving formulas.

CHAPTER XI

PROPORTIONAL LINE SEGMENTS IN CIRCLES

208. Relation between the segments of two intersecting chords. A railroad surveyor wishes to determine the radius of a **circular railway curve,** ABC, Figure 290. He measures chord AC, and segment BD, the part of the perpendicular bisector of AC intercepted by AC and arc ABC. If $AC = 200$ feet and $BD = 6$ feet, how may the radius be determined?

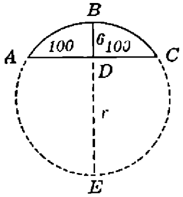


FIG. 290

If we can establish a relation between AD, DE, DC, and DB, the problem will easily be solved.

To find this relation, draw a circle, Figure 291, and chord AC, intersecting chord BE, as at D. Measure the segments AD, DE, DC, and DB, and compare $AD \cdot DC$ with $ED \cdot DB$.

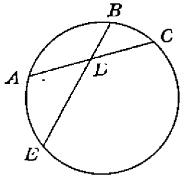


FIG. 291

Note the approximate equality of the products of the segments of each of the two chords.

To what is the difference, if any, probably due?

Show that in the foregoing problem the equality of the products is expressed by the equation $6r = 10,000$.

In general, the relation between the segments of two intersecting chords is expressed by the following theorem:

209. Theorem: ⊛ *If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other.*

State the hypothesis and the conclusion. Then prove the theorem as follows:

Proof: Draw BC and AE, Figure 292.

Prove $\triangle ADE \sim \triangle BDC$.

Show that $\frac{AD}{DB} = \frac{DE}{DC}$.

$\therefore\ AD \cdot DC = DE \cdot DB$. Why?

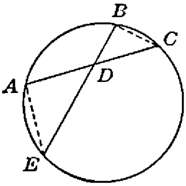


FIG. 292

EXERCISES

1. Solve the problem of § 208 by applying the theorem in § 209.

2. The segments of two intersecting chords are $x+5$ and $x-6$ of the one, and $x+2$ and $x-5$ of the other. Find x and the length of each chord.

3. A chord DC, Figure 293, cuts the chord AB at the midpoint E. ED is 4 inches longer than EC, and $AB = 16$ inches. Find the lengths of ED and EC approximately to $\frac{1}{100}$ inch.

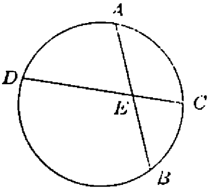


FIG. 293

4. The segments of intersecting chords are given below. Find x and the length of each chord.

	First Chord		Second Chord	
1	$x-4$	$x+8$	$x+3$	$x-4$
2	$x+2$	$x+6$	$x-4$	$x+18$
3	$2x+5$	$x+1$	$x+2$	$3x+2$
4	$2x+2$	$3x-5$	$x+1$	$x+5$

5. The distance between points A and B on a circular railroad curve is $2a$ feet, and the distance from the midpoint of the chord AB to the midpoint of the curve is b feet. Find the radius.

6. Find the radius of the circle in Exercise 5 if $a=100$, $b=4$; $a=150$, $b=5.6$.

210. Theorem: *If from a point without (outside of) a circle a tangent and secant are drawn, the tangent is a mean proportional between the entire secant (to the concave arc) and its external segment.*

State the hypothesis and the conclusion.

Proof: Draw DB and DC, Figure 294.

Show $\triangle ABD \sim \triangle ACD$.

$$\frac{AC}{AD}=\frac{AD}{AB}. \qquad \text{Why?}$$

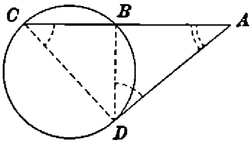


FIG. 294

EXERCISES

1. How far in one direction can a man see from the top of a mountain 2 miles above sea-level?

Let AB, Figure 295, represent the height of the mountain and let AD be the required distance.

Assuming the diameter of the earth to be 8,000 miles, the value of AD may be found by means of the preceding theorem.

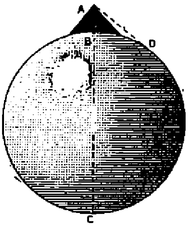


FIG. 295

2. Prove by means of the theorem in § 210 that the two tangents from an external point to a circle are equal.

3. A tangent and a secant are drawn from the same point outside of a circle. The secant measured to the concave arc is three times as long as the tangent, and the length of its external segment is 10 feet. Find the length of the tangent and secant.

4. Using Figure 296, prove that *the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

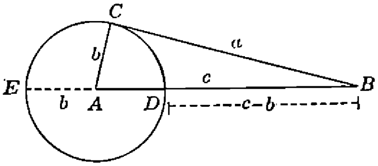


FIG. 296

Suggestions:

Let ABC be a right triangle having $\angle C = 90°$.

Show that $BE \cdot BD = \overline{BC}^2$.

Hence, $(c+b)(c-b) = a^2$, or, $c^2 = a^2 + b^2$.

5. *To divide a line segment into two parts so that the longer part is a mean proportional between the whole segment and the shorter part.*

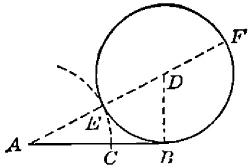


FIG. 297

Let AB be the given line segment, Figure 297.

To find the point C, such that

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

Construction: Draw $BD \perp AB$ at B, making $BD = \frac{AB}{2}$.

With D as center, and radius DB, draw circle D.

Draw AD cutting circle D at E and F.

On AB lay off $AC = AE$.

C is the required point.

Proof: $\frac{AF}{AB} = \frac{AB}{AE}$. Why?

$\frac{AF - AB}{AB} = \frac{AB - AE}{AE}$. Exercise 37, § 93.

$\frac{AF - EF}{AB} = \frac{AB - AC}{AC}$. By substitution.

$\frac{AE}{AB} = \frac{CB}{AC}$. Why?

$\frac{AC}{AB} = \frac{CB}{AC}$. Why?

$\therefore \frac{AB}{AC} = \frac{AC}{CB}$. By inversion.

211. Mean and extreme ratio.† A line segment is divided into **mean and extreme ratio** if the longer part is a mean proportional between the segment and its shorter part.

EXERCISES

1. Divide a segment 12 inches long into mean and extreme ratio, using the algebraic method.

Solution: Denote the longer segment by x.

Then $$\frac{12}{x}=\frac{x}{12-x}.$$

$$\therefore \quad 144-12x=x^2$$

Solve the equation and discard the negative root, which has no meaning in this case.

2. Divide a segment 5 inches long into mean and extreme ratio, using the geometric method. Check your result by the algebraic method.

3. If the length of a segment is denoted by m and the longer part by x, state the equation as in Exercises 1 and 2, and derive a formula by solving the equation for x.

† The current method of dividing a line in extreme and mean ratio is, according to an Arabian commentator, due to Heron of Alexandria. The theorem for dividing the line has been called by various names. Plato called it "The Section"; Lorentz (1781) called it "Continued Division."

Campanus (last half of the twelfth century) called continued division "a wonderful geometrical performance." Paciolo (1445–1514) gave it even higher esteem by writing an entire work dealing with problems in continued division and gave his work the title: *Divine Proportion.*

The peculiar mysticism of later times seized upon Paciolo's idea and went still beyond him. Ramus (1515–72) associated the divine trinity with the three segments of a continued division. Kepler (1571–1630) created a complete symbolism for his *sectio divina* ("divine section"). In the middle of the nineteenth century there arose a sort of amateurish natural philosophy that sought to subtilize mathematical laws in every branch of study. A kind of universal validity was fantastically ascribed to this continued division, and it was now christened "Golden Section."

This "Golden Section" was held to be not only the criterion for all metrical relations in nature, but it was also regarded as the "principle of beauty" in painting, architecture, and the plastic arts, as well.

212. Theorem: *If from a point without a circle two secants are drawn to the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

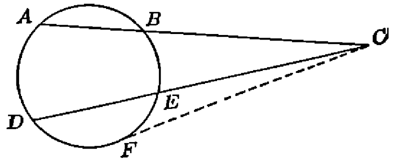


FIG. 298

Proof: From C draw CF, Figure 298, tangent to the circle.

Show by § 210 that $CA \cdot CB = \overline{CF}^2$

and $CD \cdot CE = \overline{CF}^2$.

EXERCISES

1. Two secants to the same circle from an outside point are cut by the circle into chords that are to their external segments as $\frac{5}{3}$ and $5(=\frac{5}{1})$. The first secant is 8 feet long. Find the length of the second secant.

2. The following exercises relate to two secants from an external point, as in Exercise 1. Find the length of the second secant.

	Ratio of Segments of First Secant	Ratios of Segments of Second Secant	Length of First Secant
1.	5:2	3:1	28 ft.
2.	3:1	5:2	28 ft.
3.	4:1	5:4	625 ft.
4.	4:1	4:3	25 ft.
5.	7:2	7:3	36 ft.

3. Prove that the tangents to two intersecting circles from any point on the extension of the common chord are equal.

4. Two lines drawn through the common points of two intersecting circles, Figure 299, meet the circles in A, B, C, and D, E, F, respectively. Prove $AD \parallel CF$.

Show that $\frac{GA}{GC} = \frac{GD}{GF}$.

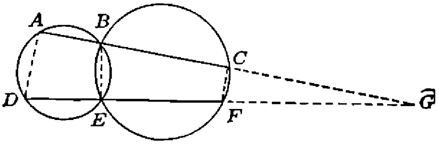


FIG. 299

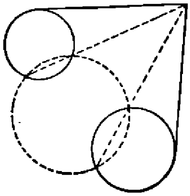


FIG. 300

5. Show how to find a point such that the tangents to two given circles are equal, Figure 300.

Summary of Chapter XI

213. Theorems. The following theorems were proved:

1. ⊛ *If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other.*

2. *If from a point without a circle a tangent and secant are drawn, the tangent is a mean proportional between the entire secant to the concave arc and the external segment.*

3. *If from a point without a circle two secants are drawn to the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

214. Constructions. The following construction was taught:

To divide a segment into mean and extreme ratio.

215. Algebraic skills. The chapter has offered opportunities for solving quadratic equations.

CHAPTER XII

REGULAR POLYGONS INSCRIBED IN, AND CIRCUMSCRIBED ABOUT, THE CIRCLE. CIRCUMFERENCE OF THE CIRCLE

Construction of Regular Polygons

216. Regular polygon. A polygon that is both *equilateral* and *equiangular* is a **regular polygon.**

217. Regular polygons in designs. Regular polygons are involved in many forms of decorative design. We use them in the tile floor, Figure 301; in the ornamental

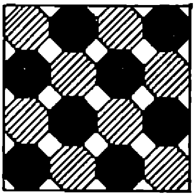


FIG. 301

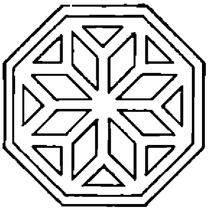


FIG. 302

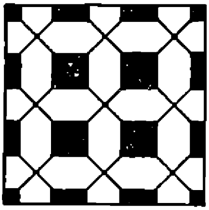


FIG. 303

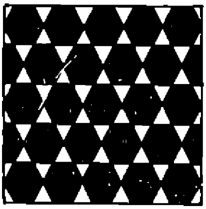


FIG. 304

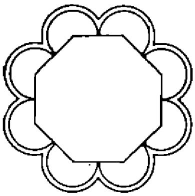


FIG. 305

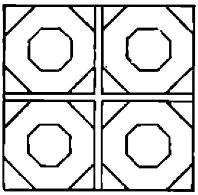


FIG. 306

window, Figure 302; in linoleum patterns, Figures 303–304; in paper doilies, Figure 305; in ceiling panels, Figure 306, floor borders and furniture designs. Find other uses of regular polygons.

It is the purpose of the first part of the chapter to show how to construct regular polygons.

EXERCISES

1. Show that an equilateral triangle is a regular polygon.

2. Draw a quadrilateral that is equilateral but not equiangular. What is such a quadrilateral called?

3. Draw an equiangular quadrilateral. What is such a quadrilateral called?

4. Draw a quadrilateral that is not equiangular and not equilateral.

5. Show that a square is a regular polygon.

6. Make a sketch of a regular pentagon; hexagon; octagon (8-side).

218. Inscribed polygon. A polygon whose vertices lie on a circle is an **inscribed polygon.** The circle is said to be *circumscribed* about the polygon.

Sketch an inscribed pentagon; hexagon.

219. Circumscribed polygon. A polygon whose sides are tangent to a circle is a **circumscribed polygon.** The circle is said to be *inscribed* in the polygon.

Sketch a circumscribed polygon.

The theorems in §§ 220 and 222 will be used when we wish to prove that an inscribed or circumscribed polygon is a regular polygon. They show that the construction of regular inscribed and circumscribed polygons depends upon the problem of dividing a circle into a given number of *equal parts.*

220. Theorem: °*If a circle is divided into equal arcs, the chords subtending these arcs form a regular inscribed polygon.*

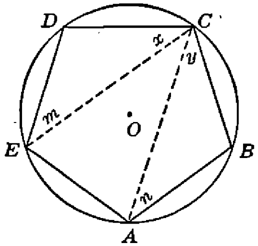


FIG. 307

Given the circle O, Figure 307, divided into equal arcs, AB, BC, CD, etc.

The polygon $ABCD$ . . . . , formed by the chords subtending these arcs.

To prove that $ABCD$ . . . . is a regular inscribed polygon.

Proof: I. Show that chords AB, BC, CD, ... are equal.

II. In triangles ABC and EDC, show that $x=y$, $m=n$ (§ 195).

$\therefore \angle D = \angle B.$ Why?

Similarly, prove that the other angles of the polygon are equal.

Hence, $ABCD$ . . . . is a regular inscribed polygon. Why?

221. Theorem: *If the midpoints of the arcs subtended by the sides of a regular inscribed polygon of n sides are joined to the adjacent vertices of the polygon, a regular inscribed polygon of $2n$ sides is formed.* Prove.

222. Theorem: °*If a circle is divided into equal arcs, the tangents drawn at the points of division form a regular circumscribed polygon.*

Given circle O, Figure 308; $\widehat{PQ} = \widehat{QR} = \widehat{RS}$, etc.; AB, BC, CD, etc., tangent to circle O, forming the circumscribed polygon $ABCD$ . . . . . .

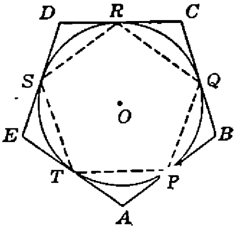


FIG. 308

To prove $ABCD$ . . . . a regular polygon.

Proof: Draw PQ, QR, RS, ... etc.

Prove that $\triangle PBQ$, QCR, RDS, etc. are congruent triangles.

$$\angle A = \angle B = \angle C, \text{ etc.}$$

Prove that $\triangle PBQ$, QCR, RDS, etc. are isosceles.

$$AP = PB = BQ = QC, \text{ etc.}$$

	$AP = BQ$	Why?
and	$PB = QC$	Why?
	$AB = BC$	Why?

Similarly, prove $BC = CD = DE$, etc.

Hence, $ABCD$ . . . . is a *regular* polygon.

223. Theorem: *If tangents are drawn to a circle at the midpoints of the arcs terminated by consecutive points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon is formed having double the number of sides.* Prove, Figure 309.

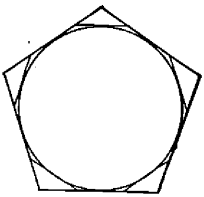


FIG. 309

224. Problem: ○*To inscribe a square in a given circle.*

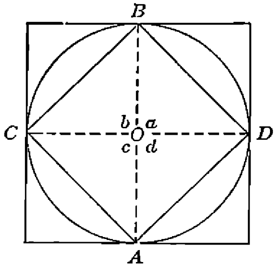


FIG. 310

Analysis: Since the square is a *regular* quadrilateral, we can inscribe a square if we can divide the circle into four equal arcs.

A circle may be divided into four equal arcs by dividing the plane around the center into four equal angles.

Since the sum of the angles around the center is 360°, each of the four equal angles must be 90°.

Hence the solution of the problem depends upon the construction of four right angles at the center.

Given circle O, Figure 310.

Required to inscribe a square in circle O.

Construction: Draw the diameter AB.

Draw diameter $CD \perp AB$.

Draw AD, DB, BC, and CA.

Then $ADBC$ is the required square.

Proof: $a=b=c=d=90$. Why?

$\overset{\frown}{AD}=\overset{\frown}{DB}=\overset{\frown}{BC}=\overset{\frown}{CA}$. Why?

$ADBC$ is a *regular* quadrilateral. Why?

$ADBC$ is a square. Why?

225. Problem: *To circumscribe a square about a given circle.*

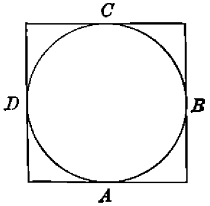


FIG. 311

Proceed as in the construction in § 224 and draw tangents at A, B, C, and D, Figure 311.

EXERCISES

1. Prove that an equilateral inscribed polygon is regular.

Suggestion: Show that the circle is divided into equal arcs. Then apply § 220.

2. Prove that an equiangular circumscribed polygon is regular.

Suggestion: Show that the circle is divided into equal arcs. Then use § 222.

3. Denoting the side of the inscribed square by a, the radius by r, prove that $a=r\sqrt{2}$.

The problem may be solved by algebra, or by trigonometry:

(*a*) Apply the theorem of Pythagoras to the sides of triangle AOD, Figure 312.

Then $r^2+r^2=a^2$.

$\therefore \quad a^2=2r^2$.

$\therefore \quad a=r\sqrt{2}$.

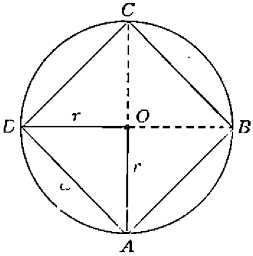


FIG. 312

(*b*) Find the required relation using the sine of 45°.

Notice that the equation $a=r\sqrt{2}$ expresses the fact that the side of the inscribed square depends upon the radius. As r varies, a varies also, but the ratio $\frac{a}{r}$ remains the same. The side a is said to *vary directly* as r.

4. Express the side a of the circumscribed square in terms of the radius r.

5. Express the perimeters of the inscribed and circumscribed squares in terms of the radius; in terms of the diameter.

6. Prove that the point of intersection of the diagonals of a square is the center of the inscribed and circumscribed circles.

7. Show how to construct regular polygons of 8, 16, 32, etc., sides.

Show that the number of sides of the polygons is expressed by the formula 2^n, where n is a positive integer equal to, or greater than 2.

226. Problem: ° *To inscribe a regular hexagon in a given circle.*

Analysis: Into how many equal arcs must the circle be divided?

How large must the central angles be that intercept these arcs?

State a simple way of constructing an angle of 60°.

Construction: With A as center and radius AO, Figure 313, draw an arc cutting the circle at B.

With B as center and the same radius draw the arc at C.

Similarly, draw arcs at D, E, and F.

Draw the polygon $ABCDEF$. This is the required hexagon.

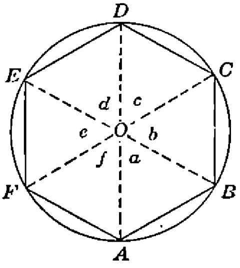


FIG. 313

Proof: Draw OA, OB, OC, etc.

Show that triangles AOB, BOC, ... EOF are equilateral.

$$\therefore \quad a=b=c=d=e=60.$$
$$a+b+c+d+e+f=360.$$
$$a+b+c+d+e \quad =300.$$
$$\therefore \quad f=60.$$

Prove that $\overset{\frown}{AB}=\overset{\frown}{BC}\ .\ .\ .\ .\ .\ =\overset{\frown}{FA}$.

Then polygon $ABCDEF$ is regular. Why?

227. Problem: *To circumscribe a regular hexagon about a given circle.*

EXERCISES

1. Express the relation between the side a of the regular inscribed hexagon and the radius r.

2. Express in terms of the radius the side of the regular circumscribed hexagon.

Solution: Draw OA and OK, Figure 314.

Show that triangle AOK is a 60°–30° right triangle.

Hence $AO = 2AK = a$.

Find the required relation between a and r, either by using the theorem of Pythagoras, or by using the tangent of 30°.

Show that the side depends upon the radius.

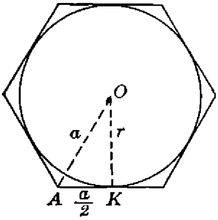


FIG. 314

3. Show that the side of a regular circumscribed hexagon varies as the radius of the circle varies, but that the ratio of the side to the radius remains constant. This fact is expressed briefly by the statement: The side *varies directly* as the radius.

4. Inscribe and circumscribe an equilateral triangle, a regular 12-side, 24-side, etc.

5. Express, in terms of the radius r, the side of the inscribed equilateral triangle.

Suggestions:

Show that OK, Figure 315, is $\frac{r}{2}$.

Obtain the required relation by using the theorem of Pythagoras or by using the tangent of 60°.

Express your result in the language of variation.

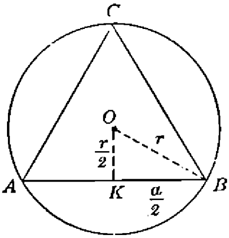


FIG. 315

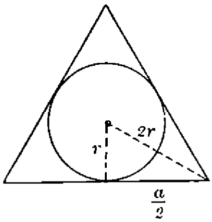


FIG. 316

6. Show that the side of the circumscribed equilateral triangle is $2r\sqrt{3}$, Figure 316.

7. Express, in terms of the radius r, the perimeters—(a) of the regular inscribed and circumscribed hexagon, (b) of the equilateral inscribed and circumscribed triangles.

Show that the perimeters vary directly as the radii.

228. Problem: *To inscribe a regular decagon in a given circle.*

Analysis: Into how many equal arcs must the circle be divided?

How large must be the central angles intercepting these arcs?

The construction of an angle of 36° depends upon the problem of dividing a segment into mean and extreme ratio (see § 210, Exercise 5).

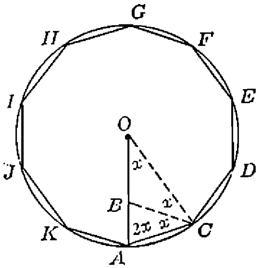


FIG. 317

Construction: Draw the radius AO, Figure 317.

Divide AO into mean and extreme ratio at B, making

$$\frac{OA}{OB}=\frac{OB}{BA}.$$

With A as center, and radius OB, draw an arc at C.
With the same radius and center C draw an arc at D.
Similarly, draw arcs at E, F, G, H, I, J, and K.
Draw AC, CD, etc.
Polygon $ACD \ldots K$ is the required polygon.

Proof: Draw BC and OC.

$$\frac{OA}{OB}=\frac{OB}{BA}. \qquad \text{By construction}$$

$$\frac{OA}{AC}=\frac{AC}{BA}. \qquad \text{By substitution.}$$

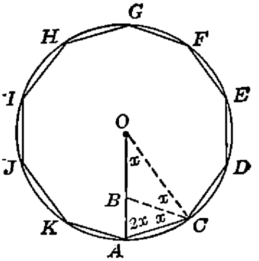


FIG. 317

$\angle CAO = \angle BAC.$

$\therefore \quad \triangle BCA \backsim \triangle AOC.$ Why?

$\therefore \quad \dfrac{BC}{OA} = \dfrac{CA}{OC}.$ Why?

$\therefore \quad \dfrac{BC}{OA} = \dfrac{CA}{OA}.$ By substitution.

$BC = CA.$ Why?

$BC = OB.$ Why?

Denote $\angle BOC$ by x.

Then $\angle BCO = x.$ Why?

Show that $\angle BCA = x.$

Show that $\angle OAC = \angle OCA = 2x.$

$\therefore \quad 2x + 2x + x = 180.$ Why?

$\therefore \quad x = 36.$

Show that polygon $ACD \ldots . K$ is a regular decagon.

EXERCISES

1. To circumscribe a regular decagon about a circle.

2. Show how to inscribe and circumscribe a regular pentagon in a given circle.

3. To inscribe and circumscribe regular polygons having 20, 40, etc., sides.

4. Express the relation between the side of the inscribed decagon and the radius of the circle.

Denote $AC = OB$ by a, Figure 318; OA, by r.

Then $\quad BA = r - a.$

Show that $\quad \dfrac{r}{a} = \dfrac{a}{r-a}.$

$$a^2 = r^2 - ra.$$

$$a^2 + ra - r^2 = 0.$$

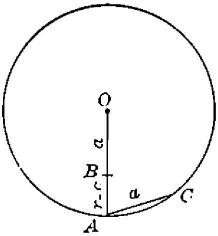


FIG. 318

Solve by means of the quadratic formula.

Then $a=\dfrac{-r\pm\sqrt{r^2+4r^2}}{2}$

or $a=\dfrac{-r\pm r\sqrt{5}}{2}=\dfrac{r}{2}(-1\pm\sqrt{5}).$

Show why the minus sign before the radical cannot be used *in this problem.*

$$a=\frac{r}{2}\left(\sqrt{5}-1\right)=\frac{r}{2}\left(1.236\right)=.618r.$$

5. Show that the side of a regular inscribed pentagon is equal to $\dfrac{r}{2}\sqrt{10-2\sqrt{5}}.$

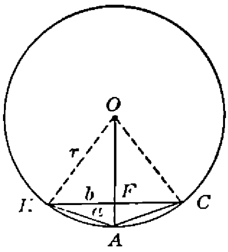


FIG. 319

Let KC, Figure 319, be the side of the pentagon, KA and AC sides of the decagon.

Denote KF by b; OK, by r; and KA, by a.

Then $\overline{OF}^2=r^2-b^2$ and $OF=\sqrt{r^2-b^2}.$ Why?

$$\therefore\quad FA=r-\sqrt{r^2-b^2}.$$

Since $\overline{KF}^2=\overline{KA}^2-\overline{FA}^2,$

$$b^2=a^2-\left(r-\sqrt{r^2-b^2}\right)^2.\quad\text{Why?}$$

Substituting for a^2 its equal, $\dfrac{r^2}{4}\left(\sqrt{5}-1\right)^2$ (Exercise 4), and solving for b, we have

$$b=\frac{r}{4}\sqrt{10-2\sqrt{5}}.$$

$$2b=\frac{r}{2}\sqrt{10-2\sqrt{5}}.$$

6. Show that an approximate value of $\sqrt{10-2\sqrt{5}}$ is $2.351+$.

7. Using the sine function, find the side of the regular inscribed pentagon; decagon.

Notice the advantage of the trigonometric method over the algebraic methods used in Exercises 4 and 5.

CARL FRIEDRICH GAUSS

CARL FRIEDRICH GAUSS

CARL FRIEDRICH GAUSS was born at Brunswick, Germany, April 30, 1777, and died at Göttingen, February 23, 1855. His father was a bricklayer and did not sympathize with the son's aspirations for an education. Coupled with this was the fact that the schools of Gauss's day were very poor; but in spite of parental disapproval and very inadequate schools he became one of the greatest mathematicians of all time.

Gauss had a marvelous aptitude for calculation, and in later years used to say, perhaps only as a joke, that he could reckon before he could talk. He owed his education to the fact that one of his teachers, named Bartels, drew the attention of the reigning Duke of Brunswick to the remarkable talents of the boy. The Duke provided for him the means of obtaining a liberal education. As a boy Gauss studied the languages with quite as much success as mathematics.

When only nineteen, Gauss, discovered a method of inscribing a regular polygon of seventeen sides in a circle. This encouraged him to pursue mathematical studies. He studied at Göttingen from 1795 to 1798. He made many of his most important discoveries while yet a student. His favorite study was higher arithmetic. In 1798 he went back to his home town of Brunswick, and for a few years earned a scanty living by private tuition.

In 1801 Gauss published a volume on higher arithmetic. His next great performance was in the field of astronomy. He invented a method for calculating the elements of a planetary orbit from three observations, by so powerful an analysis of existing data as to place him in the first rank of theoretical astronomers.

In 1807 he was appointed professor of mathematics and director of the observatory at Göttingen. He retained these offices until his death. He was devoted to his work. He never slept away from his observatory except on one occasion when he attended a scientific congress in Berlin. As a teacher he was clear and simple in exposition, and for fear his auditors might not get his train of thought perfectly he never allowed them to take notes. His writings are more difficult to follow, for he omitted the developmental details that he was so careful to supply in his lectures. His memoirs in astronomy, in geodesy, in electricity and magnetism, an electrodynamics, and in the theories of numbers and celestial mechanics are all epoch making. Most of the whole science of mathematics has undergone a complete change of form by virtue of Gauss's work.

A good description of Gauss's important work on the inscription of a regular polygon in a circle may be read in § 35 of Miller's *Historical Introduction to Mathematical Literature* (Macmillan).

The last-mentioned work (pp. 241–43), and also both Ball's and Cajori's *Histories*, give brief accounts of Gauss and his work.

8. A man has a round table-top which he wishes to change into the form of a pentagon as large as possible. The diameter of the top is $2\frac{1}{2}$ feet. What is the length of the cut required?

229. Problem: *To construct a regular 15-side in a given circle.*

Analysis: The circle must be divided into 15 equal arcs. How large are the central angles intercepting these arcs?

Notice that $24° = 60° - 36°$.

This suggests the following construction:

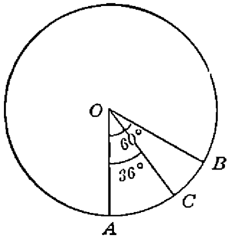


FIG. 320

Construction: At O on OA construct an angle of 60°, Figure 320.

At O on OA construct an angle of 36°, as $\angle AOC$.

Then $\angle COB = 24°$.

$\therefore$ CB may be taken as the side of the regular inscribed 15-side.

EXERCISES

1. Show how to construct regular inscribed and circumscribed polygons having 30, 60, 120. . . . sides.

Gauss (1777–1855), a German mathematician, proved that by the use of an unmarked straightedge and a compass a circle can be divided into $(2^k + 1)$ equal parts, k being a number that makes $2^k + 1$ a prime number.

Denoting $2^k + 1$ by n, we have:

For $k = 1$, $n = 3$, a prime number.

For $k = 2$, $n = 5$, a prime number.

For $k = 3$, $n = 9$, not a prime number.

For $k = 4$, $n = 17$, a prime number.

For $k = 5$, $n = 33$, not a prime number, etc.

2. Find the side of a decagon inscribed in a circle of radius 8; 10; 15; a.

3. The side of an inscribed pentagon is 18.8 inches. Find the radius of the circumscribed circle (Exercise 5, § 228).

4. The side of an inscribed decagon is 14.83 inches. Find the radius of the circumscribed circle.

5. *If at the midpoints of the arcs subtended by the sides of a given regular inscribed polygon, tangents are drawn to the circle, they are parallel to the sides of the given polygon and form a regular circumscribed polygon.*

To prove that $AB \parallel A'B'$, Figure 321, draw the radius OP' to the contact point of $A'B'$. Show that AB and $A'B'$ are both perpendicular to OP'.

To prove that $A'B'C'D'E'$ is regular, show that $\widehat{P'Q'} = \widehat{Q'R'} = \widehat{R'S'}$, etc.

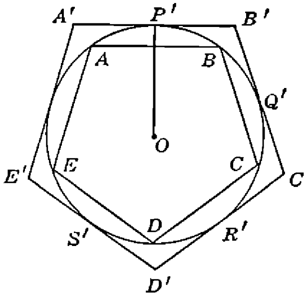


FIG. 321

6. In Figure 321, prove that points O, B, and B' are on a straight line.

Suggestion: Prove that B and B' lie on the bisector of $\angle P'OQ'$.

7. The following is a practical method of constructing the side of a regular 10-side and 5-side.

Construction: Draw the diameter AB, Figure 322.

Draw $OC \perp AB$.

Bisect OB at D.

With center at D and radius DC, draw the arc CE.

Draw the straight line CE.

The sides of triangle EOC are equal to the sides of a regular hexagon, pentagon, and decagon, respectively.

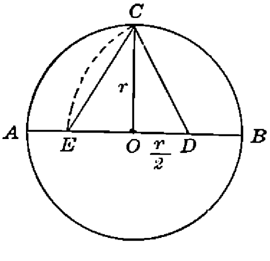


FIG. 322

Proof: I. $CO=r$ and is equal to the side of the regular inscribed hexagon.

II. $\overline{CD}^2=r^2+\frac{r^2}{4}$.

$$CD=\frac{r}{2}\sqrt{5}.$$

$$EO=ED-OD=CD-OD=\frac{r}{2}\sqrt{5}-\frac{r}{2}=\frac{r}{2}\left(\sqrt{5}-1\right).$$

Hence, EO is the side of the decagon (see § 228, Exercise 4).

III. $\overline{EC}^2=r^2+\overline{EO}^2=r^2+\frac{r^2}{4}\left(6-2\sqrt{5}\right)=\frac{4r^2+6r^2-2r^2\sqrt{5}}{4}$.

$EC=\frac{r}{2}\sqrt{10-2\sqrt{5}}$, the side of the pentagon (see § 228, Exercise 5).

230. Theorem:* *A circle may be circumscribed about any given regular polygon.*

Given the regular polygon $ABCD\ldots$, Figure 323.

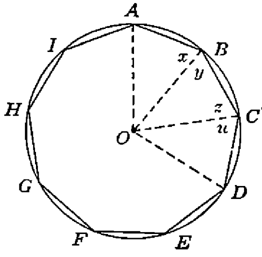


FIG. 323

To construct a circle circumscribed about $ABCD\ldots$.

Construction: Construct a circle through A, B, and C.

This is the required circle.

Proof: It is to be proved that the circle ABC passes through D, E, etc.

$x+y=z+u$.	Why?
$y=z$.	Why?
$\therefore\ x=u$.	Why?

Prove $\triangle AOB\cong\triangle COD$.

$\therefore\ AO=OD$ and the circle passes through D.

Similarly, it may be shown that the circle passes through E, F, etc.

231. Theorem: *A circle may be inscribed in any given regular polygon.*

Given the regular polygon $ABC \ldots$, Figure 324.

Required to inscribe a circle within $ABC \ldots$

Construction: Construct the center, O, of the circumscribed circle.

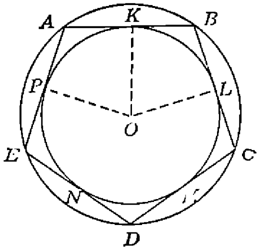


FIG. 324

Draw $OK \perp AB$.

With O as center and radius OK, draw circle $KLM \ldots$

This is the required circle.

Proof: Draw the circumscribed circle $ABC \ldots$

Draw $OP \perp AE$.

Since chord $AB =$ chord AE, it follows that $OK = OP$. Why?

Hence, circle KLM passes through P. Why?

$\therefore$ AE is tangent to the circle. Why?

Similarly, show that ED, DC, etc., are tangents to circle $KLM \ldots$

232. Theorem: *The perimeter of a regular inscribed 2n-side is greater than the perimeter of the regular n-side inscribed in the same circle.* Prove.

233. Theorem: *The perimeter of a regular circumscribed 2n-side is less than the perimeter of the regular n-side circumscribed about the same circle.* Prove.

The Circumference of the Circle

234.○ **Circumference formula.** You are familiar with the formulas $c = 2\pi r$ and $c = \pi d$, and have used them previously in solving problems involving the circumference of the circle. The following shows how the formulas may be derived by trigonometry.

Two important facts follow from the theorems in §§ 232 and 233, viz.:

1. The perimeter of the regular inscribed polygon *increases* as the number of sides *increases*.

2. The perimeter of the regular circumscribed polygon *decreases* as the number of sides *increases*.

It will be shown that by increasing the number of sides of regular inscribed and circumscribed polygons, the perimeters approach each other more and more, and that the decimal fractions expressing these two perimeters can be made to agree to a greater and greater number of decimal places.

It is easily proved that the length of a circle is greater than the perimeter of any inscribed polygon. We will assume that the length of a circle is less than the perimeter of any circumscribed polygon.

Hence, *the length of a circle lies between the lengths of the perimeters of any pair of inscribed and circumscribed polygons.*

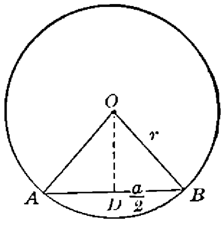


FIG. 325

Let AB, Figure 325, be the side of a regular inscribed n-side.

Draw $OD \perp AB$.

Show that $\angle DOB = \left(\frac{360}{2n}\right)^\circ$.

Let a denote the side of a regular inscribed polygon.

Show that $\sin\left(\frac{360}{2n}\right)^\circ = \frac{\frac{a}{2}}{r} = \frac{a}{2r} = \frac{a}{d}$.

Hence, $a = \left[\sin\left(\frac{360}{2n}\right)^\circ\right] d$. Why?

the perimeter, $p = n\left[\sin\left(\frac{360}{2n}\right)^\circ\right] d$. Why? . . (A)

From Figure 326 show that the perimeter P of the circumscribed polygon is given by

$$P=n\left[\tan\left(\frac{360}{2n}\right)^{\circ}\right]d \ldots\ldots.(B)$$

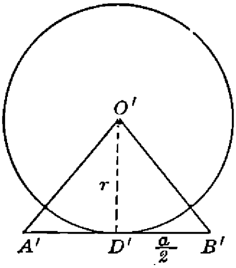


FIG. 326

By means of formulas (A) and (B) the perimeters of inscribed and circumscribed polygons may be computed, leading to the determination of approximate values of the length of the circle.

Make the computations and compare your results with the results given in the following table:

Number of Sides	Perimeter of Inscribed Polygon	Length of Circle, l	Perimeter of Circumscribed Polygon
3	$2.5980\ldots.d$	$2.5d<l<5.2d$	$5.1963\ldots.d$
4	$2.8284\ldots.d$	$2.8d<l<4.0d$	$4.0000\ldots.d$
5	$2.9390\ldots.d$	$2.9d<l\quad 3.7d$	$3.6325\ldots.d$
6	$3.0000\ldots.d$	$3.0d<l<3.5d$	$3.4644\ldots.d$
7	$3.0359\ldots.d$	$3.0d<l<3.4d$	$3.3691\ldots.d$
8	$3.0614\ldots.d$	$3.0d\quad l<3.4d$	$3.3137\ldots.d$
12	$3.1058\ldots.d$	$3.1d<l<3.3d$	$3.2153\ldots.d$
18	$3.1248\ldots.d$	$l=3.1d$, approximately	$3.1734\ldots.d$
90	$3.1410\ldots.d$	$l=3.141d$, approximately	$3.1415\ldots.d$

The foregoing table shows how the decimal fractions expressing the perimeters agree more and more closely as the number of sides of the polygon is increased.

The following table, which gives the decimal fractions to *six* places, shows the approach of the perimeters still better:

P_4 † $= 4.000000 \ldots \cdot d$	p_4 † $= 2.828427 \ldots \cdot d$
$P_6 = 3.434121 \ldots \cdot d$	$p_6 = 3.000000 \ldots \cdot d$
$P_8 = 3.313708 \ldots \cdot d$	$p_8 = 3.061467 \ldots \cdot d$
$P_{12} = 3.215390 \ldots \cdot d$	$p_{12} = 3.105828 \ldots \cdot d$
$P_{16} = 3.182598 \ldots \cdot d$	$p_{16} = 3.121445 \ldots \cdot d$
$P_{24} = 3.159659 \ldots \cdot d$	$p_{24} = 3.132623 \ldots \cdot d$
$P_{32} = 3.151725 \ldots \cdot d$	$p_{32} = 3.136548 \ldots \cdot d$
$P_{48} = 3.146086 \ldots \cdot d$	$p_{48} = 3.139350 \ldots \cdot d$
$P_{64} = 3.144118 \ldots \cdot d$	$p_{64} = 3.140331 \ldots \cdot d$
$P_{96} = 3.142714 \ldots \cdot d$	$p_{96} = 3.141032 \ldots \cdot d$
$P_{128} = 3.142224 \ldots \cdot d$	$p_{128} = 3.141277 \ldots \cdot d$
$P_{192} = 3.141873 \ldots \cdot d$	$p_{192} = 3.141452 \ldots \cdot d$
$P_{256} = 3.141750 \ldots \cdot d$	$p_{256} = 3.141514 \ldots \cdot d$
$P_{384} = 3.141662 \ldots \cdot d$	$p_{384} = 3.141557 \ldots \cdot d$

The last three perimeters agree to *three* decimal places. Thus, the length of the circle of diameter d which lies between these perimeters is determined to three decimal places. It equals $3.141 \ldots \cdot d$.

As the perimeters of the inscribed and circumscribed polygons, with increasing numbers of sides, approach each other in length, both of them approach more and more closely the length of the circle. But however close the length of the perimeter of any polygon may come to the length of the circle, there is always another polygon the perimeter of which comes still closer to the length of the circle; and for every number given as expressing the *difference* between any perimeter and the circle we can find a polygon whose perimeter differs from the circle by *less* than that number. This is expressed by saying that the perimeters of the inscribed and circumscribed polygons *approach* the circle *as a limit.*

† The subscripts indicate the number of sides of the polygons.

As is seen by the table, p. 237, the value of this limit can be expressed more and more closely by taking polygons of a greater and greater number of sides. It cannot, however, be determined exactly.

Continuing to increase the number of sides, we find in the foregoing table:

$$P_{8192} = 3.1415928\ldots \cdot d$$

and

$$p_{8192} = 3.1415926\ldots \cdot d.$$

From this it is seen that the circle, being between P_{8192} and p_{8192}, can be expressed by $C = 3.141592\ldots \cdot d$, approximately, with an error of *less than one-millionth.*

The length of the circle is therefore a *multiple* of the diameter, which, however, may not be exactly expressed in figures. The number 3.141592 . . . . by which d is multiplied, is commonly denoted by π (the first letter of the Greek word denoting *periphery* or *circumference*).

Thus,

$$\boldsymbol{C = \pi d}\ \dagger$$

and

$$\boldsymbol{C = 2\pi r}$$

are the formulas expressing the circumference of the circle in terms of the diameter and radius, respectively. For our purposes it is sufficient to use $\pi = 3.14$, or $\pi = \frac{22}{7}$, which is equal to 3.14 when carried out to *two* decimal places.

† The determination of the value and of the nature of the number π is one of the famous problems of geometry.

Ahmes took $\pi = \left(\frac{16}{9}\right)^2$.

Archimedes (212–287 B.C.) found the value of π to be such that $3\frac{10}{71} < \pi < 3\frac{10}{70}$ by finding the values of P_{96} and p_{96}.

Ptolemy (150 A.D.) calculated $\pi = 3 + \frac{8}{60} + \frac{8}{60^2} = 3.14166$.

At the end of the sixteenth century Vieta (1579 A.D.) found the value of π to 10 decimal places, and Ludolph van Ceulen (1540–1610) to 20, 32, and 35 places. The value of π has since been carried out to more than 700 decimal places, to 30 places it is as follows:

3.141592653589793238462643383279+ (see the article "Circle" in the *Encyclopedia Britannica*, 11th ed.).

It was shown by Lambert (1728–77) that the number π cannot be expressed exactly in terms of integers, and hence is not a *rational number*.

Lindemann (1882) proved that π belongs to a class of numbers called *transcendental,* numbers which do not satisfy any algebraic equation with rational coefficients.

EXERCISES

1. The length of a circle is 300 inches. Find the radius.

2. Show that *the lengths of two circles are to each other as the radii or as the diameters.*

3. The distance around one of the famous large trees in California is about 100 feet. Find the diameter.

4. The radius of a fly wheel of an engine is 9 feet. If the wheel makes 40 revolutions per minute, what is the rate, in feet, per minute of a point on its outer rim?

5. The size of a man's hat is indicated by the number of inches in the diameter of a circle of length equal to the distance measured around the head where his hat rests. What size of hat does a man need, the distance around whose head is $22\frac{3}{4}$ inches?

6. Measure the distance around your own head and calculate the size of hat you need.

7. A trick circus rider performed on a tall bicycle one turn of whose driving wheel carried the bicycle 18.8 feet forward. How tall was the wheel?

8. A circular pond is 2640.1 yards in circumference. Find the diameter.

9. Using the formulas of § 234, show that the perimeters of regular polygons of the same number of sides are to each other as the radii or as the diameters.

10. Find the length of belting needed to pass around two equal pulleys 16 inches in diameter whose centers are 60 inches apart.

11. Taking the diameter of the earth as 7,927 miles long, find the length of the equator.

12. Measure the distance around a circular object and find the diameter by means of the circumference formula.

Summary of Chapter XII

235. Geometric terms. The chapter has taught the meaning of the following terms: regular polygon, circumscribed polygon, inscribed polygon.

236. Theorems: The following theorems have been proved:

1. °*If a circle is divided into equal arcs the chords subtending these arcs form a regular inscribed polygon.*

2. *If the midpoints of the arcs subtended by the sides of a regular inscribed polygon of n sides are joined to the adjacent vertices of the polygon, a regular inscribed polygon of 2n sides is formed.*

3. °*If a circle is divided into equal arcs, the tangents drawn at the points of division form a regular circumscribed polygon.*

4. *If tangents are drawn to a circle at the midpoints of the arcs terminated by consecutive points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon is formed having double the number of sides.*

5. *If at the midpoints of the arcs subtended by the sides of a given regular inscribed polygon tangents are drawn to the circle, they are parallel to the sides of the given polygon and form a regular circumscribed polygon.*

6. **A circle may be circumscribed about any given regular polygon.*

7. *A circle may be inscribed in any given regular polygon.*

8. *The perimeter of a regular inscribed 2n-side is greater than the perimeter of the regular n-side inscribed in the same circle.*

9. *The perimeter of a regular circumscribed 2n-side is less than the perimeter of the regular n-side circumscribed about the same circle.*

10. *The circumference of a circle is determined by means of the formulas* $C=2\pi r$, $C=\pi d$.

237. Constructions. The chapter has taught the following constructions: to inscribe in, and to circumscribe about, a circle, a square, hexagon, decagon, 15-side.

Other regular inscribed and circumscribed polygons may be obtained by dividing the arcs of the circle into two or more equal parts, and then joining the points of division of the circle successively by line segments.

238. Algebraic relations. The side and perimeter of several regular inscribed or circumscribed polygons have been expressed in terms of the radius and the circumference of the circle. The side and perimeter vary directly as the radius.

CHAPTER XIII

AREAS

Areas of Quadrilaterals

239. Area of the rectangle. The rectangle is the fundamental figure by which the areas of all other rectilinear figures are measured. In your former courses you have seen that the area of the rectangle is given by the formula

$$S = b\ h,$$

S denoting the area, b the base, and h the altitude. In the form of a theorem this is stated as follows:

The area of a rectangle is equal to the product of the base by the altitude.

The formula $S = b\ h$, which was shown to hold for *rational* values of b and h, is also true when b and h are *irrational*. This may be shown as follows:

Let $b = \sqrt{12} = 3.464101 \ldots$ and $h = \sqrt{27} = 5.196152 \ldots$.

Verify the following table giving the areas of rectangles the lengths of whose sides vary, being approximations

Rectangle	b	h	$S = bh$
I	3.464	5.196	17.998944
II	3.4641	5.1961	17.99981001
III	3.46410	5.19615	17.999983215
IV	3.464101	5.196152	17.999995339352

of $\sqrt{12}$ and $\sqrt{27}$ to three, four, five, and six decimal places. Since the approximations are *rational numbers*, the formula $S = b\ h$ may be applied in each case.

It is seen from the table that the *difference between* 18 and the several areas, I, II, III, and IV *decreases*, being less than .002, .0002, .00002, .000005, respectively. By taking b and h to a greater number of decimal places, this difference will continue to decrease; in fact it can be made less than *any assigned* number, *however small.* The area is accordingly said to *approach* 18 *as a limit*, which is also the result obtained by applying the formula, i.e.,

$$S = b\,h = \sqrt{12}\cdot\sqrt{27} = \sqrt{2^2\cdot 3\cdot 3^3} = 18.$$

240. Theorem: *Parallelograms having equal bases and equal altitudes are equal in area.*

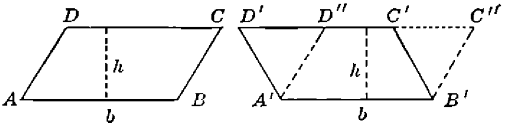


FIG. 327

Given parallelograms $ABCD$ and $A'B'C'D'$, having equal altitudes, h, and equal bases, b, Figure 327.

To prove that $ABCD = A'B'C'D'$.

Proof: Imagine $ABCD$ placed upon $A'B'C'D'$, so that AB coincides with $A'B'$. Why can this be done?

Then DC must fall in the same line as $D'C'$, for the parallelograms have equal altitudes.

Prove that $\triangle D'D''A' \cong C'C''B'$.

But $D'A'B'C'' \equiv D'A'B'C''$.

$\therefore \quad A'B'C''D'' = A'B'C'D'$.

(Equals subtracted from equals give equal remainders.)

$\therefore \quad ABCD = A'B'C'D'$. Why?

241. Theorem: ○*A parallelogram is equal to a rectangle having the same base and altitude.*

Apply the theorem in § 240.

242. Theorem: ⊛ *The area of a parallelogram is equal to the product of the base and altitude.* Prove.

i.e., show that $\boldsymbol{A = bh}$.

243. Theorem: *A triangle is equal to one-half a parallelogram having the same base and altitude.* Prove.

Use the theorem that a diagonal divides a parallelogram into congruent triangles, Figure 328.

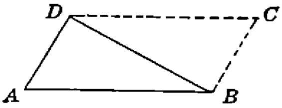


FIG. 328

244. Theorem: *Triangles having equal bases and altitudes are equal.* Prove.

EXERCISES

1. Prove the theorem of Pythagoras, using §§ 240–44:

Theorem of Pythagoras: *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the sides, including the right angle.*

Let ABC, Figure 329, be a right triangle having a right angle at C. Let S_1, S_2, and S denote the areas of the squares on the sides a, b, and c, respectively.

To prove $S = S_1 + S_2$.

Proof: Draw $CD \perp AB$, dividing S into rectangles R_1 and R_2.

Draw AE and CF.

Show that triangle EBA and square S_1 have equal bases and altitudes.

Then triangle $EBA = \frac{1}{2}S_1$. Why? (1)

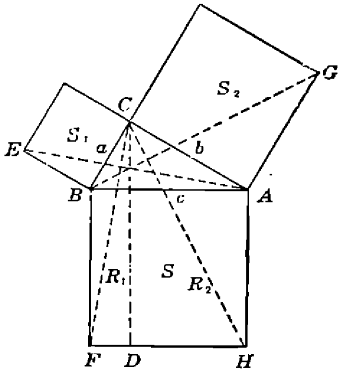


FIG. 329

Similarly, prove that

$\triangle FBC = \frac{1}{2}R_1.$ (2)

$EB = BC.$ Why?

$AB = BF.$ Why?

$\angle ABE = \angle FBC.$ Why?

$\therefore \quad \triangle ABE \cong \triangle FBC.$ (3)

From (1), (2), (3) we have $\frac{1}{2}S_1 = \frac{1}{2}R_1.$

$\therefore \quad S_1 = R_1.$ (4)

Similarly, draw BG and CH, and prove $S_2 = R_2.$

$\therefore \quad S = S_1 + S_2.$

2. Construct a square equal to a given rectangle.

Suggestion: Denote the lengths of the sides of the rectangle by b and h, and the side of the square by x.

Show that $x^2 = bh.$

$\therefore \quad x = \sqrt{bh}.$

Hence, construct the mean proportional between b and h, and on it construct a square.

3. Construct a square equal to the sum of two or more given squares.

Given x, y, z, w, the sides of given squares.

Required to construct a square equal to the sum of the given squares. Figure 330 suggests the construction.

Prove that

$c^2 = x^2 + y^2 + z^2 + w^2.$

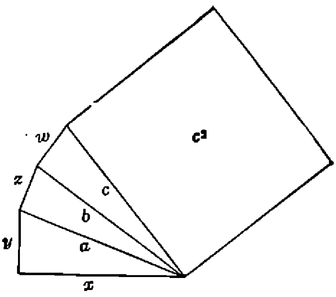


FIG. 330

4. Construct a square equal to four times a given square.

5. Construct the square root of an integral number.

Make the construction, Figure 331, on squared paper.

Measure AC, AD, AE, AF and check by extracting the square roots of 2, 3, 4, 5.

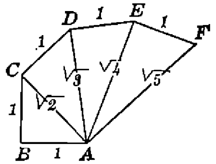


FIG. 331

6. Theorem: *The sum of the squares of two sides of a triangle is equal to twice the square of one-half of the third side increased by twice the square of the median to the third side.* Prove.

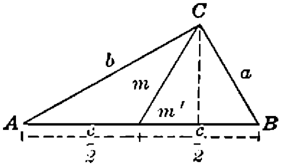


FIG. 332

Given $\triangle ABC$ having the median m to the side c, Figure 332.

To prove that $a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2$.

Proof:

$$a^2=\left(\frac{c}{2}\right)^2+m^2-2\left(\frac{c}{2}\right)m'. \qquad \text{§ 142.}$$

$$b^2=\left(\frac{c}{2}\right)^2+m^2+2\left(\frac{c}{2}\right)m'. \qquad \text{§ 142.}$$

$$a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2. \qquad \text{Why?}$$

7. Show that the length of the median to a side of a triangle may be expressed in terms of the sides of the triangle by means of the following formula:

$$m_c=\sqrt{\frac{a^2+b^2-2\left(\frac{c}{2}\right)^2}{2}}=\tfrac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

8. Find m_c when a, b, and c are, respectively,

(1) 6, 10, 8 (2) 5, 13, 12 (3) 9, 15, 12.

9. Express m_a in terms of the sides a, b, and c of the triangle ABC.

The Area of the Triangle

245. Area of a triangle in terms of base and altitude. Since all plane figures formed by straight lines may be divided into triangles, it is important to obtain formulas for computing the area of a triangle from given parts. All other figures may then be measured by means of the triangle. You are acquainted with the following formula which gives the area of a triangle in terms of the base and altitude:

Theorem: ⊛ *The area of a triangle is equal to one-half the product of the base and altitude.*

$$\triangle \boldsymbol{ABC} = \tfrac{1}{2}\boldsymbol{b\ h}.$$

To prove the formula, use § 243.

246. Area of a triangle in terms of two sides and the included angle. The area of a triangle may be expressed in terms of *two* sides and the included angle.

For, from Figure 333, $\triangle ABC = \frac{1}{2}b \cdot h$.

Since $\sin A = \dfrac{h}{c}$,

it follows that $h = c \sin A$.

By substitution, $\triangle \boldsymbol{ABC} = \frac{1}{2}\boldsymbol{bc \sin A}$.

This may be expressed as a theorem as follows:

Theorem: *The area of a triangle is equal to one-half the product of two sides by the sine of the included angle.*

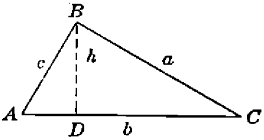


FIG. 333

247. Area of an equilateral triangle. *The area of an equilateral triangle is one-fourth the square of a side times the square root of 3*, or, in symbols, $\boldsymbol{A = \frac{a^2}{4}\sqrt{3}}$.

The area of triangle ABC, Figure 334, is given by the formula

$$\triangle ABC = \tfrac{1}{2}ah.$$

Show that $h^2 = a^2 - \dfrac{a^2}{4} = \dfrac{3a^2}{4}$.

$$h = \frac{a}{2}\sqrt{3}.$$

By substitution, $\triangle ABC = \frac{1}{2}a \cdot \dfrac{a}{2}\sqrt{3}$.

$$\therefore \triangle \boldsymbol{ABC} = \boldsymbol{\frac{a^2}{4}\sqrt{3}}.$$

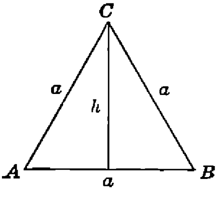


FIG. 334

EXERCISES

1. Find the areas of the following equilateral triangles, having the side equal to 12; 10; 4; 8; $c+d$; $2mn$.

2. Find the side of an equilateral triangle whose area is

$\frac{121}{4}\sqrt{3}$; 12;

$25\sqrt{3}$; $10\sqrt{3}$.

3. Prove the formula for the area of an equilateral triangle by using § 246.

248. The area of a triangle expressed in terms of the sides and the radius of the inscribed circle. The area of a triangle may be expressed in terms of the sides and the radius of the inscribed circle, as follows:

Theorem: *The area of a triangle is equal to the product of one-half the perimeter by the radius of the inscribed circle.*

Let O, Figure 335, be the center of the inscribed circle.

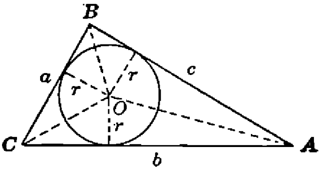


FIG. 335

Draw OA, OB, and OC, dividing triangle ABC into three triangles whose sum is $\triangle ABC$.

Show that

$$\triangle COA = \tfrac{1}{2}r \cdot b.$$
$$\triangle AOB = \tfrac{1}{2}r \cdot c.$$
$$\triangle BOC = \tfrac{1}{2}r \cdot a.$$

$$\boldsymbol{\triangle ABC = \tfrac{1}{2}r(a+b+c).}$$

It is customary to denote $\frac{1}{2}(a+b+c)$ by the symbol s. Then $\boldsymbol{\triangle ABC = rs.}$

249. The area of a triangle in terms of the sides and radius of the circumscribed circle. The relation is expressed as follows:

Theorem: *The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle.*

For, let ABC, Figure 336, be an inscribed triangle.

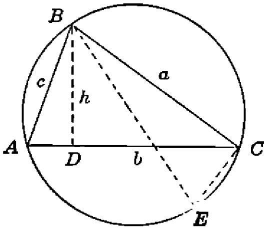


FIG. 336

Draw the diameter BE.

Draw EC.

$$\triangle ABC = \tfrac{1}{2}b \cdot h.$$

Show $\triangle BDA \sim \triangle BCE$.

Then $\dfrac{h}{a} = \dfrac{c}{2r}$. Why?

$$h = \frac{ac}{2r}.$$

By substitution, $\triangle ABC = \frac{1}{2} \cdot b \cdot \dfrac{ac}{2r}$,

or, $\boldsymbol{\triangle ABC = \dfrac{abc}{4r}}$.

EXERCISES

1. The three sides of a triangle are 14, 8, and 12. The diameter of the circumscribed circle is 14.1. Find the area of the triangle.

2. The sides of a triangle are 12, 10, and 8. The area is 39.7. Find the diameter of the circumscribed circle.

3. Denoting the area of a triangle by T, then $T = \dfrac{abc}{4r}$. Solve the equation for r.

4. Using the facts that the area of a triangle is $\frac{1}{2}bh$ and $\dfrac{abo}{2d}$, d being the diameter of the circumscribed circle, find a formula for the altitude to the side b in terms of the other sides and the diameter.

5. The angles of a right triangle are to each other as 1:2:3 and the altitude on the hypotenuse is 6 feet. Find the area.

250. The area expressed in terms of the sides. The area of a triangle may be expressed in terms of the sides alone, as shown below:

Theorem: *The area of a triangle in terms of its sides, is* $\sqrt{s(s-a)(s-b)(s-c)}$.

Given triangle ABC, Figure 337, having the sides a, b, and c.

To prove that the area of ABC is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

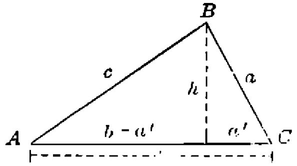


FIG. 337

Proof: Area $ABC = \frac{1}{2}b \cdot h$. (1)

Since this formula gives the area in terms of one side and the altitude, h, which is not known, it will be necessary to express h in terms of the sides and to substitute the result for h in equation (1).

$$h^2 = c^2 - (b-a')^2. \quad \text{Why?} \quad (2)$$

$$h^2 = a^2 - a'^2. \quad \text{Why?} \quad (3)$$

By comparison, $c^2 - (b-a')^2 = a^2 - a'^2$. (4)

Therefore, $c^2 - a^2 - b^2 + 2ba' = 0$. (5)

Solving for a', we find $\quad a' = \dfrac{b^2 - c^2 + a^2}{2b}$. (6)

Substituting in (3) the value of a' found in (6), we get

$$h^2 = a^2 - \left(\frac{b^2 - c^2 + a^2}{2b}\right)^2. \quad (7)$$

Equation (7) expresses h^2 in terms of the sides a, b, and c.

We could now substitute the value of h in equation (1) and have a formula for the area of ABC in terms of a, b, and c. But in order to get a more symmetrical result, the value of h^2 in (7) will first be changed *in form.*

The right side of equation (7), being the difference of two squares, may be factored thus:

$$h^2=\left(a+\frac{b^2-c^2+a^2}{2b}\right)\left(a-\frac{b^2-c^2+a^2}{2b}\right).$$

Carrying out the indicated addition and subtraction within the parentheses, we have

$$h^2=\frac{2ab+b^2-c^2+a^2}{2b}\cdot\frac{2ab-b^2+c^2-a^2}{2b}.$$

$$h^2=\frac{a^2+2ab+b^2-c^2}{2b}\cdot\frac{c^2-(a^2-2ab+b^2)}{2b}. \quad \text{Why?}$$

$$h^2=\frac{(a+b)^2-c^2}{2b}\cdot\frac{c^2-(a-b)^2}{2b}, \quad \text{Why?}$$

or $$h^2=\frac{(a+b-c)(a+b+c)}{2b}\cdot\frac{(c+a-b)(c-a+b)}{2b}. \quad (8)$$

Let $(a+b+c)=2s.$

Subtracting from both sides of this equation first $2c$, then $2a$ and then $2b$, we have

$$\left.\begin{aligned} a+b-c&=2s-2c=2(s-c).\\ b+c-a&=2s-2a=2(s-a).\\ c+a-b&=2s-2b=2(s-b).\end{aligned}\right\} \quad (9)$$

Substituting (9) in (8), we have

$$h^2=\frac{2(s-c)\cdot 2s\cdot 2(s-b)\cdot 2(s-a)}{4b^2}$$

$$=\frac{4s\cdot(s-a)(s-b)(s-c)}{b^2}.$$

Therefore,

$$h=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}. \quad \text{Why?} \quad (10)$$

Substituting (10) in (1), we have

$$ABC=\frac{1}{2}b\cdot\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore \quad \boldsymbol{ABC=\sqrt{s(s-a)(s-b)(s-c)}}. \quad (11)\dagger$$

† The law of formula (11) was introduced into mathematical texts by Heron of Alexandria in the first century B.C.

EXERCISES

1. The sides of a triangle are 3, 5, and 6. Find the area.

According to formula (11) of § 250,

the area $=\sqrt{7\cdot(7-3)(7-5)(7-6)}=\sqrt{7\cdot4\cdot2\cdot1}=2\sqrt{14}$, or 7.482, approximately.

2. The sides of a triangle are 34, 20, and 18. Find the area.

3. The sides of a triangle are 10, 6, and 8. Find the area.

4. The sides of a triangle are 90, 80, and 26. Find the area.

5. The sides of a triangle are 70, 58, and 16. Find the area.

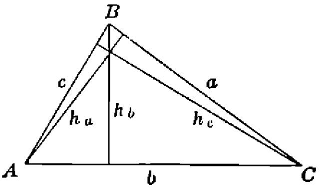


FIG. 338

251. Altitudes of a triangle. Denoting the altitudes of the triangle ABC to the sides a, b, and c by h_a, h_b, and h_c, respectively, Figure 338, show that

$$h_b=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.\dagger \qquad (1)$$

(See § 250, formula [10]).

$$h_a=\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}. \qquad (2)$$

$$h_c=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}. \qquad (3)$$

How can (2) and (3) be obtained from (1) by analogy?

† Heron (first century B.C.) expressed the altitude and area, respectively, of an equilateral triangle as $h=a(1-\frac{1}{10}-\frac{1}{30})$, and $A=a^2(\frac{1}{3}+\frac{1}{10})$.

EXERCISES

1. In the triangle ABC, $a=10$, $b=17$, $c=21$. Find h_a.

$$h_a=\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$s=\tfrac{1}{2}(a+b+c)=\tfrac{1}{2}(10+17+21)=24.$$

$s-a=14$, $s-b=7$, $s-c=3$.

Substitute these values in the formula, and

$$h_a=\frac{2}{10}\sqrt{24\cdot 14\cdot 7\cdot 3}=\frac{1}{5}\sqrt{4\cdot 3\cdot 2\cdot 2\cdot 7\cdot 7\cdot 3}$$

$$=\frac{1}{5}\sqrt{4\cdot 9\cdot 4\cdot 49}=\frac{1}{5}(2\cdot 3\cdot 2\cdot 7)=\frac{84}{5}=16\frac{4}{5}.$$

2. Find the altitudes of each of the following triangles:

(1) $a=35$, $b=29$, $c=8$.
(2) $a=70$, $b=65$, $c=9$.
(3) $a=45$, $b=40$, $c=13$.

252. Factoring the difference of two squares. In proving the formula for the area of a triangle in terms of the sides, § 250, you have factored the polynomials $2ab+b^2-c^2+a^2$ and $2ab-b^2+c^2-a^2$.

In the following you are to study further the method used in factoring such polynomials:

EXERCISES

Factor the following polynomials:

1. $a^2-2ab+b^2-c^2$.

If you group the first three terms, then $a^2-2ab+b^2-c^2$
$=(a-b)^2-c^2=(a-b+c)(a-b-c)$.

2. $x^2-6xy+9y^2-16z^2$.

3. $25x^2+16y^2-4a^2+40xy$.

4. $k^2-x^2-2xy-y^2$.

5. $1-a^2-2ab-b^2$.

6. $9m^2-a^2-4ab-4b^2$.
7. $36r^2-4+20t-25t^2$.
8. $x^2+2xy+y^2-a^2-2ab-b^2$.
9. $a^2+2a+2bc-b^2-c^2+1$.
10. $9x^2+16y^2-49a^2-4b^2+28ab+24xy$.
11. $9a^2-12ab+4b^2-16x^2-8xy-y^2$.

Areas of Polygons

253. Theorem: ® *The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.*

Suggestion: Draw AC, Figure 339. Find the areas of $\triangle ACB$ and DAC. Add the results.

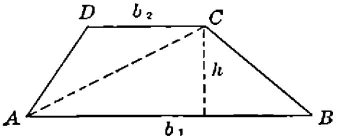


FIG. 339

254. Theorem: ® *The area of a regular inscribed polygon is equal to the product of one-half of the perimeter and the perpendicular from the center to the side (apothem).*

Draw AO, BO . . . . , Figure 340.

Denote the length of a side of the polygon by a, the perpendicular from the center to the side by h, the number of sides by n.

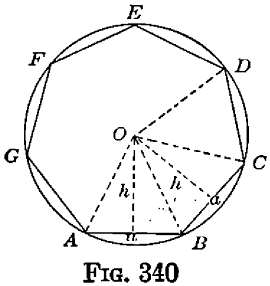


FIG. 340

Then $\qquad \triangle AOB=\frac{ah}{2}$.

$\triangle BOC=\frac{ah}{2}$, etc. $\qquad$ Why?

$\therefore \quad ABCD\ldots=\frac{nah}{2}=\frac{ph}{2}$. $\qquad$ Why?

255. Theorem: *The area of a regular circumscribed polygon is equal to the product of one-half the perimeter and the radius.*

Show, Figure 341, that

$$\triangle AOB = \frac{ar}{2}.$$

$$\triangle BOC = \frac{ar}{2}, \text{ etc.}$$

$$\therefore \quad ABCD \ldots = \frac{nar}{2} = \frac{pr}{2}.$$

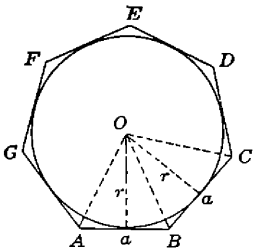


FIG. 341

EXERCISES

1. Express, in terms of the radius, the areas of the inscribed and circumscribed squares (see Exercises 3, 4, § 225).

2. The area of a square is 16 square centimeters. Find the diameters of the inscribed and circumscribed circles.

3. Prove that the area of the equilateral inscribed triangle is $\frac{3}{4}r^2\sqrt{3}$ (see Exercise 5, § 227).

Show that the area of the equilateral inscribed triangle varies as the square of the radius.

4. Prove that the area of the circumscribed equilateral triangle is $3r^2\sqrt{3}$.

Show that the area varies as the square of the radius.

5. Prove that the area of the regular inscribed hexagon is $\frac{3}{2}r^2\sqrt{3}$.

6. Prove that the area of the circumscribed regular hexagon is $2r^2\sqrt{3}$.

7. Find the area of a regular hexagon whose side is 6 inches.

8. The radius of a circle is 10. Find the area of the inscribed regular hexagon.

9. The sides of a quadrilateral are as follows: $AB=29$, $BC=8$, $CD=28$, $DA=21$, and the diagonal $AC=30$. Find the area and the distance from D to AC.

10. The diameter of a circle is 8. Find the area of the regular inscribed hexagon.

11. Prove that in the same circle the area of the regular inscribed hexagon is twice as large as that of the equilateral inscribed triangle.

256. Area of any polygon. The area of polygons may be found by dividing the polygons into triangles, as in

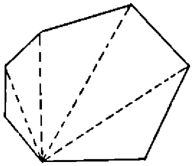


FIG. 342

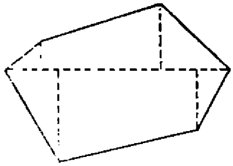


FIG. 343

Figure 342, or into triangles and trapezoids, as in Figure 343.

Proportionality of Areas

The proofs of the theorems in §§ 257 to 261 are very simple and are left to the student.

257. Theorem: *Two parallelograms are to each other as the products of their bases and altitudes,* i.e., $\frac{P}{P'}=\frac{bh}{b'h'}$.

258. Theorem: *Two parallelograms having equal bases are to each other as the altitudes,* i.e., $\frac{P_1}{P_2}=\frac{h_1}{h_2}$.

259. Theorem: *Two triangles are to each other as the products of the bases and altitudes.*

260. Theorem: *Areas of triangles having equal bases are to each other as the altitudes.*

261. Theorem: *Areas of triangles having equal altitudes are to each other as the bases.*

EXERCISES

1. Show that the area of a triangle having a fixed base varies directly as the altitude, i.e., show that $\frac{T}{h}$ remains constant as T and h vary.

2. Show that the area of an equilateral triangle varies directly as the square of the side.

262. Theorem: *The areas of two triangles that have an angle in one equal to an angle in the other are to each other as the products of the sides including the equal angles.*

Given $\triangle ABC$ and $A'B'C'$ having $C = C'$, Figure 346.

To prove that $\frac{T}{T'} = \frac{ab}{a'b'}$.

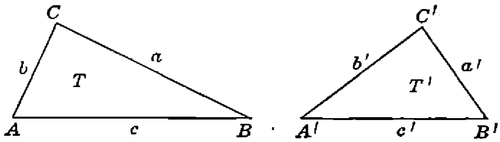


FIG. 344

Proof: $T = \frac{1}{2}ab \sin C$. Why?

$T' = \frac{1}{2}a'b' \sin C'$. Why?

$\frac{T}{T'} = \frac{ab}{a'b'}$. Why?

EXERCISES

1. Two triangles have an angle in each equal. The including sides of one are 48 and 75, those of the other triangle are 45 and 70. Find the relative areas of the triangles.

2. Two sides of a triangular building are 150 feet and 130 feet. What part of the whole building is included by 50 feet on the first side and 30 feet on the second?

3. A triangular lot extends 60 feet and 80 feet on two sides from a corner. If a building is to front 50 feet on the first side, how many feet on the second side should it occupy to cover three-fourths of the lot?

4. Two sides, a and b, of a triangle are 9 and 15, respectively. Show where a line going through a point on a and 5 units from the common vertex of a and b must intersect the side b to bisect the surface of the triangle.

263. Theorem: ⊛ *The areas of similar triangles are to each other as the squares of the homologous sides.*

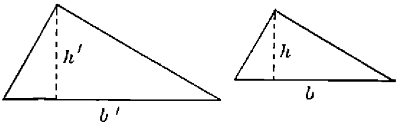


FIG. 345

Show that $\frac{T}{T'}=\frac{bh}{b'h'}=\frac{b}{b'}\cdot\frac{h}{h'}$, Figure 345.

$\frac{h}{h'}=\frac{b}{b'}$. Why?

$\therefore \quad \frac{T}{T'}=\frac{b}{b'}\cdot\frac{b}{b'}$. By substitution.

$\frac{T}{T'}=\frac{b^2}{b'^2}$. Why?

EXERCISES

1. The side of a triangle is 10 inches. Find the corresponding side of a similar triangle having twice the area.

2. Two similar triangles have two homologous sides 5 and 15, respectively. What is the ratio of the areas?

3. Bisect the surface of a triangle by a line drawn from a vertex to the opposite side.

264. Theorem: **The areas of similar polygons are to each other as the squares of the homologous sides.*

Given polygon $ABC \ldots\ldots \backsim$ polygon $A'B'C' \ldots\ldots$, Figure 346. Let P denote the area of $ABC \ldots\ldots$ and P' denote the area of $A'B'C' \ldots\ldots$.

To prove $\frac{P}{P'} = \frac{d^2}{d'^2}$.

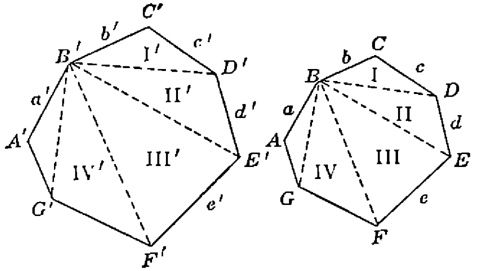


FIG. 346

Proof: Divide $ABC \ldots\ldots$ and $A'B'C' \ldots\ldots$ into triangles I, II, III, etc., and I', II', III', etc., respectively, by drawing diagonals from homologous vertices, as B and B'.

Then $I \backsim I'$, $II \backsim II'$, etc. Why?

$$\frac{I}{I'} = \frac{c^2}{c'^2}, \frac{II}{II'} = \frac{d^2}{d'^2} \ldots\ldots, \text{etc.} \qquad \text{Why?}$$

$$\frac{c}{c'} = \frac{d}{d'}. \qquad \text{Why?}$$

Show that $$\frac{c^2}{c'^2} = \frac{d^2}{d'^2} \ldots\ldots, \text{etc.} \qquad \text{Why?}$$

$$\frac{I}{I'} = \frac{II}{II'} = \frac{III}{III'} \ldots\ldots, \text{etc.} \qquad \text{Why?}$$

$$\frac{I + II + III + \ldots\ldots}{I' + II' + III' + \ldots\ldots} = \frac{II}{II'} = \frac{d^2}{d'^2} \quad (\S\ 263).$$

$$\frac{P}{P'} = \frac{d^2}{d'^2}.$$

EXERCISES

1. Two homologous sides of two similar triangles are 5 and 8. The area of the first is 150. Find the area of the second.

2. If one square is 9 times as large as another, what is the relative length of the homologous sides?

3. The area of a polygon is $6\frac{1}{4}$ times the area of a similar polygon. A side of the smaller is 4 feet. Find the length of the homologous side of the larger.

4. Show that if equilateral triangles are constructed on the sides of a right triangle, the triangle on the hypotenuse is equal to the sum of the triangles on the other two sides.

5. Similar polygons, P_1, P_2, and P_3, are drawn on the sides of a right triangle as homologous sides, Figure 347. Prove that P_3, the area of the polygon on the hypotenuse, is equal to the sum of P_1 and P_2.

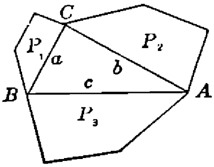


FIG. 347

Proof: $\frac{P_1}{P_3}=\frac{a^2}{c^2}$. Why?

$\frac{P_2}{P_3}=\frac{b^2}{c^2}$. Why?

$\frac{P_1+P_2}{P_3}=\frac{a^2+b^2}{c^2}$. Why?

$\therefore \quad (P_1+P_2)c^2=P_3(a^2+b^2)$. Why?

$\therefore \quad P_1+P_2=P_3$. Why?

6. The homologous sides of similar hexagons are 9 inches and 12 inches, respectively. Find the homologous side of a similar hexagon equal to their sum.

7. Draw a square equal to a given triangle.

Analysis: Since the area of the triangle is $\frac{1}{2}bh$ and since the area of the square is a^2, we must have $a^2=\frac{1}{2}bh$, where b and h are known, and a unknown. Hence the problem reduces to constructing the mean proportional between $\frac{1}{2}b$ and h.

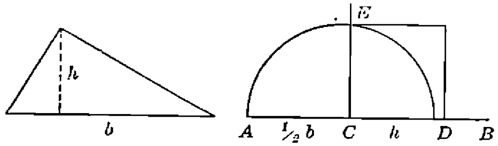


FIG. 348

Construction: On AB, Figure 348, lay off $AC=\frac{1}{2}b$ and $CD=h$.

Draw $CE \perp AD$.
Draw the semicircle on AD.
Draw a square on CE as a side. This is the required square. Prove.

8. Transform a polygon into a triangle equal to it.

Draw the diagonal AD, Figure 349.

Through E draw $EF \parallel AD$ intersecting the extension of AB in F.

Draw DF and show that $\triangle DFA = \triangle DEA$.

Show that $FBCD$ is equal to $ABCDE$.

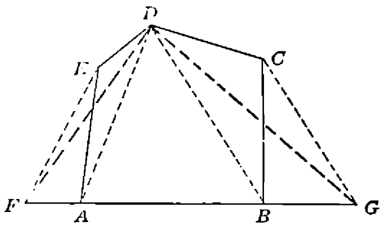


FIG. 349

This reduces the pentagon to the equivalent quadrilateral $FBCD$.

Draw the diagonal DB.
Draw $CG \parallel DB$.
Draw DG.
Show $\triangle DCB = \triangle DGB$.
Show that $FBCD = \triangle FGD$, which is the required triangle.
$\therefore ABCDE = \triangle FGD$. Why?

9. Explain how to draw a square equal to a given polygon.

265. Miscellaneous problems and exercises. Solve the following problems and exercises:

1. Bisect a parallelogram by a line drawn through a point on its perimeter.

2. Construct an equilateral triangle equivalent to a given triangle.

1. Transform the given triangle into an equal triangle having one angle 60°.

2. To determine the length of the side of the equilateral triangle, apply the theorem: two triangles having an angle in each equal are to each other as the products of the sides including the equal angles.

3. The base of a triangle is 18 feet. Find the length of a line parallel to the base which bisects the triangle.

4. A line parallel to the base of a triangle cuts off a triangle equal to $\frac{3}{4}$ of it. If one side of the triangle is 12, how far from the vertex does the line cut it?

5. Draw through a vertex of a triangle lines dividing it:

(1) Into two parts, one of which shall be (*a*) $\frac{2}{3}$, (*b*) $\frac{1}{2}$, (*c*) $\frac{5}{9}$ of the other. (2) Into three parts in the ratio of 2:3:4.

6. To bisect the surface of a triangle by a line through a given point P on the perimeter, not at the vertex of an angle, Figure 350.

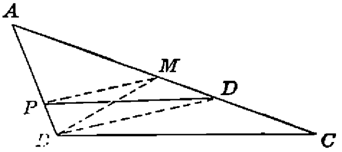


FIG. 350

Draw the median BM, also PM, $BD \parallel PM$, and PD.

Then, $\triangle PMD = \triangle PMB$. Why?

$\therefore$ $\triangle ADP = \triangle AMB = \triangle BMC = PDCB$. Why?

7. The sides of a triangle are 17, 10, and 9. The altitude of a similar triangle upon the side homologous to the side 10 in the given triangle is $14\frac{2}{5}$. Find all the sides of the second triangle.

8. The side of a square (or of any polygon, or the radius of a circle) is a. Find the side (or radius) of a similar figure k times as large.

9. The area of a rectangle is 60 and diagonal is 13. Find its dimensions.

10. The perimeter of a rectangle is 46 and the area is 120. Find its dimensions.

11. The perimeter of a rectangle is 62 and the diagonal is 25. Find its area.

12. The altitude and base of a rectangle are in the ratio of 8 to 15 and the diagonal is 34 feet. Find the area.

13. The dimensions of a rectangle are in the ratio of $2ab$ to a^2-b^2, and the diagonal is $a^2c^2+b^2c^2$. Find the area.

14. Compute the altitude upon the hypotenuse of the right triangle ABC in terms of the sides of the right angle.

15. The diagonals of a rhombus are $2x-14$ and $2x$, and a side is $x+1$. Find x.

16. The homologous sides of two similar hexagons are 9 inches and 12 inches. Find the homologous side of a similar hexagon (1) equal to their sum; (2) equal to their difference.

Area of the Circle

266. Development of the formula for finding the area of the circle. If the midpoints of the arcs subtended by the sides of a given regular inscribed polygon, as triangle ABC, Figure 351, are joined to the adjacent vertices of the polygon, a regular inscribed polygon, $AFBECD$, is formed having twice as many sides as the given polygon.

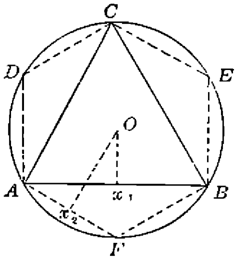


FIG. 351

The perimeter of the second polygon is greater than that of the first. Why?

If the process of doubling the number of sides is continued, the perimeter increases as the number of sides increases. It can be made to differ from the length of the circle by less than *any quantity however small.* The perimeter is said to *approach* the circle *as a limit.*

The apothem OX *approaches* the radius *as a limit.*

The area of the polygon *approaches* the area of the circle *as a limit.*

If tangents are drawn at the midpoints of the arcs, terminated by consecutive points of contact of the sides of a given regular circumscribed polygon, as $ABCD$, Figure 352, a regular circumscribed polygon, as $EFGHIKLM$, is formed having twice as many sides as the given polygon (see § 223).

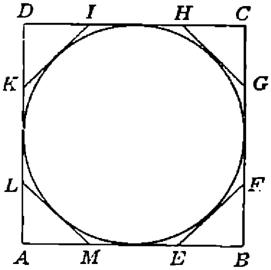


FIG. 352

The perimeter of the second polygon is less than that of the first. Why?

If the process of doubling the number of sides is continued, the perimeter decreases as the number of sides increases. It can be made to differ from the length of the circle by less than *any quantity, however small,* thus approaching the circle *as a limit.*

The area of the polygon *approaches* the area of the circle *as a limit.*

The area of the circle is the *common limit* approached by the areas of the inscribed and circumscribed regular polygons, as the *number* of sides *increases* indefinitely.

These areas are given by the formulas:

$\frac{ph}{2}$ and $\frac{Pr}{2}$, respectively (see §§ 254, 255).

As the number of sides of the polygons is increased indefinitely, $\frac{ph}{2}$ approaches $\frac{cr}{2}$ as a limit, for p approaches c, and h approaches r.

$\frac{Pr}{2}$ approaches $\frac{cr}{2}$ as a limit, for P approaches c.

Hence, the *common limiting value*, $\frac{cr}{2}$, expresses the area of the circle.

In words, this may be stated as follows:

Theorem: *The area of a circle is one-half the product of the length of the circle and the radius, i.e., area of circle is given by the formula*

$$\boldsymbol{A=\tfrac{1}{2}cr.}\dagger$$

Since, $c=2\pi r$, it follows that the *area of a circle is given by*

$$\boldsymbol{A=\pi r^2.}$$

267. Theorem: *The area of a sector of a circle is equal to one-half the product of the radius and the length of the arc of the sector.*

We have seen in § 192 that central angles have the same measure as the intercepted arcs, and that two central angles are to each other as the intercepted arcs (§ 192 Exercise 7).

Hence, $\frac{a}{b}=\frac{a'}{b'}$, Figure 353.

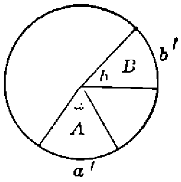


FIG. 353

Similarly, we may show that *equal* central angles include *equal* sectors and that two sectors are to each other as their central angles.

Hence, $\frac{a}{b}=\frac{A}{B}$.

† A proof of the theorem is not attempted, as this is considered beyond the province of secondary-school work.

Denote by a the number of degrees in a central angle, and consider the circle as an arc whose central angle is 360°.

Then, $\frac{a}{360}=\frac{a'}{2\pi r}$, and $\mathbf{a'=\frac{\pi r a}{180}}$. (A)

Similarly, $\frac{A}{\pi r^2}=\frac{a}{360}$. Why?

$$A=\frac{\pi r^2 a}{360}=\tfrac{1}{2}\cdot\frac{\pi r a}{180}\cdot r.$$ Why?

or $\mathbf{A=\tfrac{1}{2}a'\cdot r}$ (*B*)

268. Area of a segment. The *area of a segment* ACB, Figure 354, may be found by subtracting the area of triangle, AOB, from the area of the sector, $AOBC$, the area of triangle AOB being computed by means of the formula $T=\frac{1}{2}r^2 \sin O$, § 246; or

by $T=\frac{1}{2}a\sqrt{r^2-\frac{a^2}{4}}$, § 245.

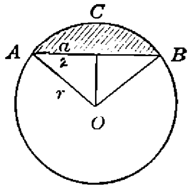


FIG. 354

Hence the area of a segment is given by the following formulas:

(1) $S=\frac{1}{2}a'r-\frac{1}{2}r^2 \sin X$, where X is the central angle subtended by the chord a. Or by

(2) $S=\frac{1}{2}a'r-\frac{1}{2}a\sqrt{r^2-\frac{a^2}{4}}$,

where a is the length of the chord, a' the length of the arc, and r the radius of the circle.

EXERCISES

1. The area of a circle is 64. Find the diameter and length.

2. Find the diameter of a circle whose area is 1 square inch; 1 square foot; 1 square yard.

3. What is the area of the ring formed by two concentric circles, Figure 355, whose radii are 5 inches and 6 inches, respectively; a inches and b inches, respectively?

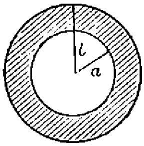


FIG. 355

4. The circumference of a circle is 50 inches. What is the area?

5. The area of a circle is 616 square inches. How many degrees are there in an angle at the center that intercepts an arc 11 inches long?

6. The radius of a circle is 100 feet. The length of the arc of a sector is 25 feet. Find the area of the sector.

Use formula (B), § 267.

7. Show that the area of a circle varies as the square of the radius.

8. The radius of a sector is 9 inches; its area is 72 square inches. Find the length of the arc.

9. The area of a sector is a square foot, and the radius is r feet long. Find the length of the arc.

10. The radius of a circle is 8 inches. Find the area of a sector with arc 36°.

Suggestion: Make use of the fact that the area of the sector is $\frac{1}{10}$ of the area of the circle.

11. Find the area of the segment whose arc is 36° in a circle of radius 12 inches.

Suggestion: When finding the area of the triangle, notice that the base of the triangle is the side of a regular 10-side, Exercise 4, § 228, or use the formula $\frac{1}{2}ab \sin C$.

12. Find the area of a segment of arc 72°, in a circle of radius 20.

13. The area of a circle is 15,400 square inches. Find the area of a segment whose arc is 60°.

14. Find the area of the segment of a circle whose radius is 10 feet, if the chord of the segment is equal to the radius.

15. A circular sector is used for making a funnel. The radius of the sector is 6 inches, and the central angle is 135°. Find the area of the tin used in the sector.

16. Prove that the areas of two circles are to each other as the squares of the radii.

17. Prove that the areas of two circles are to each other as the squares of the diameters.

18. What is the ratio of the areas of two circles whose radii are 5 inches and 10 inches?

19. The areas of two circles are in the ratio 2 to 4. What is the ratio of the diameters?

20. The radii of two circles are to each other as 3:5, and their combined area is 3,850. Find the radii of the two circles.

21. The radii of two circles are to each other as 7:24, and the radius of a circle whose area is equal to their sum is 50. Find the radii of the first two circles.

22. The radii of two circles are 25 and 24. Find the radius of a circle equivalent to their difference.

23. The area of one of three circles is equal to the sum of the other two, and their radii are x, $x-7$, $x+1$. Find x.

24. The difference of two circles whose diameters are $x+2$ and x is equivalent to a circle whose diameter is $x-7$. Find x.

25. Show that if semicircles are drawn on the sides of a right triangle, the area of the semicircle on the hypotenuse is equal to the sum of the areas of the semicircles on the two sides of the right angle.

26. Semicircles are drawn on the sides of a right triangle, Figure 356. Show that the sum of the areas of lunes I and II is equal to the area of the right triangle (theorem of Hippocrates, 430 B.C.).

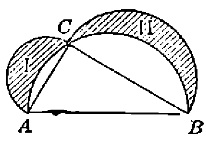


FIG. 356

Summary of Chapter XIII

269. Theorems. The following theorems were proved in this chapter:

1. *Parallelograms having equal bases and equal altitudes are equal.*

2. ○*A parallelogram is equal to a rectangle having the same base and altitude.*

3. *A triangle is equal to one-half a parallelogram having the same base and altitude.*

4. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the sides including the right angle.*

5. *In a triangle the sum of the squares of two sides is equal to twice the square of one-half of the third side increased by twice the square of the median to the third side.*

6. ⊛*The area of a triangle is equal to one-half the product of the base and altitude,*

$$A = \tfrac{1}{2} b\, h.$$

7. *The area of a triangle is equal to one-half the product of two sides by the sine of the included angle,*

$$A = \tfrac{1}{2} ab \sin C.$$

8. *The area of a triangle is equal to one-half the perimeter times the radius of the inscribed circle,*

$$A = \tfrac{1}{2} p\, r.$$

9. *The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle,*

$$A = \frac{abc}{4r}.$$

10. *The area of a triangle is expressed in terms of the sides by the formula*

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

11. *The area of an equilateral triangle is one-fourth the square of a side times the square root of* 3,

$$A = \frac{a^2}{4}\sqrt{3}.$$

12. *The area of a rectangle is equal to the product of the base and the altitude.*

13. ⊛ *The area of a parallelogram is equal to the product of the base and the altitude.*

14. ⊛ *The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.*

15. ⊛ *The area of a regular inscribed polygon is equal to the product of one-half the perimeter and the apothem.*

16. *The area of a regular circumscribed polygon is equal to the product of one-half the perimeter and the radius.*

17. *The area of a circle is one-half the product of the length of the circle and the radius,* i.e., $A = \frac{1}{2}cr.$

18. *The area of a circle is given by the formula* $A = \pi r^2.$

19. *The area of a sector is given by the formula* $A = \frac{1}{2}a'r.$

20. *The area of a segment of a circle is given by the formulas:* $A = \frac{1}{2}a'r - \frac{1}{2}a\sqrt{r^2 - \frac{a^2}{4}}$, or

$$A = \frac{1}{2}a'r - \frac{1}{2}r^2 \sin X.$$

21. *Two parallelograms are to each other as the products of the bases and altitudes.*

22. *Two parallelograms having equal bases are to each other as the altitudes.*

23. *Two triangles are to each other as the products of the bases and altitudes.*

24. *Areas of triangles having equal bases (altitudes) are to each other as the altitudes (bases).*

25. *The areas of two triangles having an angle in one equal to an angle in the other are to each other as the products of the sides including the equal angles.*

26. ⊛*The areas of similar triangles are to each other as the squares of the homologous sides.*

27. **The areas of similar polygons are to each other as the squares of the homologous sides.*

28. *The areas of two circles are to each other as the squares of the radii.*

270. Constructions. The following problems of construction were taught:

1. *To construct a square equal to the sum of two or more given squares.*

2. *To construct the square root of an integral number.*

3. *To transform a polygon into a triangle.*

4. *To construct a square equal to a given rectangle.*

5. *To draw a square equal to a given triangle.*

CHAPTER XIV †

INEQUALITIES

Axioms of Inequality

271. Review and extension of the axioms and theorems of inequality previously established.

1. *Any magnitude is greater than a part of itself.*

This axiom is to be applied only when the magnitudes and their parts are all *positive.* For, let the segment AC, Figure 357, be considered *positive.* Then CB is *negative* and $AC+CB=AB$. For this reason AC and CB may be called *parts* of AB. One of these parts, AC, is greater in magnitude than AB.

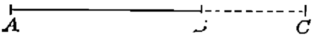


FIG. 357

2. *The sums obtained by adding unequals to equals are unequal in the* **same** *order as the unequal addends.*

For example,	$8>3.$
and	$4=4.$
Hence,	$12>7.$

3. *The sums obtained by adding unequals to unequals in the same order are unequal in the* **same** *order.*

For example,	$9>2$
and	$4>3.$
Hence,	$13>5.$

4. *If three magnitudes are so related that the first is greater than the second and the second greater than the third, the first is greater than the third.*

†Chapter XIV is optional.

Thus, if $a>b$ and $b>c$, then $a+b>b+c$.

Subtracting b from both sides, $a>c$.

In obtaining the last inequality the following axiom is used:

5. *If equals are subtracted from unequals, the remainders are unequal in the* **same** *order as the unequal minuends.*

For example, $10>4$.

and $3=3$.

Hence, $7>1$.

6. *The differences obtained by subtracting unequals from equals are unequal in the order* **opposite** *to that of the subtrahend.*

For example, $12=12$

and $8>2$.

Hence, $4<10$.

7. *The products obtained by multiplying unequals by positive equals are unequal in the* **same** *order as the multiplicands.*

For example,

$$\begin{array}{r} 10<15. \\ 2=2. \\ \hline 20<30. \end{array}$$

8. *The products obtained by multiplying unequals by negative equals are unequal in the order* **opposite** *to that of the multiplicands.*

For example,

$$\begin{array}{r} 12<15. \\ -3=-3. \\ \hline -36>-45. \end{array}$$

9. *The quotients obtained by dividing unequals by positive equals are unequal in the* **same** *order as the dividends.*

For example,

$$\begin{array}{r} 20<30. \\ 2=\ 2. \\ \hline 10<15. \end{array}$$

10. *The quotients obtained by dividing unequals by negative equals are unequal in the order opposite to that of the dividends.*

For example,

$$\begin{array}{r} 50>\ \ 40. \\ -\ 2=-\ 2. \\ \hline -25<-20. \end{array}$$

11. *The shortest distance between two points is the straight line segment joining the points.*

The following theorems express inequalities:

12. *The sum of two sides of a triangle is greater than the third side, and their arithmetical difference is less than the third side.*

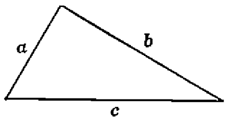


FIG. 358

The first part of this theorem follows directly from 11.

The second part follows from 5. For, let $a+b>c$, Figure 358.

Then $c<a+b$.

Subtracting a from both sides, $c-a<b$.

Similarly, show that $b-a<c$; that $c-b<a$.

13. *If two sides of a triangle are unequal, the angles opposite them are unequal, the greater angle lying opposite the greater side.*

For, let $CA > CB$, Figure 359.

Lay off $CD = CB$.

Draw DB.

Then $\angle B > m$. Why?

$m = m'$. Why?

$\angle B > m'$. Why?

$m' > \angle A$. Why?

$\angle B > \angle A$. Why?

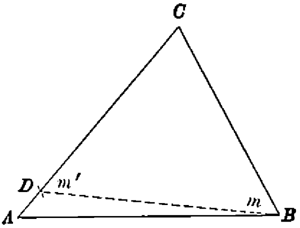


FIG. 359

14. *If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle.*

For, let $\angle A < \angle B$, Figure 360.

Lay off $\angle ABD = \angle BAD$.

Then $AD = DB$.

$CD + DB > CB$.

$CA > CB$.

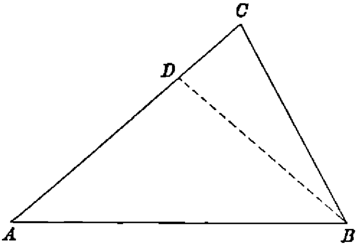


FIG. 360

15. *The shortest distance from a point to a line is the perpendicular from the point to the line.* Prove.

16. *Any point outside of the perpendicular bisector of a line segment is unequally distant from the endpoints* (Exercise 1, § 99).

17. *Any point not on the bisector of an angle is not equidistant from the sides of the angle* (Exercise 2, § 99).

Solution of Problems by Means of Inequalities

272. Many problems lead to relations expressed as inequalities. These *inequalities* may then be solved by using the **axioms of inequality** in the same way as *equations* are solved by using the **axioms of equality.** The following exercises will show the solution of problems by means of inequalities:

EXERCISES

1. Express relations which hold between the sides of the triangle in Figure 361.

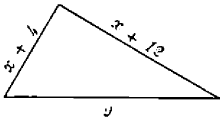


FIG. 361

2. For what values of x do the relations in Exercise 1 hold?

Solution:

$x+4+9>x+12.$ Why?

$\therefore \quad 13>12.$ Why?

$\therefore \quad x=$ *any value*, i.e., any value of x will satisfy the inequality.

$9+x+12>x+4.$ Why?

$21>4.$ Why?

$x=$ *any value.*

$$x+12+x+4>9.$$
$$2x+16>9.$$
$$2x>-7.$$
$$\therefore \quad x>-3\tfrac{1}{2}.$$

$\therefore$ Any value of x *greater than* $-3\frac{1}{2}$ will satisfy all three inequalities. Why?

3. For what values of x may the following expressions represent the lengths of the sides a, b, and c, of a triangle?

a	$x-5$	$2x+3$	$x+5$	7	$2x$
b	$x+7$	$2x+2$	$8-x$	$x-3$	5
c	16	21	1	9	$4x-7$

4. Two sides of a triangle are 9 and 24 inches. Between what limits must the third side be?

Let x denote the third side.

Then $x+9>24$. Why?

$x+24>9$. Why?

$9+24>x$. Why?

Find the values of x satisfying all three inequalities.

5. There are \$50 in the treasury of a club. The club wants to buy furniture costing between \$80 and \$90. How much should be raised?

Let x be the number of dollars to be raised, etc.

6. A twentieth-century limited train wants to make the distance between New York and Chicago (1,000 miles approximately) in less than 20 hours. During the first five hours it goes at the rate of 45 miles per hour. During the next 7 hours it goes at the rate of 57 miles per hour. How fast should it go thereafter to cover the distance within the desired time?

7. A's record average speed on a 2-mile run is 6 miles per hour, and B's is 5$\frac{3}{4}$ miles. How many feet can A afford to give B as a handicap?

8. Prove that *the diameter of a circle is longer than any other chord of that circle.*

Show that $AB=CO+OD>CD$, Figure 362.

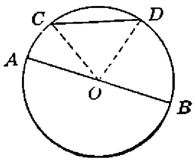


FIG. 362

9. Prove the following:

(*a*) The distance between the centers of two circles which lie *entirely outside* of each other is greater than the sum of the radii, Figure 363.

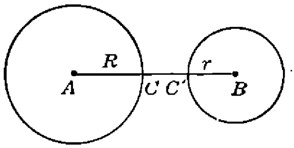


FIG. 363

(*b*) The distance between the centers of two circles touching each other *externally* is equal to the sum of the radii, Figure 364.

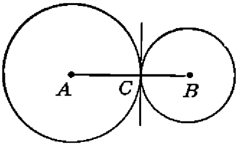


FIG. 364

(*c*) The distance between the centers of two *intersecting circles* is less than the sum of the radii, but greater than the difference, Figure 365.

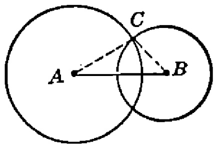


FIG. 365

(*d*) The distance between the centers of two circles touching each other *internally* is equal to the difference of the radii, Figure 366.

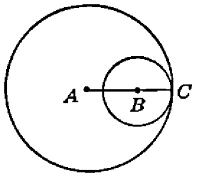


FIG. 366

(*e*) The distance between the centers of two circles, one of which lies *entirely within* the other, is less than the difference of the radii, Figure 367.

10. Prove that *an exterior angle of a triangle is greater than either of the remote interior angles.*

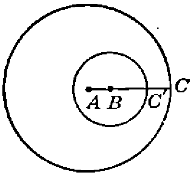


FIG. 367

11. Prove that the sum of the diagonals of a quadrilateral is less than the perimeter, but greater than the semiperimeter.

12. In Figure 368, prove that x is greater than y.

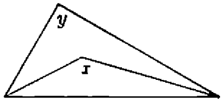


FIG. 368

13. The lengths of the diagonals, Figure 369, are denoted by $5x+4$ and $4x-31$. By means of the relations in Exercise 11, determine the integral values of x.

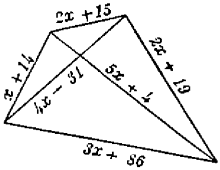


FIG. 369

14. The line joining a vertex of a triangle to the midpoint of the opposite side is a *median* of the triangle.

Prove that the median to one side of a triangle is less than one-half of the sum of the other two sides.

Suggestions: In Figure 370 extend BD, making $DE=BD$, and draw EC.

Then $BE<BC+CE$.

Prove $CE=BA$.

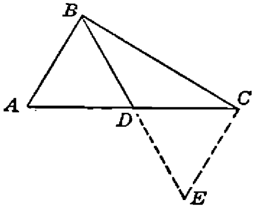


FIG. 370

15. Two towns are located at A and B respectively, Figure 371. Determine a point P on the edge of a river, XY, so that the distances from P to A and B may be piped with the least amount of pipe.

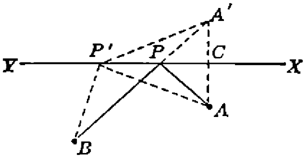


FIG. 371

Suggestions: Draw $AA' \perp XY$ and make $CA'=CA$.

Draw BA' meeting XY at P.

P is the required point.

Show that $BP'A>BPA$, P' being any other point on the edge of the river.

273. Theorem: *If two oblique line segments drawn to a line from a point on the perpendicular to the line have unequal projections, the oblique line segments are unequal.*

Let $EA \perp BC$, and $AF > AD$, Figure 372.

Prove that $EF > ED$.

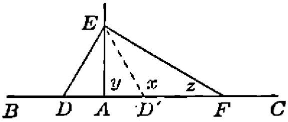


FIG. 372

Proof: Lay off $AD' = AD$ and draw ED'.

Then $x > y$.

Since $y = 90$,

$\therefore$ $x > 90$.

$\therefore$ $z < 90$.

$\therefore$ $x > z$.

$\therefore$ $EF > ED'$

and $EF > ED$.

274. Theorem: *Two unequal oblique line segments drawn to a line from a point on a perpendicular to the line have unequal projections.*

Given $CB > CA$, $CD \perp AB$, Figure 373.

To prove that $DB > DA$.

Proof (indirect method):

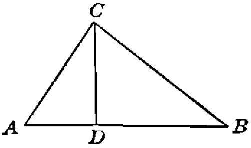


FIG. 373

1. Assume $DB = DA$, then $CB = CA$. Why?
This contradicts the hypothesis.
$\therefore$ The assumption is wrong and $DB \neq DA$.

2. Assume $DB < DA$.
Then show that $CB < CA$.
This is impossible—Why?—and DB is not less than DA.

3. Since DB is not equal to DA and not less than DA, it follows that $DB > DA$.

275. Theorem: *If from a point inside a triangle line segments are drawn to the end points of one side, the sum of these line segments is less than the sum of the other two sides.*

Given $\triangle ABC$, Figure 374, and a point P inside the triangle.

To prove that
$AP+PC<AB+BC$.

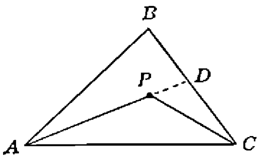


FIG. 374

Proof: Prolong AP until it intersects BC at some point, as D.

We now have:

$$AP+PD<AB+BD. \qquad \text{Why?}$$
$$PC<PD+DC. \qquad \text{Why?}$$

Adding, $AP+PD+PC<AB+BD+PD+DC$.

Subtracting PD from both sides,

$$AP+PC<AB+BD+DC.$$
$$AP+PC<AB+BC. \qquad \text{Why?}$$

EXERCISES

1. Prove that the sum of the three line segments joining a point inside of a triangle with the vertices is less than the perimeter of the triangle but greater than its semiperimeter, Figure 375.

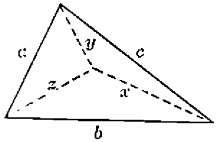


FIG. 375

2. Determine between what limits x must lie in Figures 376 and 377.

What values could x have if we require it to be an integer.

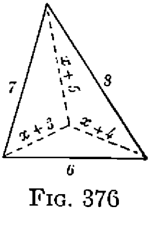


FIG. 376

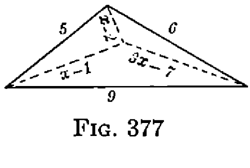


FIG. 377

3. *Construct a triangle ABC, the sides a and b, and the angle A, opposite one of them, being given.*

Construction: On line AB, construct an angle equal to angle A. On one side of this angle, as AC, lay off $AD=b$.

With D as center and radius a, draw a circle.

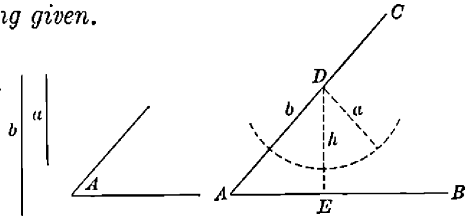


FIG. 378

This circle will either intersect AB in two points, Figures 380 to 382, or it will touch AB, Figure 379, or it will not meet AB at all, Figure 378. We will consider the case where $\angle A$ is acute.

1. If $a<h$, the length of the perpendicular from D to AB, the circle will not meet AB, and there is *no* triangle satisfying the given conditions, i.e., *no* solution of the problem exists, Figure 378.

2. If $a=h$ the circle will touch AB and there is *one* solution of the problem, i.e., $\triangle ADE$, Figure 379.

3. If $a>h$, and $<b$, the circle intersects AB in *two* points F and F'. There are *two* solutions, $\triangle ADF$ and $\triangle ADF'$, Figure 380.

4. If a is equal to b the circle will meet AB in A and in another point F. There is *one* solution, i.e., $\triangle ADF$, Figure 381.

5. If $a>b$, the circle will meet AB in two points F and F', but only $\triangle ADF$ satisfies the conditions of the problem, Figure 382.

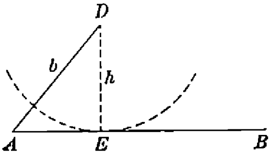


FIG. 379

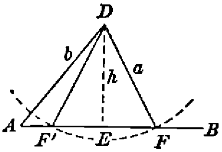


FIG. 380

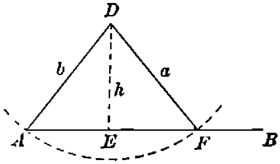


FIG. 381

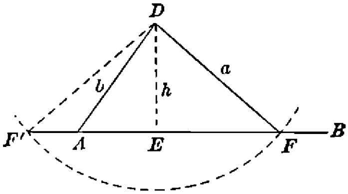


FIG. 382

4. Express trigonometrically the length of the perpendicular, h, in terms of b and A.

Suggestion: Find sin A from the right triangle ADE.

276. Theorem: *In the same circle or in equal circles, unequal chords are unequally distant from the center of the circle, the shorter chord lying at the greater distance; and conversely, chords unequally distant from the center are unequal, the chord at the greater distance being the shorter chord.*

Given $\odot P = \odot Q$, Figure 383.
Chord $AB >$ chord DE, $PP' \perp AB$, $QQ' \perp DE$.

To prove $PP' < QQ'$.

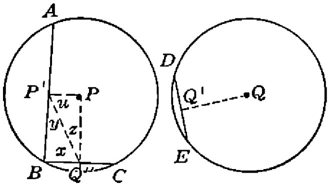


FIG. 383

Proof: Place $\odot Q$ on $\odot P$, so that Q falls on P, D on B, and chord DE in the position BC; then Q' will take a position as at Q''.

Draw $P'Q''$.

		$AB > DE$,	Why?
	or	$AB > BC$.	
		$PP' \perp AB$.	Why?
		$P'B = \frac{1}{2}AB$.	Why?
		$QQ' \perp DE$,	Why?
	or	$PQ'' \perp BC$.	
	$\therefore$	$BQ'' = \frac{1}{2}BC$.	Why?
Then		$P'B > BQ''$.	Why?
		$x > y$.	Why?
Since		$x + z = y + u$.	Why?
		$z < u$.	Why?
		$PP' < PQ''$.	Why?
		$PP' < QQ'$.	Why?

Conversely, **Given** $\odot P = \odot Q$, Figure 383, $PP' \perp AB$; $QQ' \perp DE$; $PP' < QQ'$.

To prove that $AB > DE$.

Proof: Proceed with the steps of the foregoing demonstration in the opposite order.

EXERCISES

1. Triangles are to be constructed with the following parts.

Without constructing the triangle, tell the number of solutions in each case by comparing the lengths of a, b, and h, as found by the formulas in Exercise 4, § 275.

1. $b = 145$.	$a = 178$.	$A = 41°$.
2. $a = 6$.	$b = 3.5$.	$A = 63°$.
3. $a = 140$.	$b = 170$.	$A = 40°$.
4. $b = 28$.	$a = 23$.	$A = 65°$.

2. Construct the triangles in Exercise 1 and see if the constructions verify the results obtained from the formula.

3. Discuss Exercise 3, § 275, for angle A, a right angle; for angle A, an obtuse angle.

4. Prove that, in the same circle, a side of a regular inscribed decagon is less than a side of a regular inscribed pentagon; but that the side of the decagon is greater than half the side of the regular pentagon.

5. Show that the greater the number of sides of a regular inscribed polygon, the shorter is the length of one of its sides.

6. Prove that the distance from the center of a circle to a side of a regular inscribed polygon is greater, the greater the number of sides of the polygon.

277. Theorem: *If two sides of one triangle are equal to two sides of another triangle, but the angle included between the two sides in the first is greater than the angle included by the corresponding sides in the second, then the third side in the first triangle is greater than the third side in the second.*

Given $\triangle ABC$ and DEF, Figure 384.

$$AB = DE;\ BC = EF;\ \angle B > \angle E.$$

To prove that $AC > DF$.

Proof: Place $\triangle DEF$ on $\triangle ABC$ so that DE falls on AB, D on A, E on B, and EF on the same side of AB as BC. Then EF must fall within $\angle ABC$. Why?

For the position of F there are three possibilities:

I. F falls *below* AC, as at F', Figure 384.

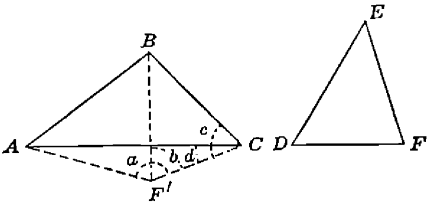


FIG. 384

Then	$a > b$.	Why?
	$b = c$.	Why?
	$a > c$.	Why?
	$c > d$.	Why?
$\therefore$	$a > d$.	Why?
$\therefore$	$AC > AF'$ and $AC > DF$.	Why?

II. F falls *on* AC, as at F'', Figure 385.

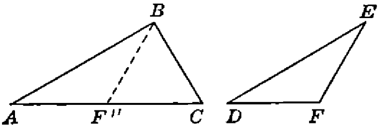


FIG. 385

Then $AC > AF''$. Why?
$AC > DF$. Why?

III. F falls *above* AC, as at F''', Figure 386.

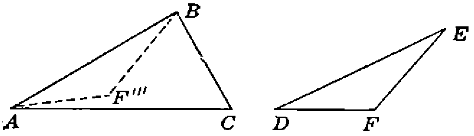


FIG. 386

Then $AF''' + F'''B < AC + CB$. Why?
$F'''B = CB$. Why?

$\therefore$ $AF''' < AC$. Why?
and $DF < AC$. Why?

278. Theorem: *If two sides of one triangle are equal to two sides of another triangle, the third side of the first triangle being greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second triangle.*

Given $\triangle PQR$ and XYZ, Figure 387.
$PQ = XY$; $QR = YZ$; $PR > XZ$.
To prove that $\angle Q > \angle Y$.

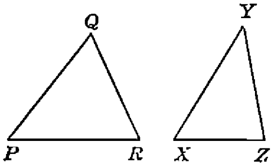


FIG. 387

Analysis: If $Q = Y$, what is known about the triangles; about PR and XZ?

Hence, can $Q = Y$ if $PR > XZ$, as here given?

What do we know about PR and XZ if $Q < Y$?

Then, is $Q < Y$, if $PR > XZ$, as here given?

How, then, must angles Q and Y compare, if $PR > XZ$?

Give full proof, using the indirect method.

279. Theorem: *In the same circle or in equal circles, the arcs subtended by unequal chords are unequal in the same order as the chords; and conversely, chords subtending unequal arcs are unequal in the same order as the arcs.*

Given $\odot A = \odot B$, Figure 388. $CD > EF$.
To prove arc $CD >$ arc EF.

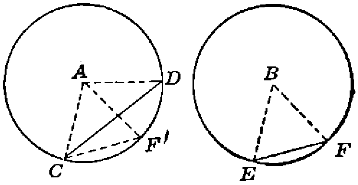


FIG. 388

Proof: Draw radii AC, AD, BE, and BF.

Show that

$\angle CAD > \angle EBF$.

Place $\odot B$ on $\odot A$, so that EB falls on CA, E on C, B on A, and F on the same side of AC as D.

Then BF must come between AD and AC, as in position AF'. Why?

Hence $\widehat{EF}$ comes in the position $\widehat{CF'}$, and F' falls on the circle between C and D.

Then	arc $CF' <$ arc CD.	Why?
Also,	arc $CF' =$ arc EF.	Why?
$\therefore$	arc $EF <$ arc CD.	Why?

Conversely, **Given** $\odot A = \odot B$, Figure 388, $\widehat{CD} > \widehat{EF}$.

To prove chord $CD >$ chord EF.

Proof: Draw radii AC, AD, BF, and BE, and place $\odot B$ on $\odot A$ so that EB coincides with CA.

Since $\widehat{CD} > \widehat{EF}$, the point F will fall between C and D, as at F', and the line BF will come on the same side of AD as AC, as in position AF'.

Then we have	$\angle CAF' < \angle CAD$.	Why?
Also	$\angle CAF' = \angle EBF$	Why?
and	$\angle CAD > \angle EBF$.	Why?

Show that $CD > EF$.

EXERCISES

1. The length of the chords AB and BC, Figure 389, being $6x-14$ and $4x+20$, respectively, and the lines PP' and PP'' being 16 and 10, determine x and the chords.

We have $\quad P'B=3x-7.$ $\quad$ Why?

$P''B=2x+10.$ $\quad$ Why?

Then $\quad (3x-7)^2+16^2=\overline{PB}^2$ $\quad$ Why?

and $\quad (2x+10)^2+10^2=\overline{PB}^2.$ $\quad$ Why?

$\therefore \quad (3x-7)^2+16^2=(2x+10)^2+10^2,$

or $\quad 9x^2-42x+49+256=4x^2+40x+100+100.$

$5x^2-82x+105=0.$

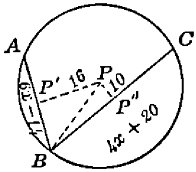


FIG. 389

$$\therefore \quad x=\frac{82\pm\sqrt{82^2-4\cdot5\cdot105}}{10}.$$

$$x=\frac{82\pm68}{10}=15, \text{ or } [1\tfrac{2}{5}]$$

Then $\quad AB=76.$

$CB=80.$

How is the truth of the theorem in § 279 illustrated by these answers?

2. The length of the lines AB and BC, PP' and PP'', Figure 389, being denoted by l_1, l_2, d_1, and d_2, respectively, de-

	l_1	l_2	d_1	d_2
1......	$2a-7$	$4a-14$	2	1
2......	6	12	$u+11$	$3u+4$
3......	$x+3$	$x+5$	6	4
4......	$4t+14$	$10t-2$	6	3

termine the unknown number in each of the following cases. In every case test by § 279.

Summary of Chapter XIV

280. Axioms. The principles of inequalities studied in the preceding chapters were reviewed and extended.

The use of the principles of inequalities in the solution of problems was shown.

281. Theorems. The following theorems were proved:

1. *The diameter of a circle is larger than any other chord of the circle.*

2. *An exterior angle of a triangle is greater than either of the remote interior angles.*

3. *If two oblique line segments drawn to a line from a point on a perpendicular to the line have unequal projections, the oblique line segments are unequal.*

4. *Two unequal oblique line segments drawn to a line from a point on a perpendicular to the line have unequal projections.*

5. *If from a point inside a triangle, line segments are drawn to the endpoints of one side, the sum of these line segments is less than the sum of the other two sides.*

6. *In the same or in equal circles unequal chords are unequally distant from the center, the shorter chord lying at the greater distance; and the converse of this theorem.*

7. *If two sides of one triangle are equal to two sides of another triangle, but the angle included between the two sides of the first is greater than the angle included between the corresponding sides in the second, then the third side in the first is greater than the third side in the second; and the converse of this theorem.*

8. *In the same or equal circles, the arcs subtended by unequal chords are unequal in the same order as the chords, and the converse of this theorem.*

282. Construction. The following construction was taught:

To construct a triangle ABC, the sides a and b and the angle A, opposite one of them, being given.

TABLE OF SINES, COSINES, AND TANGENTS OF ANGLES FROM 1°–89°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	59	.7547	.6561	1.1504
5	.0872	.9962	.0875	**50**	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	**55**	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	**60**	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	.3420	.9397	.3640	**65**	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	**70**	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	**75**	.9659	.2588	3.7321
31	.5250	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	**80**	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	**85**	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0157	57.2900
45	.7071	.7071	1.0000	**90**	1.0000	.0000	——

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	**90**	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.641

SYMBOLS

$=$	is equal to	Ⓢ	circles
$>$	is greater than	$\therefore$	hence, therefore
$<$	is less than	$\because$	since
$\parallel$	parallel, is parallel to	$\equiv$	identical, is identical to
$\perp$	perpendicular, is perpendicular to	$\doteq$	approaches
$\backsim$	similar, is similar to	$+$	plus
$\cong$	congruent, is congruent to	$-$	minus
$\angle$	angle	$\neq$	does not equal
$\angle$s	angles	rt $\angle$	right angle
▱	parallelogram	$\triangle$	triangle
▭	rectangle	$\triangle$s	triangles
$\odot$	circle	$\frown$	arc

FORMULAS

$a^2+b^2=c^2$: relation between the sides of a right triangle.

$a^2+b^2\pm 2ab'=c^2$: relations between the sides of an oblique triangle.

$c=2\pi r=\pi d$: circumference of a circle.

$b\cdot h$: area of a parallelogram, of rectangle.

a^2: area of a square.

$\frac{1}{2}bh$, $\frac{1}{2}ab \sin C$, $\frac{1}{2}r(a+b+c)$, $\frac{abc}{4R}$, $\sqrt{s(s-a)(s-b)(s-c)}$: area of a triangle.

$\frac{1}{4}a^2\sqrt{3}$: area of an equilateral triangle.

$\frac{1}{2}h(b+b')$: area of a trapezoid.

$\frac{1}{2}p\ a$: area of a regular inscribed polygon.

$\frac{1}{2}p\ r$: area of a regular circumscribed polygon.

$\frac{1}{2}cr=\pi r^2$: area of a circle.

$\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$, $\tan A=\frac{a}{b}$, $\sin^2 A+\cos^2 A=1$,

$\tan A=\frac{\sin A}{\cos A}$.

SUPPLEMENTARY EXERCISES

CHAPTER I

11. 1. Name the opposite (vertical) angles in Figure 1.

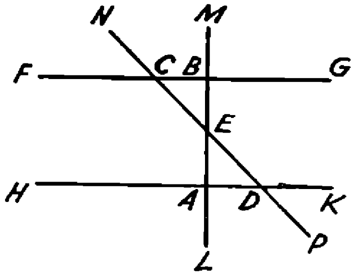


FIG. 1

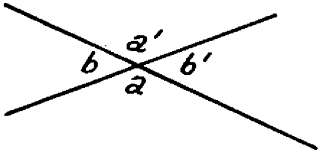


FIG. 2

2. In Figure 2 find a' if $a=125$; b' if $b=50$; b if $b'=30$; a if $a'=110$.

12. 1. Find the complement of $(90-x)^\circ$; $(a-15)^\circ$; $\left(\frac{m}{n}\right)^\circ$.

2. One angle is five times as large as another and the two angles are complements. Find each.

3. The complement of an angle is twice as large as the angle. How large is it?

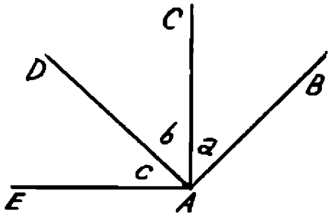


FIG. 3

4. Angles BAD and CAE, Figure 3, are right angles. Express a and c in terms of b. If b is 35, find a and c.

5. Draw several illustrations of complementary angles.

16. 1. If $a=b$, Figure 4, compare c and d.

2. $DB \perp AC$, Figure 5, and $a=m$. State relations between m and n; a and b; n and b.

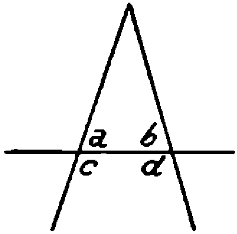


FIG. 4

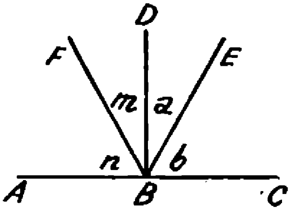


FIG. 5

3. If $x=x'$, Figure 6, show that $y=y'$.

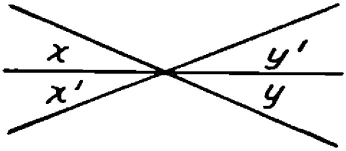


FIG. 6

17. 1. The angles of a triangle are in the ratio 1:2:3. Find each.

2. The angles of a triangle are equal. Find each.

3. Show that a triangle cannot have more than one right angle.

4. How large are the angles formed by the bisectors of two angles of an equiangular triangle?

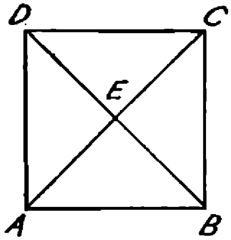


FIG. 7

19. 1. DB and AC, Figure 7, bisect each other and are perpendicular. Show that the four triangles are congruent.

2. Point E, Figure 8, is the midpoint of DC, and $ABCD$ is a square. Show $\triangle ADE \cong \triangle BCE$.

3. BD, Figure 9, bisects angle ABC. C and A are equally distant from B. Show $\triangle EBC \cong \triangle EBA$.

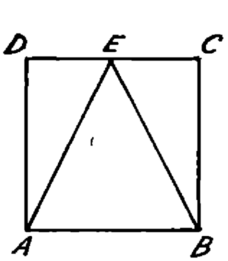


FIG. 8

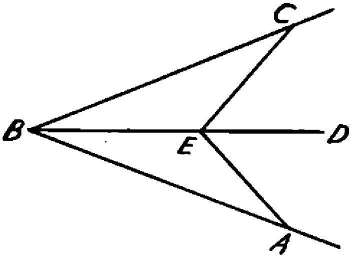


FIG. 9

23. 1. Triangles ABC and DEF, Figure 10, are isosceles. $AC = DF$ and $\angle C = \angle F$. Show that the triangles are congruent.

2. In the isosceles triangle, Figure 11, $\angle ACE = \angle BCE$. Show that $\triangle ADC \cong \triangle BDC$.

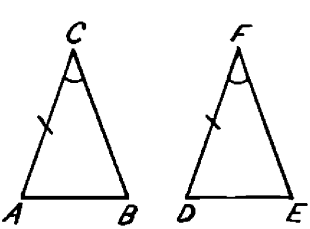


FIG. 10

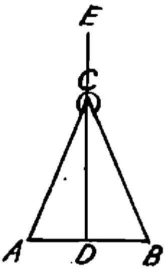


FIG. 11

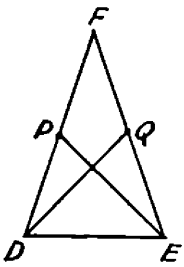


FIG. 12

3. $\triangle DFE$, Figure 12, is isosceles, and P and Q are the midpoints of the equal sides. Show $\triangle DPE \cong \triangle DQE$.

4. In Figure 13 show $\triangle AEC \cong \triangle BED$.

5. Triangles BAD and BCD, Figure 14, are isosceles. Show $\angle ABC = \angle ADC$.

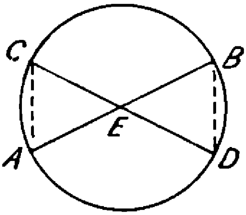


FIG. 13

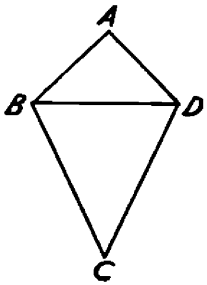


FIG. 14

24. 1. In Figure 15 show $\triangle ABE \cong \triangle DCE$, if $AE = ED$ and $\angle D = \angle A$.

2. To find the distance across a stream, Figure 16, a surveyor lays off $CE = EA$ and $\angle C = \angle A$. Show that $CD = BA$.

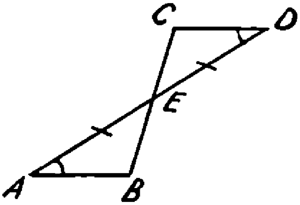


FIG. 15

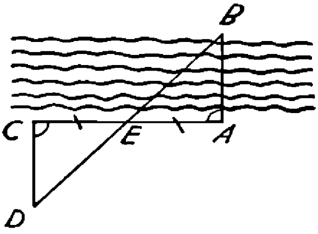


FIG. 16

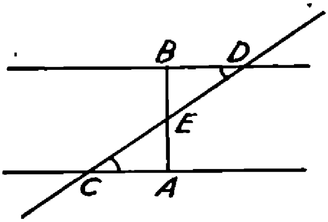


FIG. 17

3. $\angle D = \angle C$, Figure 17, and $DE = EC$. Show that $BE = EA$.

4. $AB = DC$, Figure 18, and $m = n$. Show $\triangle ABC \cong \triangle ADC$.

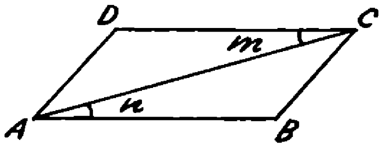


FIG. 18

5. A line is drawn perpendicular to the bisector of an angle. Show that the triangles formed are congruent.

28. 1. $BC = BA$, Figure 19, and $CD = DA$. Show $\triangle BCD \cong \triangle BAD$.

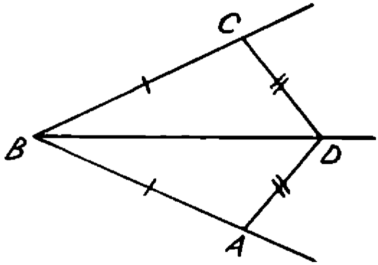


FIG. 19

2. Show that a wooden frame formed by three rods bolted together as in Figure 20 cannot change shape.

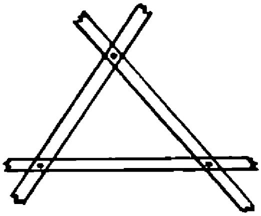


FIG. 20

3. Farmer Jones observed that the front gate, Figure 21, was beginning to sag. How could he prevent further sagging?

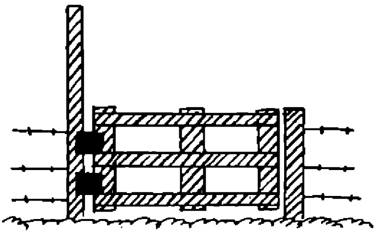


FIG. 21

4. Show that a line bisecting the vertex angle and the base of an isosceles triangle divides it into two congruent triangles.

5. Show that a line joining two opposite vertices of a square divides it into congruent triangles.

29. 1. Construct an angle of 30°.

2. Construct an angle of 45°.

3. Construct an angle of 135°.

30. 1. Divide a line segment into four equal parts.

2. Construct a line segment one and a half times as long as a given line segment.

3. Draw an equilateral triangle. Find the midpoints of the sides and connect them by straight lines. Show that the four triangles formed are congruent.

32. 1. Draw the perpendicular bisectors of the sides of a triangle.

2. Construct a square when the perimeter is given.

3. Construct an isosceles triangle whose base and altitude are given.
4. At one of the endpoints of a segment construct a perpendicular to the segment.
5. Draw an obtuse triangle. Construct a perpendicular from each vertex to the opposite side.
6. In a right triangle construct the perpendicular bisectors of the sides.
7. From the midpoint of the base of an isosceles triangle construct a perpendicular to each of the equal sides.

33. 1. One of the acute angles of a right triangle is given, construct the other.
2. Construct an isosceles triangle, having given the base and the vertex angle.
3. Construct an angle twice as large as a given angle.
4. Construct an angle equal to the sum of three given angles.

38. 1. Prove that the lines drawn from the midpoint of the base of an isosceles triangle perpendicular to the equal sides are equal.

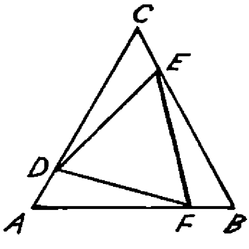


FIG. 22

2. Triangle ABC, Fig. 22, is equilateral. $AD=CE=BF$. Prove that $\triangle DEF$ is equilateral.

3. Triangles ABC and ADC, Figure 23, are isosceles. Prove that BD bisects angles B and D.
4. In Figure 24, $AD=DC$ and BD bisects angle D. Prove $AB=BC$.

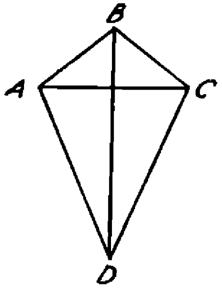


FIG. 23

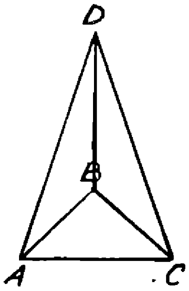


FIG. 24

5. Prove the theorem on page 33 for the triangles in Figure 25.

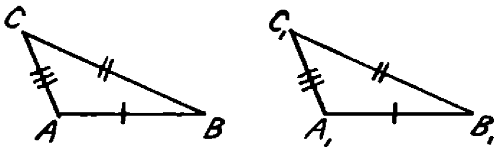


FIG. 25

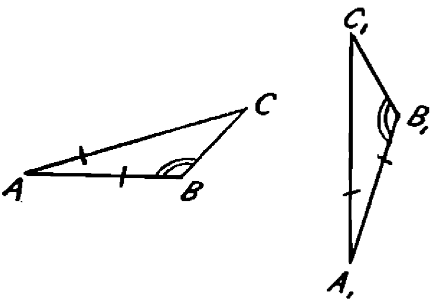


FIG. 26

6. Prove the theorem on page 32 for Figure 26.

7. Let AD and BE be the bisectors of two angles of an equilateral triangle ABC. Prove that $\triangle BDA \cong \triangle BEA$.

8. Prove that lines of trisection of one angle of an equilateral triangle form two congruent triangles with the sides of the triangle.

CHAPTER II

51. 1. Prove that if a line cuts one of several parallel lines it must cut all of the parallel lines if far enough extended.

2. Prove that two angles of a triangle cannot be right angles.

3. Construct a line parallel to a given line, using the fact that two lines perpendicular to the same line are parallel.

4. Show that lines AB and CD, Figure 27, drawn with the right triangle are parallel.

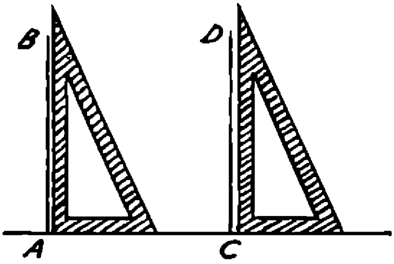


FIG. 27

5. Show that the opposite sides of a rectangle are parallel.

55. 1. If $\angle D = \angle A$, Figure 28, prove $DE \parallel AB$.

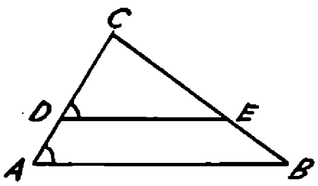


FIG. 28

2. $\angle CBA = \angle DAE$, Figure 29. Prove that the bisectors of the angles are parallel.

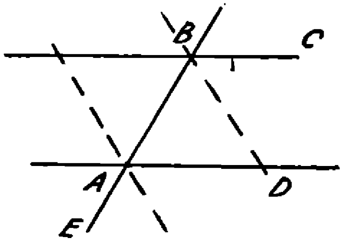


FIG. 29

3. $AE = EC$, Figure 30, and $DE = EB$. Prove $DC \parallel AB$.

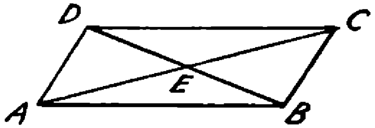


FIG. 30

4. If $AD = BC$, Figure 30, and $AB = DC$, prove $AB \parallel DC$.

5. Two isosceles triangles are drawn on the same base so as to form a quadrilateral. Prove that the line joining the vertices bisects two angles of the quadrilateral and is perpendicular to the line joining the other two vertices.

58. 1. Using Figure 31, prove that the sum of the angles of a triangle is 180°.

2. If $AC \parallel DB$ and $AE = EB$, Figure 32, prove that AB bisects CD.

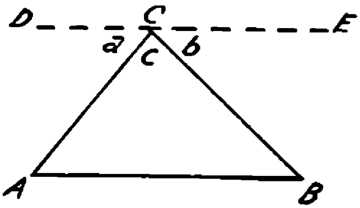


FIG. 31

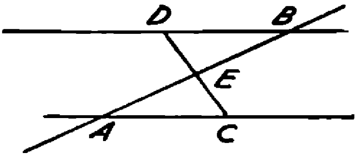


FIG. 32

3. If $ED \parallel AB$, Figure 33, prove $\angle E = \angle B$, $\angle D = \angle A$.

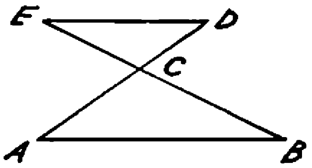


FIG. 33

59. 1. If two parallel lines are cut by a transversal, the angle formed by the bisectors of the interior angels on the same side is 90°. Prove.

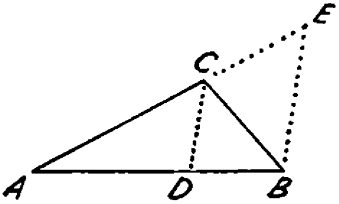


FIG. 34

2. CD, Figure 34, bisects $\angle ACB$, and $BE \parallel DC$. Prove $CE = CB$.

3. In $\triangle ACB$, Figure 35, $\angle C=90°$ and $CD \perp AB$. Prove $\angle DCB=\angle CAB$ and $\angle DBC=\angle ACD$.

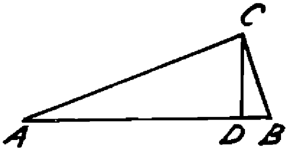


FIG. 35

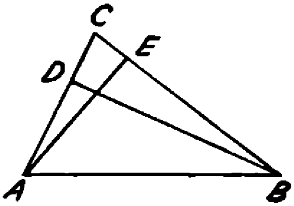


FIG. 36

4. In $\triangle ABC$, Figure 36. $BD \perp AC$, and $AE \perp BC$. Prove $\angle CAE=\angle CBD$.

5. An acute angle of a triangle is given twice as large as another. Divide the triangle into two isosceles triangles.

6. In Figure 37, $AB \| CD \| EF$, $BD=DF$, and AH and CG are parallel to BF. Prove $\triangle ACH \cong \triangle CEG$.

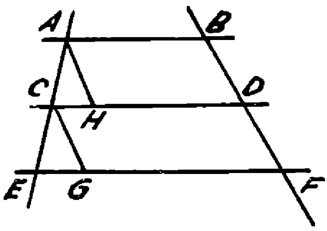


FIG. 37

7. Let ABC be an obtuse isosceles triangle with C as vertex. Extend AC to D, making an isosceles triangle CBD such that $CB=BD$. Extend AB to E. Prove $\angle EBD=3A$.

CHAPTER III

69. 1. Prove that the sum of the angles of a parallelogram is 360°.

2. One of the angles of a parallelogram is 48°. Find the others.

3. From any point of the base of an isosceles triangle draw lines parallel to the equal sides. Prove that the perimeter of the parallelogram is equal to the sum of the equal sides of the triangle.

4. In the parallelogram, Figure 38, $DE \perp AB$ and $CF \perp AB$. Prove $\triangle AED \cong \triangle BFC$.

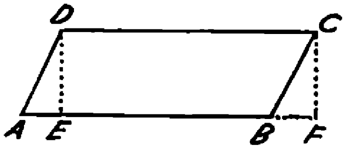


FIG. 38

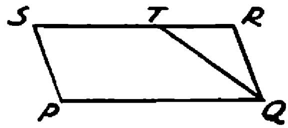


FIG. 39

5. In the parallelogram, Figure 39, QT bisects $\angle Q$. Prove $\triangle TRQ$ isosceles.

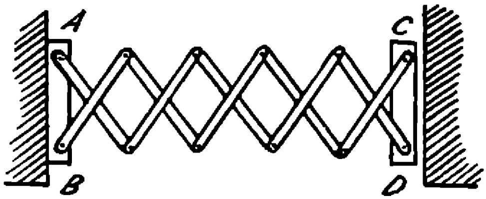


FIG. 40

73. 1. Figure 40 is the picture of a gate which may be opened by pushing CD toward AB. Prove that CD remains parallel to AB as it moves toward it.

2. Draw a parallelogram and divide each diagonal into four equal parts. Prove that the quadrilateral formed by joining the midpoints of the halves of diagonals is a parallelogram.

3. In triangle ACB, Figure 41, D and E are the midpoints of two of the sides and $EF = ED$. Prove that $ABFD$ is a parallelogram.

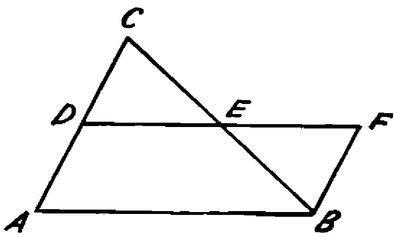


FIG. 41

4. In a parallelogram $ABCD$ extend AD to E and CB to F so that $DE = BF$. Prove $AFCE$ a parallelogram.

5. The consecutive angles of a parallelogram are to each other as 2:3. Find each angle.

6. Through any point on the diagonal of a parallelogram lines are drawn parallel to the sides. Prove that the parallelograms not intersected by the diagonal are equal.

7. The angle formed by the bisectors of the exterior angles A and B of a triangle is the supplement of double the angle C.

8. From the vertex C of $\triangle ABC$ draw the altitude CD and the bisector CE of angle C. Prove that $\angle DCE$ is equal to $\frac{A-B}{2}$.

9. From the vertex A of a right triangle draw the altitude AD and the median AE to the hypotenuse. Prove that $\angle DAE$ is equal to the difference of the two acute angles of $\triangle ABC$.

10. Prove that the vertex angle of an isosceles triangle is twice as large as the angle formed by the base and the altitude drawn to one of the equal sides.

11. Two parallelograms have one side in common. Prove that the sides opposite the common side are parallel to each other.

12. Prove that two vertices of a triangle are equally distant from the median drawn from the third vertex.

13. Through any point D on the bisector of angle ABC the line DE is drawn parallel to BC, meeting BA at E. Prove $\triangle BED$ isosceles.

14. A diagonal of a parallelogram is extended the same distance through both vertices. If the endpoints are joined to the other vertices of the parallelogram, prove that the quadrilateral thus formed is a parallelogram.

77. 1. The diagonals of an isosceles trapezoid form pairs of congruent triangles. Prove.

2. Prove that by extending the equal sides of an isosceles trapezoid until they meet you form two isosceles triangles.

3. Prove that a parallelogram whose diagonals are equal is a rectangle or square.

4. Draw a parallelogram $ABCD$. On AC lay off $AE = CF$ and join D, E, B, and F. Prove $DEBF$ a parallelogram.

5. If a rectangle is not a square and the bisectors of the four angles are drawn, the quadrilateral thus formed is a square.
6. If the diagonal of a parallelogram bisects the vertex angle, the parallelogram is a rhombus.
7. Construct a rhombus having one angle equal to 60°.
8. If the diagonals of a parallelogram are perpendicular to each other, the figure is a square or rhombus.
9. State how the square and rhombus differ and in what ways they are alike.
10. The line joining the midpoints of the bases of an isosceles trapezoid is perpendicular to the bases. Prove.
11. If the diagonals of a trapezoid are equal, the trapezoid is isosceles. Prove.
12. Construct a line equidistant from two given points and passing through a third given point.

CHAPTER IV

87. 1. If $AD = DH$, Figure 42, $HF = FC$, and if DE, HB, and FG are perpendicular to AC, prove $BH = ED + GF$.

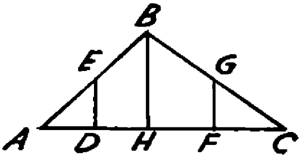


FIG. 42

2. In the parallelogram, Figure 43, E and F are the midpoints of DC and AB. Prove that DB is trisected.

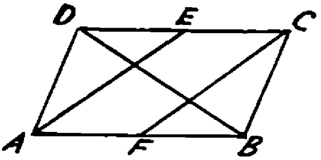


FIG. 43

3. If the line joining the vertex A of a triangle to the midpoint of the opposite side is extended its own length to D and if a line is drawn from D to a second vertex, it is parallel to the side opposite that vertex. Prove.

90. 1. Construct an equilateral triangle if the perimeter is given.

2. The lines joining the midpoints of the sides of a rhombus form a rectangle.

3. Let D be any point on AB, Figure 44, and let E, F, and G be the midpoints of AD, AC, and CD. Prove that $EFGD$ is a parallelogram.

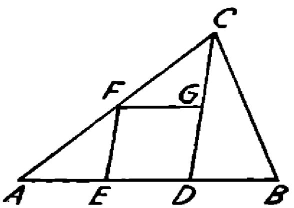


FIG. 44

4. If a line joins the midpoints of two sides of a triangle, it bisects all lines drawn to the third side from the opposite vertex. Prove.

5. The line joining the midpoints of the non-parallel sides of a trapezoid bisects the diagonals. Prove.

92. 1. Divide a segment in the ratio 1:2:3.

2. A line joining points D and E on the sides AC and BC of $\triangle ABC$ divides the sides into segments 10, 6, 5, and 3 inches long. Show that $DE \parallel AB$.

3. In $\triangle ABC$, Figure 45, D is the midpoint of AB. DF and DE bisect angles CDB and CDA. Prove $EF \parallel AB$.

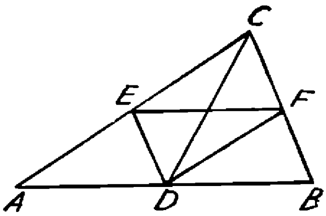


FIG. 45

4. If two vertices of a parallelogram are joined by lines to the midpoints of the opposite sides, prove that the two lines trisect the diagonal drawn through the other two vertices of the parallelogram.

CHAPTER V

96. 1. Find the locus of the vertices of triangles having the same line as base and the corresponding median of fixed length.

2. Find the locus of vertices of triangles having the same base and the corresponding altitude of fixed length.

3. Find the locus of midpoints of line segments parallel to a side of a triangle and terminated by the other sides.
4. Prove that the midpoints of two opposite sides of a general quadrilateral and the midpoints of the diagonals are vertices of a parallelogram.
5. Find the locus of the midpoints of equal chords of a circle.
6. Find the locus of centers of circles tangent to a given line at a given point.
7. Find the locus of centers of circles passing through a fixed point and having a fixed radius.
8. Locate the points equidistant from A and B, Figure 46, and 1 inch from C.

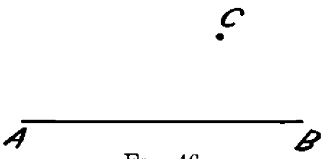


FIG. 46

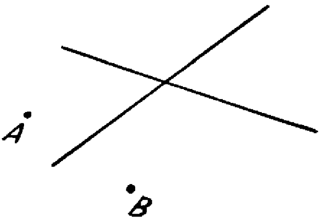


FIG. 47

9. Locate the points equidistant from A and B, Figure 47, and equidistant from the two intersecting lines.
10. Draw the locus of the centers of circles tangent to two intersecting straight lines.
11. Find the locus of centers of all circles tangent to a given circle at a given point.
12. Find a point at a given distance from a given point and equally distant from two intersecting lines.

13. Find a point equally distant from two intersecting lines and at a fixed distance from a third line.

99. 1. Prove that the perpendicular bisector of a chord of a circle passes through the center.

2. The vertices of all isosceles triangles having the same base lie on a straight line. Prove.

3. Construct an isosceles triangle, having given the base and altitude.

4. Construct an isosceles triangle, having given the altitude and one of the equal sides.

5. Find a point within an angle equidistant from both sides and at a fixed distance from them.

6. A point moves and is always twice as far from a fixed point as from a fixed line. Plot the locus of the moving point.

7. Prove that the bisector of an angle of a square passes through the opposite vertex.

111. 1. Prove that if the perpendicular bisectors of two sides of a triangle intersect on the third side the triangle is a right triangle.

2. If two medians of a triangle are equal, the triangle is isosceles. Prove.

3. The perpendicular bisector of the base of an isosceles trapezoid passes through the point of intersection of the non-parallel sides. Prove.

4. If from the vertex of an isosceles triangle two lines are drawn cutting the base in points that are equally distant from the endpoints, they form equal angles with the equal sides. Prove.

5. Prove that the angle formed by the bisectors of the base angles of an isosceles triangle is the supplement of a base angle.

6. If one base of an isosceles trapezoid is equal to one of the non-parallel sides, the diagonals bisect the angles adjacent to the other base. Prove.

7. If from the endpoints of a side of a triangle perpendiculars are drawn to the median to that side, prove that they are equal.

8. If the exterior angles at two vertices of a triangle are bisected, the bisectors form an angle which is the complement of half the interior angle at the third vertex. Prove.

9. The angles of a triangle are 74°, 56°, and 50°. Find the three angles whose common vertex is the point of intersection of the bisectors of the angles of the triangle.

10. If one of the equal sides of an isosceles triangle ABC is extended its own length through the vertex B to point D, prove that $DA \perp AC$.

11. Prove that the midpoints of the diagonals of a trapezoid lie on the line determined by the midpoints of the non-parallel sides.

12. If the medians drawn from two vertices of a triangle are extended their own length, prove that the endpoints and the third vertex lie on the same straight line.

CHAPTER VI

121. 1. In the trapezoid, Figure 48, prove $\triangle DEC \backsim \triangle AEB$.

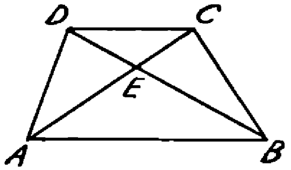


FIG. 48

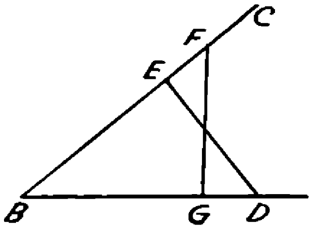


FIG. 49

2. F and D, Figure 49, are any two points on the sides of an angle. FG and DE are drawn perdicular to the sides. Prove $\triangle BFG \backsim \triangle BDE$.

3. Prove that the diagonals of a trapezoid divide each other proportionally.

4. Let D be any point on BC, Figure 50, and let E, F, and G be midpoints of DB, DA, and DC. Prove $\triangle GEF \backsim \triangle CBA$.

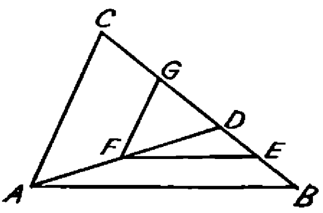


FIG. 50

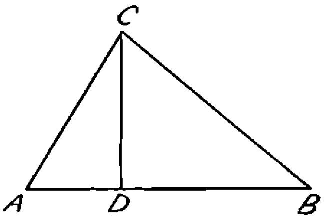


FIG. 51

5. Divide one side of a triangle in the ratio of the other two.

133. 1. $CD \perp AB$, Figure 51. Prove $\overline{AD}^2 + \overline{BC}^2 = \overline{BD}^2 + \overline{AC}^2$.

2. Construct the mean proportional between two given segments a and b, using the theorem of § 132.

   Suggestion: On an indefinite line lay off $AB=a$, the longer of the given segments. Draw a semicircle on AB as diameter. On segment BA lay off $BD=b$. At D draw $DC \perp AB$. Then CB is the required mean proportional. Prove.

3. The bases of a trapezoid are 10 and 12 inches long and the altitude is 6 inches. Produce the non-parallel sides until they meet, and compute the altitudes of the triangles thus formed.
4. Construct a segment equal to $\sqrt{6}$.
5. The hypotenuse of a right triangle is 15 inches and one of the other two sides is 12 inches. Find the altitude drawn to the hypotenuse.

CHAPTER VII

139. 1. Prove that the sum of the squares of the diagonals of a rhombus is equal to the sum of the squares of the four sides.

2. Express the altitudes of an equilateral triangle in terms of the side, and the side in terms of the altitude.
3. Find the altitude of an isosceles triangle in terms of the base and one of the equal sides.
4. Find the length of a diagonal path of a lot 33 by 125 feet.
5. A rope fastened to the top of a flag pole is 4 feet longer than the height of the pole. When stretched tight it touches the ground 12 feet from the foot of the pole. How high is the pole?

CHAPTER VIII

163. 1. Find the height of a pole if the angle of elevation from a marked point is 42° and the distance from that point to the foot of the pole 45 feet.

2. The diagonal of a rectangle makes an angle of 50° with a side which is 28 feet long. Find the other side of the rectangle.

3. The shadow of a vertical pole is 30 feet long, and the angle of elevation of the sun is 51°. Find the height of the pole.

4. If the length of the side of a regular pentagon is 28 inches, find the distance from the center to the side.

5. A 16-foot ladder leans against the wall of a house, making an angle of 72° with the ground. How far from the ground is the top point of the ladder?

6. Find the height of a building whose shadow is 20 feet when the angle of elevation of the sun is 65°.

7. A ship sailing north observes a lighthouse due west. After sailing 8 miles farther, the lighthouse is observed 40° west of south. How far is the ship from the lighthouse?

CHAPTER IX

172. 1. Draw a circle whose center is at point *A* and which passes through point *B*.

2. Draw a circle which passes through two given points. How many circles may be drawn through two points?

3. Draw a circle which passes through three given points not on a straight line. How many circles may be drawn through three points?

4. If a circle is divided into three equal arcs and the points of division are joined, prove that an equilateral triangle is formed.

5. Why is it impossible to draw a circle through three points on a straight line?

178. 1. If a circle is divided into six equal arcs and radii are drawn to the points of division, how large are the central angles?

2. Prove that the bisector of a central angle bisects the arc intercepted by its sides.

3. Prove that the arcs cut off by two intersecting diameters of a circle are equal in pairs.

4. Draw an arc and bisect it.

5. Prove that a line which passes through the mid-points of two parallel chords of a circle also passes through the center of the circle.

6. If the perpendiculars drawn from a point A on a circle to two radii are equal, prove that point A bisects the intercepted arc.

7. Through a point A inside of a circle draw a chord which shall be bisected at A.

8. Find the locus of centers of parallel chords.

9. If two angles of an inscribed triangle are equal, prove that the arcs subtended by the opposite sides are equal.

10. If $CE \parallel$ chord DB, Figure 52, and if AB is a diameter, prove that arc AD is bisected.

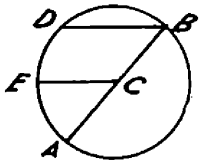


FIG. 52

11. If a central angle is made to vary from 0° to 180°, what are the corresponding changes of the intercepted arc?

179. 1. Prove that the bisector of the angle formed by two equal chords passes through the center of the circle.

2. If the angle between two chords is bisected by the diameter passing through the point of intersection, prove that the chords are equal.

3. Find the locus of midpoints of equal chords of a circle.

4. If through the endpoints of a diameter two parallel chords are drawn, prove that they are equal.

5. If $GO=OF$, Figure 53, and $HO=OE$, prove that $CD=AB$.

6. Prove that the sides of an inscribed square are equally distant from the center.

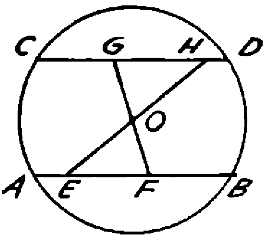


FIG. 53

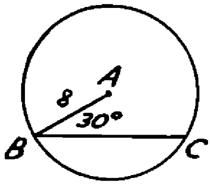


FIG. 54

7. Find the distance of the center A, Figure 54, from the chord BC.

180. 1. If in two concentric circles tangents are drawn to the small circle, prove that the chords cut off by the large circle are equal.

2. Prove that the line bisecting the angle between two tangents to a circle passes through the center.

3. Prove that the tangents drawn from a point outside of a circle are equal.

4. Prove that the line joining the points of tangency of two parallel tangents passes through the center of the circle.

5. If the sides of a quadrilateral are tangent to a circle, prove that the sums of the opposite sides are equal.

6. Construct a tangent parallel to a chord of a circle.

7. Prove that the sum of the hypotenuse of a right triangle and the diameter of the inscribed circle is equal to the sum of the sides of the right angle.

8. Two circles are tangent to each other at point A. BC is a common external tangent. Prove $\angle BAC = 90°$.

9. If AB is a diameter of circle C, Figure 55, and EF a tangent, prove that the projections of AC and CB on EF are equal.

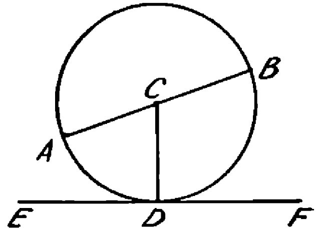


FIG. 55

10. Construct a line tangent to a given circle and parallel to a given line.

11. Prove that the bisectors of the angles of a circumscribed quadrilateral are concurrent.
12. Show that a moving straight line may be made to have in common with a fixed circle two points; one point; no point.

181. 1. $AB \parallel CD$, Figure 56. Prove $\angle COA = \angle DOB$.

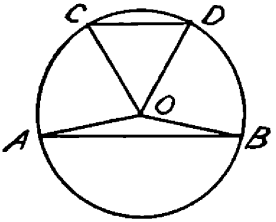


FIG. 56

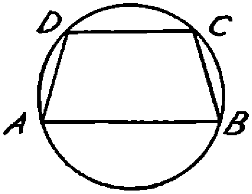


FIG. 57

2. Prove that the trapezoid, Figure 57, is isosceles.
3. Prove that if a circle is inscribed in an equilateral triangle the radius is one-third as long as the altitude.
4. If a chord is made to vary from zero to the length of a diameter, how does its arc vary? If the arc is doubled, is the chord doubled? Is the central angle doubled?
5. If the bisector of the angle formed by two secants passes through the center of the circle, prove that the circle cuts off equal chords.

182. 1. If two circles are tangent externally, prove that the two tangents drawn to the circles from any point on the common tangent are equal.
2. Draw two circles that have no common external tangent.
3. Prove that the diameter drawn to the point of tangency bisects all chords parallel to the tangent.

4. If a chord of a circle is equal to the radius, prove that its arc is one-sixth of the circle.

5. If a diameter is projected upon a secant, the projections of the endpoints of the diameter are equally distant from the center. Prove.

6. Let circle C be tangent to circle B at A, Figure 58, and AB a radius of circle B. Prove that any chord of circle B passing through the point of contact is bisected by circle C.

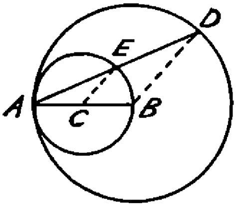


FIG. 58

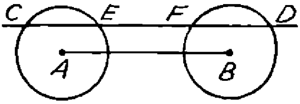


FIG. 59

7. Let circles A and B, Figure 59, be equal and let CD be parallel to the line of centers AB. Prove $CE = FD$.

186. 1. In circle A, Figure 60, $DE = FG$, and DE and FG are perpendicular to BC. Prove $AE = AG$.

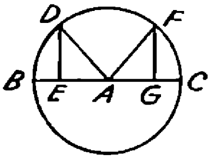


FIG. 60

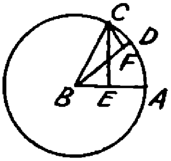


FIG. 61

2. Let BA, BC, and BD be radii of circle B, Figure 61. If $CE \perp BA$ and $CF \perp BD$, prove. $\angle FCE = \angle DBA$.

3. The distance between the centers of two non-intersecting circles is 16 inches. The radii of the circles are 8 and 6 inches. What is the length of the common external tangent?

CHAPTER X

195. 1. How large are the inscribed angles whose arcs are 46°, 22°, 90°, 110°, 131°?

2. How large are the arcs intercepted by inscribed angles of 12°, 23°, 40°, 100°?

3. Find the angles of a triangle of the arcs subtended by the sides are 94°, 110°, and 156°.

4. If two chords are equal and intersect within the circle, the segments of one are equal to the segment of the other. Prove.

5. If a circle is drawn with one side of an isosceles triangle as diameter, prove that the circle bisects the base.

6. If two opposite sides of an inscribed quadrilateral are equal, prove that the other sides are parallel.

7. Prove that the opposite angles of an inscribed quadrilateral are supplementary.

197. 1. Find the angle between a tangent and a chord drawn through the point of contact, if the intercepted arc is 98°.

2. If at the endpoints of a chord tangents are drawn to a circle, prove that they make equal angles with the chord.

3. Prove that the bisector of an angle formed by a tangent and a chord drawn at the point of con-

tact passes through the midpoint of the intercepted arc.

4. A secant drawn through the point of tangency of two tangent circles meets the circles at A and B. Prove that the tangents at A and B are parallel.
5. If AB, Figure 62, is tangent to the circle, prove $\triangle ABD \backsim \triangle ACB$.

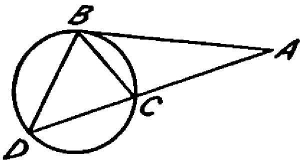


FIG. 62

6. If a circle is drawn on the hypotenuse of a right triangle as diameter, prove that it passes through the vertex of the right angle.
7. If two sides of an inscribed triangle are equal to the radius, find the number of degrees in each angle.
8. Chords AC and DE, Figure 63, intersect and make $BC = BD$. Prove $BE = BA$.

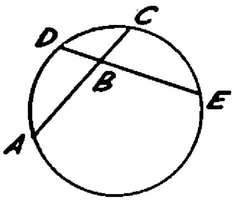


FIG. 63

9. If two chords intercept equal arcs and do not intersect within the circle they are parallel. Prove.

10. Prove that a parallelogram inscribed in a circle is a rectangle.

201. 1. Extend the diameter AB the length of the radius to C. Draw a tangent at A and from C draw two tangents to the circle meeting the first tangents at D and E. Prove that $\triangle CDE$ is equilateral.

2. Draw AB and AC tangents to circle O. Draw the diameter BOD. Draw DC. Prove $DC \parallel OA$.

3. Prove that the angle of a regular inscribed pentagon is trisected by the diagonals drawn from the vertex.

4. $OD \perp AB$, Figure 64, which is a side of an inscribed triangle. Prove that $\angle C = \angle AOD$.

5. Prove that the radius of a circle inscribed in an equilateral triangle is one-third of the altitude.

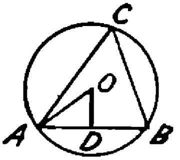


FIG. 64

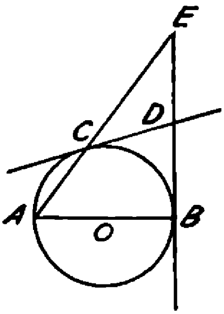


FIG. 65

6. AB is a diameter of circle O, Figure 65. AC is a chord. BD is a tangent at B and CD is a tangent at C. AC and BD meet at E. Prove that $CD = DE$.

CHAPTER XI

215. 1. Prove $\triangle ADB \sim \triangle CEB$, Figure 66.

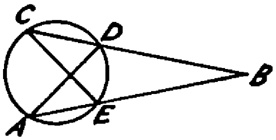


FIG. 66

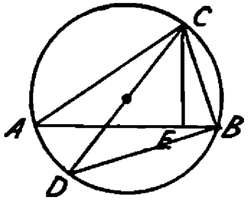


FIG. 67

2. $CE \perp AB$, Figure 67, and CD is a diameter. Prove $\triangle AEC \sim \triangle DBC$.

3. Chord AB, Figure 68, is 10 inches and the diameter CD is 18 inches. Find OE.

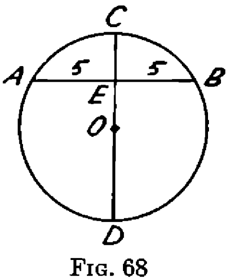


FIG. 68

4. Two circles intersect at A and B. At A tangents AC and AD are drawn to the circles. Draw AB, CB, and BD. Prove $\overline{AB}^2 = CB \cdot BD$.

5. Find the diameter of a circle if the length of the tangent from A is 10 inches and the external segment of a secant passing from A through the center is 4 inches.

6. If the chord AB, Figure 68, is 10 inches long and 3 inches from the center, find the length of the radius.

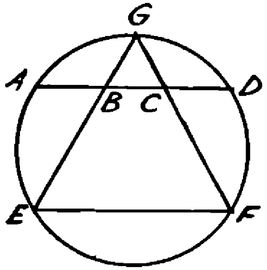


FIG. 69

7. In Figure 69, $AB=BC=CD$ and $EB=FC$. Prove $GB=GC$.

CHAPTER XII

226. 1. Find the side of a regular circumscribed hexagon whose radius is 12 feet.

2. Find the ratio of the radii of the circles inscribed in, and circumscribed about, an equilateral triangle.

3. Show that the side of a regular inscribed octagon expressed in terms of the radius of the circle is $r\sqrt{2-\sqrt{2}}$.

4. Prove that the angle at the center of a regular polygon is equal to the exterior angle of the polygon.

5. Prove that the ratio of the sides of the inscribed and circumscribed equilateral triangles is 1:2.

6. Find the sizes of the angles formed by the diagonals of a regular hexagon.

7. Prove that regular polygons having the same number of sides are similar.

8. Prove that the radius of a regular inscribed polygon drawn to a vertex bisects the angle.

9. Prove that an equiangular inscribed polygon having an odd number of sides is regular.

10. Prove that the bisectors of the angles and the perpendicular bisectors of the sides of a regular polygon are concurrent.

231. 1. A circle is inscribed in a rhombus whose side is 10 inches and equal to the shorter diagonal. Find the radius of the circle.

2. Find the central angle of a regular pentagon, a regular octagon, a regular pentadecagon.

3. Prove that an angle of a regular polygon is the supplement of the angle at the center.

4. Prove that the product of the segments of one of two intersecting diagonals of a regular polygon is equal to the product of the segments of the other.

5. Prove that the length of the side of a regular circumscribed octagon in terms of the radius is $2r(\sqrt{2}-1)$.

6. Find the ratio of the apothem of an equilateral triangle to the radius of the circumscribed circle.

7. Find the number of sides of a regular polygon whose angle is 120°.

8. Prove that the apothem of an inscribed equilateral triangle is one-half as long as the radius.

234. 1. If the radius of a circle is doubled, how is the circumference changed?

2. If the radius of a circle is increased by 4, how is the circumference changed?
3. How long is an arc of 22° if the radius of the circle is 6 inches?
4. Construct a circle whose circumference is twice as large as that of a given circle.

CHAPTER XIII

244. 1. Prove that the diagonals divide a parallelogram into four triangles having equal area.

2. If the midpoints of two adjacent sides of a parallelogram are joined, prove that the triangle formed is one-eighth of the parallelogram.
3. Prove that the parallelogram formed by joining the midpoints of the sides of a given parallelogram is one-half as large as the given parallelogram.
4. Find the area of a rhombus in terms of the diagonals by showing that the rhombus is one-half of the rectangle formed by drawing lines through the vertices parallel to the diagonals.

252. 1. The side of an equilateral triangle is 12 inches. What will be the length of the side of an equilateral triangle twice as large?

2. The sides of an isosceles triangle are 12, 12, and 8 inches. Find the area.
3. Find the area of an equilateral triangle whose altitude is 8 inches.
4. Prove that a median divides a triangle into two triangles having equal area.
5. Find the area of a rhombus whose diagonals are 12 and 18 inches.

6. Find the area of a parallelogram if one angle is 30° and the including sides are 8 and 12 inches.
7. Prove that the triangles into which the diagonals divide a parallelogram are equal.
8. Find the diagonals of a rhombus whose area is 72 square feet and whose side is 8 feet.
9. Using one side of a triangle as a base, construct a rhombus equal to the triangle.
10. Prove that the area of a square on the diameter of a circle is four times the area of an inscribed isosceles right triangle.
11. Find the ratio of an isosceles right triangle to the equilateral triangle drawn on the hypotenuse.
12. Let D and E, Figure 70, be the midpoints of AC and BC and let F be any point on AB. Prove that $ABC = 2$ times $AFED$.

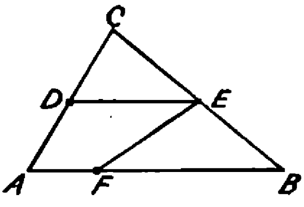


FIG. 70

13. From the endpoints of the shorter base of a trapezoid lines are drawn parallel to the non-parallel sides forming two parallelograms. Prove that they are equal.

256. 1. Prove that the area of a rhombus is one-half the product of the diagonals.

2. Find the area of a rhombus whose perimeter is 64 and whose shorter diagonal is 12.

3. Prove that the six triangles into which the medians divide a triangle have equal areas.

4. Let one of the diagonals of a quadrilateral be divided into three equal parts and let the points of trisection be joined to the two opposite vertices. Prove that the given quadrilateral is divided into three equal quadrilaterals.

5. Divide a square into four equal triangles by drawing lines through one of the vertices.

263. 1. The longest side of a triangle is 6 feet. Find the longest side of a similar triangle nine times as large.

2. If two sides of one triangle are equal to two sides of another and if the included angles are supplementary, prove that the triangles are equal in area.

3. The altitude of one equilateral triangle is equal to the side of another. Find the ratio of the altitudes.

4. The ratio of the areas of two similar polygons is 1:4. Find the ratio of the corresponding sides.

5. The ratio of the areas of two similar triangles is $\frac{1}{9}$. The sides of one are 27, 36, and 48 inches. Find the sides of the other.

6. Through any point on one side of a triangle draw a line dividing the triangle into equal parts.

265. 1. Find the locus of vertices of triangles of equal area and having a common base.

2. The midpoint of one of the non-parallel sides of a trapezoid is joined to the opposite vertices, forming a triangle. Prove that the triangle is equal to one-half of the trapezoid.

3. Transform a parallelogram into a rhombus having the same base as the parallelogram.

4. Construct a square equal to a given trapezoid.

5. Draw a line parallel to one side of a triangle which bisects the triangle.

6. Find the side of an equilateral triangle whose area is $100\sqrt{3}$.

7. Prove that the area of an inscribed equilateral triangle is one-half as large of that of the inscribed regular hexagon.

8. If a quadrilateral is formed by a diameter of a circle, the two tangents drawn at the endpoints, and a third tangent, prove that the area is one-half the product of the diameter and the side opposite.

266. 1. $AB = BC$, Figure 71. Show that the area of the shaded surface is one-half the area of the large semicircle.

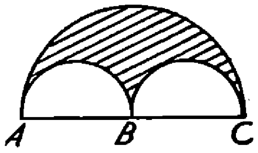


FIG. 71

2. If the area of one circle is twice as large as that of another, find the ratio of the radii.

3. If the radius of a circle is doubled, how is the area changed?

4. The diameter of one circle is equal to the radius of another. Compare the areas.

5. The altitude of an equilateral triangle is 16 inches. Find the ratio of the areas of the inscribed and circumscribed circles.

6. If the diameter of a circle is six times as great as that of another, find the ratio of the areas of the circles.

268. 1. A square is inscribed in a circle whose radius is 10 inches. What is the area of the segment of the circle cut off by one side of the square?

2. Find the area of the shaded surface in Figure 72.

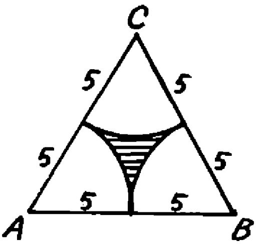


FIG. 72

3. Prove that the side of an inscribed equilateral triangle cuts off a segment whose area is $\frac{(4\pi - 3\sqrt{3})R^2}{12}$, where R is the radius of the circle.

4. Prove that the ratio of two similar sectors is equal to the ratio of the squares of the radii.

5. The angle of a sector is 72° and the area is 16π. Find the radius of the circle.

INDEX

(References are to pages)

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